THE DIFFUSION EQUATION

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1. Heat Conduction in an Interval

We shall describe the diffusion of heat energy through a long thin rod G with uniform cross section S. As before, we identify G with the open interval (a, b), and we assume the rod is perfectly insulated along its length. Let $u(x, t)$ denote the temperature within the rod at a point $x \in G$ and at time $t > 0$. Our previous experience shows that the *heat flux* $q(x, t)$ at the point x, *i.e.*, the flow rate to the right per unit area, is proportional to the temperature gradient $\frac{\partial u}{\partial x}$. The precise statement of this experimental fact is *Fourier's law* of heat conduction,

$$q(x,t) = -k(x)\frac{\partial u}{\partial x}(x,t), \tag{1}$$

and this equation defines the *thermal conductivity* $k(x)$ of the material at the point $x \in (a, b)$. Since heat flows in the direction of decreasing temperature, we see again that the minus sign is appropriate.

The amount of heat stored in the rod within a section $[x, x + h]$ with $h > 0$ is given by

$$\int_x^{x+h} c(s)\rho(s)Su(s,t)\,ds,$$

where $c(\cdot)$ is the *specific heat* of the material, and $\rho(\cdot)$ is the volume-distributed *density*. The specific heat provides a measure of the amount of heat energy required to raise the temperature of a unit mass of the material by a degree.

Now, equating the rate at which heat is stored within the section to the rate at which heat flows into the section plus the rate at which heat is generated within this section, we arrive at the *conservation of energy* equation for the section $[x, x + h]$

$$\frac{\partial}{\partial t}\int_x^{x+h} c(s)\rho(s)Su(s,t)\,ds = S\big(q(x,t) - q(x+h,t)\big) + \int_x^{x+h} f(s,t)S\,ds$$

where $f(x, t)$ represents the rate at which heat is generated per unit volume. This heat generation term is assumed to be a known function of space and time. Dividing this by Sh and letting $h \to 0$ yields the *conservation of energy* equation

$$c(x)\rho(x)\frac{\partial u}{\partial t}(x,t) + \frac{\partial q}{\partial x}(x,t) = f(x,t). \tag{2}$$

Finally, by substituting the Fourier law (1) into the energy conservation law (2), we obtain the one–dimensional heat conduction equation

$$c(x)\rho(x)\frac{\partial u}{\partial t}(x,t) - \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}(x,t)\right) = f(x,t). \tag{3}$$

This is also known as the *diffusion equation.*

If we assume the material properties $c(\cdot), \rho(\cdot)$, and $k(\cdot)$ are constants, then equation (3) may be written in the form

$$\frac{\partial u}{\partial t}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = \frac{1}{c\rho} f(x,t), \qquad x \in G,\ t > 0, \tag{4}$$

where $\alpha^2 \equiv \frac{k}{c\rho}$ is called the *thermal diffusivity* of the material and is a measure of the rate of change of temperature of the material. For example, if the material is made out of a substance with a high thermal conductivity and low heat capacity $c\rho$, then the material will react very quickly to transient external conditions.

1.1. **Initial and Boundary conditions.** Since the heat equation is second–order in space and first–order in time, one may expect that in order to have a well–posed problem, two boundary conditions and one initial condition should be specified. We shall see that this is true here.

We want to find a solution of (3) which satisfies an *initial condition* of the form

$$u(x,0) = u_0(x), \quad a < x < b,$$

where $u_0(\cdot)$ is given. Following the outline in Chapter 1, we describe some examples of appropriate *boundary conditions.* Each of these will be illustrated with a condition at the right end point, $x = b$, and we note that another such condition will be prescribed at the left end point, $x = a$.

1. Dirichlet Boundary Conditions

One can specify the value of the temperature at an end point:

$$u(b,t) = d_b(t), \quad t > 0\,,$$

where $d_b(\cdot)$ is given. This type of boundary condition describes *perfect contact* with the boundary value, and it arises when the value of the end point temperature is known (usually from a direct measurement). Such a condition arises when one sets the boundary temperature to a prescribed value, for example, $d_b(t) = 0$ when the end point is submerged in ice-water.

