POWER SERIES

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1. SEQUENCES

We denote by $\lim_{n\to\infty} a_n = a$ that the **limit** of the sequence $\{a_n\}$ is the number a. By this we mean that for any $\varepsilon > 0$ there is an integer N such that $|a_n - a| < \varepsilon$ for all integers $n \geq N$. This makes precise the statement that as n gets large, the error $|a_n - a|$ gets small. This is also denoted by "$a_n \to a$", and we say the sequence is **convergent**.

Example 1.1. *For the sequence $a_n = \frac{1}{n}$ we have $\lim_{n\to\infty} a_n = 0$. That is, the sequence $\{\frac{1}{n}\}$ is convergent and $\lim_{n\to\infty} \frac{1}{n} = 0$.*

Limits can be taken into sums, multiples and quotients (when the denominator has a non-zero limit), such as in the two following examples.

Example 1.2. *For the sequence $a_n = \frac{n}{n+1}$, we have*

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+\lim_{n\to\infty}\frac{1}{n}} = 1.$$

Or directly we can compute

$$|\frac{n}{n+1} - 1| = \frac{1}{n+1} \to 0.$$

Example 1.3. *For the sequence $a_n = \frac{3n^2-1}{n^2+n+1}$ we have*

$$a_n = \frac{3-\frac{1}{n^2}}{1+\frac{1}{n}+\frac{1}{n^2}},$$

so we have $\lim_{n\to\infty} a_n = \frac{3-0}{1+0+0} = 3$. Since we know the limit, we can check directly for convergence from the crude estimate $|a_n - 3| < \frac{3}{n} + \frac{4}{n} \to 0$.

In fact, for any function $F(\cdot)$ which is continuous at the point a, If $a_n \to a$, then $F(a_n) \to F(a)$. The two preceding examples follow from the respective cases $F(x) = \frac{1}{1+x}$ and $F(x) = \frac{3-x^2}{1+x+x^2}$ and $a_n = \frac{1}{n}$.

Note that for any number $h \geq 0$ and integer $n \geq 1$ we have

$$(1+h)^n \geq 1 + hn.$$

This estimate is useful to identify limits of powers.

Example 1.4. *To evaluate the limit $n \to \infty$ of the sequence $a_n = p^n$, if $|p| < 1$, we set $|p| = \frac{1}{(1+h)}$ for some $h > 0$ to see that*

$$|a_n| = \frac{1}{(1+h)^n} < \frac{1}{1+nh} \to 0,$$

and if $|p| > 1$, we set $|p| = h + 1$ with $h > 0$ to get $|p^n| = (1+h)^n \geq 1 + hn \to \infty$ as $n \to \infty$. In summary, we have

$$\lim_{n\to\infty} p^n = \begin{cases} 0 & \textit{if } |p| < 1, \\ 1 & \textit{if } p = 1, \end{cases}$$

and the sequence does not converge otherwise.

Example 1.5. *For the sequence of fractional powers,*

$$a_n = p^{\frac{1}{n}},$$

we proceed similarly. If $p > 1$, then set $p^{\frac{1}{n}} = 1 + h_n$ so that $p = (1+h_n)^n > 1 + nh_n$ and $0 < h_n < \frac{p-1}{n}$. This shows that $\lim\limits_{n\to\infty} p^{\frac{1}{n}} = 1$. If $0 < p < 1$, then $p = \frac{1}{q}$, $q > 1$, so $\lim\limits_{n\to\infty} p^{\frac{1}{n}} = \frac{1}{\lim\limits_{n\to\infty} q^{\frac{1}{n}}} = \frac{1}{1} = 1$. In summary, we have $\lim\limits_{n\to\infty} p^{\frac{1}{n}} = 1$ for all $0 < p$.

Here's a more delicate one that will arise in applications.

Example 1.6. *For the sequence $a_n = n^{\frac{1}{n}}$, we write $\sqrt{a_n} = (\sqrt{n})^{\frac{1}{n}} = 1 + h_n$, and then $\sqrt{n} = (1+h_n)^n \geq 1 + nh_n$, so that we have $h_n \leq \frac{\sqrt{n}-1}{n} < \frac{1}{\sqrt{n}} \to 0$. This shows that*

$$1 \leq a_n \leq 1 + 2h_n + h_n^2 \leq 1 + \frac{2}{\sqrt{n}} + \frac{1}{n},$$

and we see that $a_n \to 1$. That is, $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$.