2. Neumann Boundary Conditions

One can specify the heat flux into the rod at an end point:

$$k\frac{\partial u}{\partial x}(b,t)S = f_b(t), \quad t > 0\,,$$

where $f_b(\cdot)$ is given. This type of boundary condition corresponds to a known heat *source* $f_b(\cdot)$ at the end. The homogeneous case $f_b(t) = 0$ occurs at an *insulated* end point.

3. Robin Boundary Conditions

The heat flux is assumed to be lost through the end at a rate proportional to the difference between the inside and outside temperatures:

$$k\frac{\partial u}{\partial x}(b,t)S + k_b(u(b,t) - d_b(t))S = f_b(t), \quad t > 0\,.$$

Such a boundary condition arises from a *partially insulated* end point, and it corresponds to Newton's law of cooling at the end point $x = b$. This is just the discrete form of

Fourier's law. Here, both $d_b(\cdot)$ and $f_b(\cdot)$ are given functions. The first is the outside temperature and the second is a heat source concentrated on the end. Note that the first two boundary conditions can be formally obtained as extreme cases of the Robin boundary condition; that is, as $k_b \to \infty$, $u(b,t) \to d_b(t)$, which formally yields the Dirichlet boundary condition, and as $k_b \to 0$ we similarly obtain the Neumann boundary condition. Thus, this third type of boundary condition is an interpolation between the first two types for intermediate values of k_b.

4. Dynamic Boundary Conditions

As in the discrete case, another type of boundary condition arises when we assume the end of the rod has an *effective specific heat* given by $c_0 > 0$: this is a *concentrated capacity* at the end. For example, the end of the rod can be capped by a piece of material whose conductivity is very high, so the entire piece is essentally at the same temperature, or the end is submerged in an insulated container of well-stirred water. If we assume the heat energy is supplied to the concentrated capacity by the flux from the interior of the rod and supplemented by a given heat source located at that end point, $f_b(t)$, then we are led to the boundary condition

$$c_0 u_t(b,t) + k u_x(b,t) S = f_b(t), \qquad t > 0.$$

Implicit in this boundary condition is the assumption that we have "perfect" contact between the end of the rod and the concentrated capacity. If we permit some insulation between the end point and the concentrated capacity, then the temperature $u_c(\cdot)$ of the concentrated capacity is different from that of the endpoint of the rod, and the boundary condition is

$$\begin{cases} c_0 \frac{du_c(t)}{dt} + k_b(u_c(t) - u(b,t)) = f_b(t), & t > 0, \\ k\frac{\partial u}{\partial x}(b,t) + k_b(u(b,t) - u_c(t)) = 0. \end{cases}$$

Here we have an extra condition, but we have also introduced an additional unknown, so this pair of equations should be regarded as a single constraint. Note that, as $k_b \to \infty$, $u(b,t) \to u_c(t)$, and this partially insulated dynamic boundary condition formally yields the "perfect" contact dynamic boundary condition.

Each of the preceding boundary conditions must be supplemented with an additional boundary condition at the other endpoint, $x = a$, and the two boundary conditions need not be of the same type. These two boundary constraints together with the initial condition on a solution of (3) will comprise a well–posed problem.

5. Nonlocal Boundary Conditions

If the rod is bent around and the ends are joined to form a large ring, then at the endpoints we must match the temperature and the flux:

$$u(a,t) = u(b,t), \quad t > 0,$$
$$k\frac{\partial u}{\partial x}(a,t)S = k\frac{\partial u}{\partial x}(b,t)S\,.$$

These are called *periodic* boundary conditions. Note that again we have a total of two boundary constraints for the problem, but here the constraints depend on the solution at more than a single point.

Another example arises in the case that we submerge the entire rod into a bath of well-stirred water and assume the end points of the rod are in perfect contact with the water. Then we obtain the *dynamic and nonlocal* boundary conditions

$$u(a,t) = u(b,t) = u_c(t), \quad t > 0,$$
$$c_0 \dot{u}_c(t) + k(b)\frac{\partial u}{\partial x}(b,t) - k(a)\frac{\partial u}{\partial x}(a,t) = 0,$$

where the common value of the endpoint temperatures $u_c(t)$ is unknown. Note that we have *not* introduced an additional unknown, so these two equations provide the two boundary conditions.

Exercise 1. *Assume the rod G is submerged in a perfectly insulated container of well-stirred water. The rod is partially insulated along its length, so there is some limited heat exchange with the surrounding water along the length, $a < x < b$. Also, assume the rod is in perfect contact with the water at the end points. Find an initial-boundary-value problem which models this situation.*

2. The Eigenfunction Expansion Method

We shall illustrate the method of *separation of variables* by obtaining the eigenfunction expansion of the solution of an initial-boundary-value problem with Dirichlet boundary conditions. The same method works for the other boundary conditions.

Example 1. *Suppose the rod $G = (0, \ell)$ is perfectly insulated along its length and made of an isotropic material with thermal diffusivity α^2. Assuming no internal heat sources or sinks,* i.e., *$f(x,t) = 0$, suppose both ends of the rod are held at a fixed temperature of zero and the initial temperature distribution is given by $u_0(x)$. The initial-boundary-value problem for this scenario is*

$$\frac{\partial u}{\partial t}(x,t) = \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t), \quad 0 < x < \ell, \quad t > 0, \tag{5a}$$

$$u(0,t) = 0, \ u(\ell,t) = 0, \qquad t > 0, \tag{5b}$$

$$u(x,0) = u_0(x), \quad 0 < x < \ell. \tag{5c}$$

Exercise 2. *Let $u(x,t)$ be a solution of the initial-boundary-value problem and show that*

$$\frac{d}{dt}\int_0^\ell u^2(x,t)\,dx \le 0.$$

Show that this implies there is at most one *solution of the problem.*

We begin by looking for a non-null solution of the form

$$u(x,t) = X(x)T(t). \tag{6}$$

Substituting (6) into (5a) and dividing by u yields

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}, \qquad \text{for all } x \text{ and } t.$$

Note that the left side of this last equation is exclusively a function of t, while the right side of is exclusively a function of x. The only way this equation can hold for all values of x and t is for each side to equal a common constant. Denoting this constant by $-\lambda$ leads to the pair of ordinary differential equations

$$\begin{aligned} T'(t) &+ \lambda\alpha^2 T(t) = 0, \quad t > 0, \\ X''(x) &+ \lambda X(x) = 0, \quad 0 < x < \ell. \end{aligned}$$

The boundary conditions given in (5b) imply that $X(0) = X(\ell) = 0$.

Note that if $X(\cdot)$ and $T(\cdot)$ are solutions of these respective equations, then it follows directly that their product is a solution of (5a). The first of these ordinary differential equations has the solution $T(t) = e^{-\lambda\alpha^2 t}$. Thus, it remains to find a non-null solution of the boundary-value problem

$$\begin{cases} X''(x) + \lambda X(x) = 0, \quad 0 < x < \ell, \\ X(0) = 0, \ X(\ell) = 0. \end{cases} \tag{7}$$

This is a *"regular" Sturm-Liouville boundary-value problem* and we will see later that such problems have very special properties. Since this is a linear equation with constant coefficients, we can explicitly write down all possible solutions, and they depend on the sign of λ. First we check that for the cases of $\lambda < 0$ and $\lambda = 0$, the only solution of the boundary-value problem (7) is the null solution. For the case of $\lambda > 0$, we get the general solution of the differential equation in the form

$$X(x) = C_1 \sin(\sqrt{\lambda}x) + C_2 \cos(\sqrt{\lambda}x),$$

and then from the boundary conditions we see that necessarily

$$C_2 = 0, \quad \text{and} \quad C_1 \sin(\ell\sqrt{\lambda}) = 0,$$

respectively. Since $\sin(\ell\sqrt{\lambda}) = 0$ has solutions $\lambda_n = (n\pi/\ell)^2$, this does *not* force $C_1 = 0$. These specific values for λ are called the *eigenvalues* of the regular Sturm-Liouville problem (7), and the solutions to (7), namely, multiples of