Even though the base n is growing in this example, the fractional power $\frac{1}{n}$ still brings the sequence to 1. And this happens even with higher powers of the base: for any fixed integer $M \geq 1$ we have

$$\lim_{n\to\infty} (n^M)^{\frac{1}{n}} = \lim_{n\to\infty} n^{\frac{M}{n}} = \big(\lim_{n\to\infty} n^{\frac{1}{n}}\big)^M = 1.$$

Cauchy Criterion. If the series $\{a_n\}$ is convergent with $\lim_{n\to\infty} a_n = a$, then for any pair of integers m, n we get $|a_m - a_n| \leq |a_m - a| + |a_n - a|$, so for both of m, n sufficiently large, it follows that $|a_m - a_n|$ is arbitrarily small. The sequence is **Cauchy** if for any $\varepsilon > 0$ there is an integer N such that for all pairs of integers $m,\ n \geq N$ we have $|a_m - a_n| < \varepsilon$. In particular, we just noted that every convergent sequence is Cauchy. A fundamental property of the real numbers is that every Cauchy sequence is convergent. This provides a useful test for convergence that does not depend on knowing the limit of the sequence.

2. SERIES

Let $\{a_n\}$ be a sequence. Then define a new sequence $\{s_n\}$ by

$$s_n = \sum_{m=1}^{n} a_m = a_1 + a_2 + \cdots + a_n,\ n \geq 1.$$

This is the *sequence of partial sums* of $\{a_n\}$ or the **series** $\sum_{n=1}^{\infty} a_n$, and a_n is the **n-th term** of the series. If the sequence $\{s_n\}$ is convergent, we say the series $\sum_{n=1}^{\infty} a_n$ converges and denote its limit also by $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$. It follows from the Cauchy test for convergence of the sequence $\{s_n\}$ that the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if for any $\varepsilon > 0$ there is an N such that

$$|s_m - s_n| = |a_{n+1} + a_{n+2} + \cdots + a_m| < \varepsilon$$

for all $m \geq n \geq N$. Finally, we note that if the series converges, then we necessarily have $a_n = s_n - s_{n-1} \to s - s = 0$, so the sequence of terms $\{a_n\}$ converges to 0.

The most important example is the **geometric series** obtained from the terms $a_n = p^n$.

Example 2.1. *The sequence of partial sums is*

$$s_m = 1 + p^1 + p^2 + p^3 + \cdots + p^m,\ m \geq 1.$$

Then we compute $s_m - ps_m = 1 - p^{m+1}$ *to get*

$$s_m = \begin{cases} m+1 & \text{if } p = 1, \\ \frac{1-p^{m+1}}{1-p} & \text{if } p \neq 1. \end{cases}$$

This shows that the series converges to the limit $\sum_{n=0}^{\infty} p^n = \frac{1}{1-p}$ *if* $|p| < 1$, *and it is not convergent otherwise.*

A series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent. The Cauchy criterion for convergence together with the inequality

$$|a_{n+1} + a_{n+2} + \cdots + a_m| \leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_m|$$

shows that absolute convergence implies convergence of the series. The series is **conditionally convergent** if it is convergent but *not* absolutely convergent. Examples will be given below.

Convergence Tests.

Theorem 2.1. Comparison Test: *If there is a constant $C \geq 0$ for which $|a_n| \leq Cb_n$ for all n sufficiently large, and if $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.*

Proof. Use the estimate above with the Cauchy test for convergence of the two series. □

For example, by comparing with the geometric series, it follows that if there is a constant $C \geq 0$ and integer $N \geq 1$ for which $|a_n| \leq Cp^n$ for all $n \geq N$ for some $0 \leq p < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Sufficient conditions are given by the following.

Corollary 2.2. Ratio Test: *If $|\frac{a_{n+1}}{a_n}| \leq p < 1,\ n \geq N$, or if $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| \leq p < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.*

Corollary 2.3. Root Test: *If $|a_n|^{\frac{1}{n}} \leq p < 1,\ n \geq N$, or if $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} \leq p < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.*

Integral Test. Our last criterion for convergence of series with non-negative terms is obtained by comparing with an improper integral. Suppose the function $f(\cdot)$ is continuous, decreasing, and $f(x) \geq 0$ for all $x \geq 0$. Then for $n \leq x \leq n+1$ we have $f(n) \geq \int_n^{n+1} f(x)dx \geq f(n+1)$ so we obtain

$$\sum_{n=1}^{m} f(n) \geq \int_1^{m+1} f(x)dx \geq \sum_{n=1}^{m} f(n+1)$$

This shows that the improper integral $\int_1^{\infty} f(x)dx$ converges if and only if the series $\sum_{n=1}^{\infty} f(n) = f(1) + \sum_{n=1}^{\infty} f(n+1)$ converges. This criterion is the **integral test**.