$$X_n(x) = \sin(\sqrt{\lambda_n}x),$$

are the corresponding *eigenfunctions*. If we combine these with the corresponding time-dependent solutions $T_n(t) = e^{-\lambda_n\alpha^2 t}$, we obtain solutions $e^{-\lambda_n\alpha^2 t}\sin(\sqrt{\lambda_n}x)$ of (5a) and (5b). From the superposition principle we obtain a large class of solutions of (5a) and (5b) in the form

$$u(x,t) = \sum_{n=1}^{N} A_n e^{-\lambda_n\alpha^2 t} X_n(x), \tag{8}$$

for any integer N. We check directly that (8) satisfies (5a) and (5b) for any choice of the coefficients $\{A_n\}$. In order to satisfy the initial condition (5c), the coefficients must be chosen to satisfy

$$u_0(x) = \sum_{n=1}^{N} A_n X_n(x). \tag{9}$$

Now this is a *severe* restriction on the initial data, but we shall find that we can go to a corresponding series with $N = +\infty$, and then there is essentially *no restriction* on the initial data! This will follow from the observation that the corresponding coefficients in (8) have exponentially decaying factors that make the series converge extremely rapidly for $t > 0$.

Let's take a preliminary look at the boundary-value problem (7). We have denoted its non-null solutions by $X_n(\cdot),\ \lambda_n,\ n \geq 1$ First we compute

$$(\lambda_m - \lambda_n)\int_0^\ell X_m(x)X_n(x)\,dx = -\int_0^\ell \big(X_m''(x)X_n(x) - X_m(x)X_n''(x)\big)\,dx$$
$$= -\int_0^\ell \frac{d}{dx}\big(X_m'(x)X_n(x) - X_m(x)X_n'(x)\big)\,dx = 0\,.$$

Since $\lambda_m \neq \lambda_n$ for $m \neq n$, this shows that the eigenfunctions $X_n(\cdot)$ are orthogonal with respect to the scalar-product $(\cdot,\cdot) \equiv \int_0^\ell\big(\cdot,\cdot\big)\,dx$ on the linear space of continuous functions on the interval $[0,\ell]$. By replacing each such $X_n(\cdot)$ by the function obtained by dividing it by the corresponding *norm* $\|X_n(\cdot)\| = (X_n(\cdot),X_n(\cdot))^{\frac{1}{2}}$, we obtain an *orthonormal* set of functions in that space. That is, we have

$$(X_m(\cdot),X_n(\cdot)) = \delta_{mn} \text{ for } m,n \geq 1\,,$$

where we have scaled the eigenfunctions to get the *normalized eigenfunctions*

$$X_n(x) = \sqrt{\frac{2}{\ell}}\sin(\frac{n\pi}{\ell}x).$$

Now it is clear how to choose the coefficients A_n in (9): take the scalar product of that equation with $X_m(\cdot)$ to obtain

$$(u_0(\cdot),X_m(\cdot)) = A_m, \quad m \geq 1\,.$$

Thus we obtain

$$u(x,t) = \sum_{n=1}^{N} e^{-\lambda_n\alpha^2 t}(u_0(\cdot),X_n(\cdot))X_n(x), \tag{10}$$

when $u_0(\cdot)$ is appropriately restricted. We shall see below that we can go to the corresponding series with $N = +\infty$ as indicated with essentially no restriction on $u_0(\cdot)$.

Exercise 3. *Given $m,n \in \mathbb{N}$, show directly that*

(a) $\displaystyle\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = \begin{cases} 0, & m \neq n, \\ \frac{1}{2}, & m = n. \end{cases}$

(b) $\displaystyle\int_0^1 \cos(m\pi x)\cos(n\pi x)\,dx = \begin{cases} 0, & m \neq n, \\ \frac{1}{2}, & m = n. \end{cases}$

(c) $\displaystyle\int_0^1 \sin(m\pi x)\cos(n\pi x)\,dx = 0\,.$

The technique used in Example 1 is called the *method of separation of variables.* Since it depends on superposition, this technique is appropriate for solving linear initial-boundary-value problems with homogeneous boundary conditions. It can easily be modified to solve initial-boundary-value problems that contain *constant* non-homogeneous boundary conditions. In particular, for the problem

$$\begin{cases} u_t(x,t) = \alpha^2 u_{xx}(x,t), & 0 < x < \ell, t > 0, \\ u(0,t) = d_0,\ u(\ell,t) = d_\ell, & t > 0, \\ u(x,0) = u_0(x), & 0 < x < \ell, \end{cases} \tag{11}$$

define $w(x,t) = u(x,t) - \big(d_0 \frac{\ell - x}{\ell} + d_\ell \frac{x}{\ell}\big)$ and transform the above problem into an equivalent initial-boundary-value problem for $w(x,t)$. Note that $w(0,t) = w(\ell,t) = 0$, and so now we have a problem with homogeneous boundary conditions as in Example 1.

Exercise 4. *Compute the solution of* (11) *for the case of* $u_0(\cdot) = 0$*,* $d_0 = 0$ *and* $d_\ell = 1$*.*

However, if the boundary values are given by a pair of time dependent functions, $d_0(t)$, $d_\ell(t)$, then we are led to a non-homogeneous partial differential equation. More generally, we can start with a *non-homogeneous* initial-boundary-value problem of the form

$$\begin{aligned} u_t(x,t) &= \alpha^2 u_{xx}(x,t) + f(x,t), & 0 < x < \ell, \quad t > 0, \\ u(0,t) = d_0(t),\ u(\ell,t) &= d_\ell(t), & t > 0, \\ u(x,0) &= u_0(x), & 0 < x < \ell\,, \end{aligned}$$

and then define $w(x,t) = u(x,t) - \big(d_0(t)\frac{\ell - x}{\ell} + d_\ell(t)\frac{x}{\ell}\big)$ to transform the above problem into an equivalent initial-boundary-value problem of the form

$$\begin{aligned} w_t(x,t) &= \alpha^2 w_{xx}(x,t) + \tilde{f}(x,t), & 0 < x < \ell, \quad t > 0, \\ w(0,t) = 0,\ w(\ell,t) &= 0, & t > 0, \\ w(x,0) &= w_0(x), & 0 < x < \ell, \end{aligned}$$

where $w_0(x) = u_0(x) - \big(d_\ell(0)\frac{x}{\ell} + d_0(0)\frac{\ell - x}{\ell}\big)$ and $\tilde{f}(x,t) = f(x,t) - \big(d'_\ell(t)\frac{x}{\ell} + d'_0(t)\frac{\ell - x}{\ell}\big)$. Thus, by such a change of variable, we can always reduce the initial-boundary-value problem to the form with homogeneous boundary conditions.

Example 2. *Find the* eigenfunction expansion *of the solution of the initial-boundary-value problem*

$$u_t(x,t) = \alpha^2 u_{xx}(x,t) + f(x,t), \qquad 0 < x < \ell, \quad t > 0, \tag{14a}$$

$$u(0,t) = 0,\ u(\ell,t) = 0, \qquad t > 0, \tag{14b}$$

$$u(x,0) = u_0(x), \qquad 0 < x < \ell, \tag{14c}$$

with non-homogeneous partial differential equation and *homogeneous boundary conditions.* Recall that for $f(x,t) = 0$, the solution to this case was given by equation (8). For the problem with a non-homogeneous partial differential equation (14a), we look for the solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x). \tag{15}$$

If we assume that both $f(x,t)$ and $u_0(x)$ also have eigenfunction expansions given by

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) X_n(x) \quad \text{and} \quad u_0(x) = \sum_{n=1}^{\infty} u_0^n X_n(x),$$

respectively, where

$$f_n(t) \equiv \int_0^\ell f(\zeta,t) X_n(\zeta)\, d\zeta \quad \text{and} \quad u_0^n \equiv \int_0^\ell u_0(\zeta) X_n(\zeta)\, d\zeta,$$

then substituting each of these expansions into equation (14a) yields

$$\sum_{n=1}^{\infty} \left[\dot{u}_n(t) + \lambda_n \alpha^2 u_n(t)\right] X_n(x) = \sum_{n=1}^{\infty} f_n(t) X_n(x).$$