Example 2.2. *By taking the function $f(x) = \frac{1}{x^{\alpha}}$, we find the series $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ converges if and only if $\alpha > 1$.*

Alternating Series. The series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ with $b_n \geq 0$ is an **alternating series**. That is, successive terms $a_n = (-1)^{n+1}b_n$ alternate sign. If also the terms are decreasing, $b_{n+1} \leq b_n$, then we can arrange the terms in two ways

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1}b_n &= (b_1 - b_2) + (b_3 - b_4) + (b_5 - b_6) + \dots \\ &= b_1 - (b_2 - b_3) - (b_4 - b_5) - (b_6 - b_7) - \dots\end{aligned}$$

to see that

$$\begin{aligned} s_2 \leq s_4 \leq s_6 \leq \dots \text{ and} \\ s_1 \geq s_3 \geq s_5 \geq \dots . \end{aligned}$$

These show that the odd terms are decreasing and they lie above the even terms which are increasing. Finally we note that $s_{2^n+1} - s_{2^n} = b_{2^n+1}$, so if $\lim_{n\to\infty} b_n = 0$, then these two sequences converge to the common value which is $\lim_{n\to\infty} s_n$. We summarize this as

Theorem 2.4. *An alternating series* $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ *with* $b_n \geq 0$, $b_{n+1} \leq b_n$ *and* $\lim_{n\to\infty} b_n = 0$ *is convergent.*

Example 2.3. *The* **harmonic series**

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

is convergent (alternating series) but not absolutely convergent (by the integral test).

3. Sequences and Series of Functions

Let $\{f_n(\cdot)\}$ be a sequence of functions on a set of numbers S. This sequence is **pointwise convergent** to a function $f(\cdot)$ on S if

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ for every } x \in S.$$

That is, for every $x \in S$ and $\varepsilon > 0$, there is an integer N for which $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$. (The integer N depends on ε and on x.)

The sequence $\{f_n(\cdot)\}$ is **uniformly convergent** to $f(\cdot)$ on S if for every $\varepsilon > 0$ there is an integer N for which $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ and for all $x \in S$. (The integer N depends on ε.)

Example 3.1. *The sequence* $f_n(x) = x^n$ *converges pointwise to* $f(x) = 0$ *on the set* $S = (-1, 1)$. *The convergence is uniform on* $S = (-p, p)$ *for any* p *with* $0 < p < 1$. *See Example 1.4.*

Example 3.2. *The sequence* $f_n(x) = x^{\frac{1}{n}}$ *converges pointwise to* $f(x) = 1$ *on* $(0,1)$*, but the convergence is* not uniform*. The convergence is uniform on any set of the form* $S = (\alpha, 1)$ *with* $0 < \alpha < 1$ *or* $S = [\alpha, 1]$*. See Example 1.5.*

Theorem 3.1. *If the sequence* $\{f_n(\cdot)\}$ *is* **uniformly convergent** *to* $f(\cdot)$ *on* S *and if each* $f_n(\cdot)$ *is continuous, then the limit* $f(\cdot)$ *is continuous on* S*.*

Proof. Let $\lim_{n\to\infty} f_n(\cdot) = f(\cdot)$ uniformly on S and $x_0 \in S$. Let $\varepsilon > 0$. Uniform convergence implies there is an N for which $|f_N(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in S$. Continuity of f_N implies that there is a $\delta > 0$ such that $|f_N(x) - f_N(x_0)| < \frac{\varepsilon}{3}$ for all $x \in S$ with $|x - x_0| < \delta$. But then we have

$$|f(x) - f(x_0)| \le |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| < \varepsilon$$

for all $x \in S$ with $|x - x_0| < \delta$. □

Theorem 3.2. *If the sequence of continuous functions* $\{f_n(\cdot)\}$ *is uniformly convergent to the (continuous) function* $f(\cdot)$ *on* $S = [a, b]$*, then we have*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Proof. Let $\varepsilon > 0$. There is an N for which $n \ge N$ implies $|f_n(x) - f(x)| < \frac{\varepsilon}{b-a}$ for all $x \in [a, b]$. Then

$$|\int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx| \le \int_a^b |f(x) - f_n(x)|\,dx < \varepsilon$$

for $n \ge N$. □

Corollary 3.3. *If the sequence of continuous derivatives* $\{f_n'(\cdot)\}$ *is uniformly convergent to the (continuous) function* $g(\cdot)$ *on* $S = [a, b]$*, and if the sequence* $\{f_n(\cdot)\}$ *converges pointwise to* $f(\cdot)$ *on* $S = [a, b]$*, then* $f(\cdot)$ *is differentiable and* $f' = g$*.*