By equating the coefficients of the series given in this last equation, we are led to the initial-value problems

$$\dot{u}_n(t) + \lambda_n \alpha^2 u_n(t) = f_n(t), \qquad t > 0 \tag{16a}$$

$$u_n(0) = u_0^n. \tag{16b}$$

The solution to (16) is

$$u_n(t) = e^{-\lambda_n \alpha^2 t} u_0^n + \int_0^t e^{-\lambda_n \alpha^2 (t-\tau)} f_n(\tau)\, d\tau$$

Now, if we use this in (15), we find that the solution to our non-homogeneous initial-boundary-value problem (14) is

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\lambda_n \alpha^2 t} u_0^n X_n(x) + \sum_{n=1}^{\infty} \int_0^t e^{-\lambda_n \alpha^2 (t-\tau)} f_n(\tau) X_n(x)\, d\tau, \tag{17}$$

with the coefficients u_0^n and $f_n(\cdot)$ computed as above.

This formula has the tyical structure of the solution of an initial-value problem. That is, if we use the first term in this representation to define a family of operators $E(t),\ t \geq 0$, on the space of functions on the interval $[0,\ell]$ by

$$[E(t)u_0](x) = \sum_{n=1}^{\infty} e^{-\lambda_n \alpha^2 t} \big(u_0, X_n\big) X_n(x)\,, \tag{18}$$

then the solution (17) takes the form

$$u(\cdot,t) = E(t)u_0 + \int_0^t E(t-\tau) f(\tau)\, d\tau\,. \tag{19}$$

In particular, the operator $E(t)$ in this specific case is an *integral operator*

$$\begin{aligned}
[E(t)u_0](x) &= \sum_{n=1}^{\infty} e^{-\lambda_n \alpha^2 t} \int_0^\ell (u_0(s)\,\sin(\frac{n\pi}{\ell}s))\,ds \frac{2}{\ell}\sin(\frac{n\pi}{\ell}x) \\
&= \int_0^\ell \frac{2}{\ell}\big(\sum_{n=1}^{\infty} e^{-\lambda_n \alpha^2 t}\sin(\frac{n\pi}{\ell}s)\sin(\frac{n\pi}{\ell}x)\big)u_0(s)\,ds \\
&= \int_0^\ell G(x,s,t)u_0(s)\,ds
\end{aligned}$$

for which the kernel

$$G(x,s,t) = \frac{2}{\ell}\big(\sum_{n=1}^{\infty} e^{-\lambda_n \alpha^2 t}\sin(\frac{n\pi}{\ell}s)\sin(\frac{n\pi}{\ell}x)\big)$$

is the *Green's function* for the problem.

Exercise 5. *Compute the solution of* (5a) *with initial condition* $u(x,0) = 0$ *and the boundary conditions* $u(0,t) = 0$ *and* $u(\ell,t) = t$.

Example 3. *For the situation of Example 1, suppose the left end of the rod is insulated while the right end has a heat loss given by* $-u_x(\ell,t) = k_\ell\, u(\ell,t)$ *where* $k_\ell \geq 0$. *The initial-boundary-value problem for this situation is given by*

$$u_t(x,t) = \alpha^2 u_{xx}(x,t), \qquad 0 < x < \ell, \quad t > 0, \tag{20a}$$

$$u_x(0,t) = 0,\ \ k_\ell\, u(\ell,t) + u_x(\ell,t) = 0, \qquad t > 0, \tag{20b}$$

$$u(x,0) = u_0(x), \qquad 0 < x < \ell. \tag{20c}$$

We seek a solution in the form

$$u(x,t) = X(x)T(t),$$

where the boundary conditions imply $X'(0) = 0$ and $k_\ell\, X(\ell) + X'(\ell) = 0$. The method of separation of variables leads us to the time-dependent problem

$$T'(t) + \lambda\alpha^2 T(t) = 0, \quad t > 0,$$

and the boundary-value problem

$$X''(x) + \lambda X(x) = 0, \qquad 0 < x < \ell, \tag{21a}$$

$$X'(0) = 0,\ \ k_\ell\, X(\ell) + X'(\ell) = 0. \tag{21b}$$

For values of $\lambda < 0$, we find that there are no non-null solutions. For the case of $\lambda = 0$, only if $k_\ell = 0$ do we get a non-zero solution, and this is $X_0(x) = (\frac{1}{\ell})^{\frac{1}{2}}$ with $\lambda_0 = 0$. But for $\lambda > 0$, the general solution to (21a) is

$$X(x) = C_1 \sin(\lambda^{\frac{1}{2}}x) + C_2\cos(\lambda^{\frac{1}{2}}x),$$

and the boundary conditions (21b) imply that

$$C_1 = 0, \quad \text{and} \quad k_\ell\, C_2\cos(\ell\lambda^{\frac{1}{2}}) - C_2\lambda^{\frac{1}{2}}\sin(\ell\lambda^{\frac{1}{2}}) = 0.$$

Since we are only interested in non-null solutions, the latter equation is equivalent to solving $\lambda^{\frac{1}{2}}\tan(\ell\lambda^{\frac{1}{2}}) = k_\ell$. That is, we have $\tan(\ell\lambda^{\frac{1}{2}}) = \frac{k_\ell}{\lambda^{\frac{1}{2}}}$. The tangent function is π-periodic, so we obtain a sequence $\lambda_n,\ n \geq 0$, of solutions to this equation, and the corresponding eigenfunctions are given by

$$X_n(x) = \left(\frac{2}{\ell}\right)^{\frac{1}{2}} \cos(\lambda_n^{\frac{1}{2}} x), \quad n \geq 1.$$

Note that the eigenvalues belong to intervals determined by $\ell(\lambda_n)^{\frac{1}{2}} \in [n\pi,\ n\pi + \frac{1}{2}\pi],\ n \geq 0$, and that for small $\frac{k_\ell}{(\lambda_n)^{\frac{1}{2}}}$ we have

$$\lambda_n \approx \left(\frac{n\pi}{\ell}\right)^2,$$

so the eigenvalues are asymptotically close to those of the preceding example. Combining these results with the time-dependent solutions $T_n(t) = e^{-\lambda_n \alpha^2 t}$ and using the orthogonality of the eigenfunctions, we find solutions of (20a) in the form

$$u(x,t) = \sum_{n=0}^{\infty} (u_0(\cdot), X_n(\cdot)) e^{-\lambda_n \alpha^2 t} X_n(x),$$

and it is understood that the sum starts at $n = 1$ if $k_\ell > 0$.

Exercise 6. *Compute the solution of* (5a) *with initial condition* $u(\cdot, 0) = u_0(\cdot)$ *and the boundary conditions* $u(0,t) = 0$ *and* $u_x(\ell, t) = 0$.

Exercise 7. *Consider the problem*

$$u''(x) + u'(x) = f(x)$$
$$u'(0) = u(0) = \frac{1}{2}[u'(\ell) + u(\ell)],$$

where $f(x)$ *is a given function.*

(a) *Is the solution unique?*
(b) *Does a solution necessarily exist, or is there a condition that* $f(x)$ *must satisfy for existence?*

Exercise 8. *Let the rod* G *be defined over the interval* $(0,1)$*, and suppose its lateral surface is perfectly insulated along its length. Furthermore, let's assume the material properties of the rod are constant, its thermal diffusivity is* α^2*, and there are no internal heat sources or sinks. Assuming both ends of the rod are insulated and the initial temperature distribution in the rod is given by* $u_0(x) = x$*, find the temperature distribution* $u(x,t)$ *within rod* G*.*

Exercise 9. *Given the same setup as in the previous example, find the temperature distribution* $u(x,t)$ *within rod* G*, under the following conditions:*

i. *The thermal diffusivity* α^2 *of the rod is some known constant.*
ii. *The left end of the rod is held at the fixed constant temperature* $u(0,t) = T_L$*, while the right end is held at the fixed constant temperature* $u(1,t) = T_R$*.*
iii. *The initial temperature distribution within the rod is given by* $u_0(x)$*.*

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