Proof. For each $x \in (a, b]$ we have $\int_a^x f_n'(s)\,ds = f_n(x) - f_n(a)$, and taking limits yields $\int_a^x g(s)\,ds = f(x) - f(a)$ □

Let $\{f_n(\cdot)\}$ be a sequence of functions on the set S. As before, we define a new sequence $\{s_n(\cdot)\}$ by

$$s_n(\cdot) = \sum_{m=1}^{n} f_m(\cdot) = f_1(\cdot) + f_2(\cdot) + \cdots + f_n(\cdot),\ n \ge 1,\ x \in S.$$

This is the *sequence of partial sums* of $\{f_n(\cdot)\}$ or the **series** $\sum_{n=1}^{\infty} f_n$. If the sequence $\{s_n\}$ is pointwise (or uniformly) convergent, we say the series $\sum_{n=1}^{\infty} f_n$ converges pointwise (or uniformly, respectively) and denote its limit also by $\sum_{n=1}^{\infty} f_n(x) = \lim_{n\to\infty} s_n(x)$.

Example 3.3. *Taking the sequence* $f_n(x) = x^n$, *we obtain the* **geometric series** $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ *on* $(-1, 1)$. *The series is absolutely convergent, pointwise on* $(-1, 1)$, *and uniformly on any subinterval* $[a, b] \subset (-1, 1)$.

4. Power Series

Definition 4.1. *An infinite series of the form*

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \dots \tag{4.1}$$

is a **power series** *in* x *about the point* x_0.

This is a series of functions constructed from the terms $f_n(x) = a_n(x-x_0)^n$ for $n \geq 0$. The geometric series resulted from the particular choice of coefficients $a_n = 1$ and $x_0 = 0$.

Theorem 4.2. *If the power series* (4.1) *converges at the point* $x = x_0 + r$, *then it converges absolutely at any point* x *with* $|x - x_0| < |r|$, *and for any* p *with* $0 < p < |r|$ *the convergence is uniform on those* x *with* $|x - x_0| \leq p$.

Proof. Since the series $\sum_{n=0}^{\infty} a_n r^n$ is convergent, we have $\lim_{n\to\infty} a_n r^n = 0$, so there is an integer N such that $|a_n r^n| < 1$ for all $n \geq N$. Thus, for all $n \geq N$ we have $|a_n(x-x_0)^n| = |a_n r^n||\frac{x-x_0}{r}|^n < |\frac{x-x_0}{r}|^n$, so by the comparison test we see that the series (4.1) converges absolutely for all x with $|x - x_0| < |r|$. Moreover, these estimates show the convergence is uniform for $|x - x_0| \leq p$ for any $p < |r|$. □

It follows that the set of points at which the series converges is either the single point 0, an interval $(x_0 - R, x_0 + R)$, possibly containing either endpoint, or the entire number line $\mathbb{R} = (-\infty, \infty)$. The number R is the *radius of convergence*, and we set $R = 0$ in the first case and $R = \infty$ in the last.

Theorem 4.3. *Let* $R > 0$ *be the radius of convergence of the power series* (4.1). *Then the function* $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ *is infinitely differentiable and its derivative is given by the power series* $f'(x) = \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1}$ *with the same radius of convergence.*

Proof. Let $|x - x_0| < R$ and choose ξ in the interval of convergence, ie., $|\xi - x_0| < R$, with $\frac{R}{2} < |\xi - x_0|$ and $|x - x_0| = p|\xi - x_0|$ with $0 \leq p < 1$. Then the differentiated series is bounded by

$$|na_n(x - x_0)^{n-1}| = |na_n(\xi - x_0)^{n-1}|p^{n-1} \leq \frac{2}{R}Cnp^{n-1}$$

since the convergent series $\sum_{n=0}^{\infty} a_n(\xi - x_0)^n$ has bounded terms. The series $\sum_{n=0}^{\infty} np^{n-1}$ converges by the limit ratio test, so the differentiated series converges for $|x - x_0| < R$ by the comparison test. □

Corollary 4.4 (Taylor's formula). *If the power series* (4.1) *has radius of convergence $R > 0$ and its limit is the function $f(x)$, that is,*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{4.2}$$
$$= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \dots,\ |x - x_0| < R,$$

then the n-th derivative of the sum $f(x)$ at x_0 is given by

$$f^{(n)}(x_0) = n! a_n,\ n \geq 0.$$

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