THE WAVE EQUATION

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1. Longitudinal Vibrations

We describe the longitudinal vibrations in a long narrow cylindrical rod of cross section area S. The rod is located along the $x-$axis, and we identify it with the interval $[a, b]$ in $\mathbb{R}$. The rod is assumed to stretch or contract in the horizontal direction, and we assume that the vertical plane cross-sections of the rod move only horizontally. Denote by $u(x, t)$ the *displacement* in the positive direction from the point $x \in [a, b]$ at the time $t > 0$. The corresponding *displacement rate* or *velocity* is denoted by $v(x, t) \equiv u_t(x, t)$.

Let $\sigma(x, t)$ denote the local *stress*, the force per unit area with which the part of the rod to the right of the point x acts on the part to the left of x. Since force is positive to the right, the stress is positive in conditions of *tension*. For a section of the rod, $x_1 < x < x_2$, the total (rightward) force acting on that section due to the remainder of the rod is given by

$$\big(\sigma(x_2, t) - \sigma(x_1, t)\big)S\,.$$

If the density of the rod at x is given by $\rho > 0$, the momentum of this section is just

$$\int_{x_1}^{x_2} \rho\, u_t(x, t)\, S\, dx\ .$$

If we let $F(x, t)$ denote any external applied force per unit of volume in the positive $x-$direction, then we obtain from Newton's second law that

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho\, u_t(x, t)\, S\, dx = \big(\sigma(x_2, t) - \sigma(x_1, t)\big)S + \int_{x_1}^{x_2} F(x, t)\, S\, dx$$

for any such $x_1 < x_2$. For a sufficiently smooth displacement $u(x, t)$, we obtain the *conservation of momentum* equation

$$\rho\, u_{tt}(x, t) - \sigma_x(x, t) = F(x, t)\ , \qquad a < x < b,\ t > 0\ . \tag{1}$$

The *stress* $\sigma(x, t)$ is determined by the type of material of which the rod is composed and the amount by which the neighboring region is stretched or compressed, i.e., on the *elongation* or *strain*, $\varepsilon(x, t)$. In order to define this, first note that a section $[x, x+h]$ of the rod is deformed by the displacement to the new position $[x+u(x), (x+h)+u(x+h)]$. The *elongation* is the limiting increment of the change in the length due to the deformation as given by

$$\lim_{h\to 0}\frac{[u(x+h)+(x+h)]-[u(x)+x]-h}{h} = \frac{d\,u(x)}{dx}\,,$$

so the strain is given by $\varepsilon(x, t) \equiv u_x(x, t)$.

The relation between the stress and strain is a *constitutive law*, usually determined by experiment, and it depends on the type of material. In the simplest case, with small displacements, we find by experiment that $\sigma(x,t)$ is proportional to $\varepsilon(x,t)$, i.e., that there is a constant k called *Young's modulus* for which

$$\sigma(x,t) = k\,\varepsilon(x,t)\ .$$

The constant k is a property of the material, and in this case we say the material is purely *elastic*. A rate-dependent component of the stress-strain relationship arises when the force generated by the elongation depends not only on the magnitude of the strain but also on the speed at which it is changed, i.e., on the *strain rate* $\varepsilon_t(x,t) = v_x(x,t)$. The simplest such case is that of a *visco-elastic* material defined by the linear constitutive equation

$$\sigma(x,t) = k\,\varepsilon(x,t) + \mu\,\varepsilon_t(x,t)\ ,$$

in which the material constant μ is the *viscosity* or internal friction of the material. Finally, if we include the effect of the transverse motions of the rod that result from the elongations under conditions of constant volume or mass, we will get an additional term to represent the transverse inertia. If the constant P denotes *Poisson's ratio*, and r is the average radius of that cross section, then the corresponding *stress-strain* relationship is given as before by

$$\sigma(x,t) = k\,\varepsilon(x,t) + \mu\,\varepsilon_t(x,t) + \rho\,r^2\,P\,\varepsilon_{tt}(x,t)\,.$$

In terms of displacement, the total stress is

$$\sigma(x,t) = ku_x(x,t) + \mu u_{xt}(x,t) + \rho\,r^2\,Pu_{xtt}(x,t)\,. \tag{2}$$

The partial differential equation for the longitudinal vibrations of the rod is obtained by substituting (2) into (1).

1.1. **Initial and Boundary conditions.** Since the momentum equation is second–order in time, one may expect that in order to have a well–posed problem, two initial conditions should be specified. Thus, we shall specify the *initial conditions*

$$u(x,0) = u_0(x), \quad u_t(x,t) = v_0(x), \qquad a < x < b\,,$$

where $u_0(\cdot)$ and $v_0(\cdot)$ are the initial displacement and the initial velocity, respectively.

We list a number of typical possibilities for determining the two boundary conditions. Each of these is illustrated as before with a condition at the right end, $x = b$, and we note that another such condition will also be prescribed at the left end, $x = a$.

1. The displacement could be specified at the end point:

$$u(b,t) = d_b(t), \quad t > 0\,.$$

This is the *Dirichlet* boundary condition, or boundary condition of *first type*. It could be obtained from observation of the endpoint position, or it could be imposed directly on the endpoint. The homogeneous case $d_b(t) = 0$ corresponds to a *clamped end*.

2. The horizontal force on the rod could be specified at the end point:

$$\sigma(b,t) = f_b(t), \quad t > 0\,.$$

For the purely elastic case, $\sigma = ku_x$, this is the *Neumann* boundary condition, or boundary condition of *second type.* The homogeneous condition with $f_b(t) = 0$ corresponds to a *free end.*

3. The force on the end is determined by an elastic constraint, a restoring force proportional to the displacement:

$$\sigma(b,t) + k_0(u(b,t) - d_b(t)) = f_b(t), \quad t > 0\,.$$

For the purely elastic case this is the *Robin* boundary condition, or boundary condition of *third type.* Here both $d_b(\cdot)$ and $f_b(\cdot)$ are prescribed. The first is a prescribed displacement of the spring reference, and the second is a horizontal force concentrated on the right end point. For $k_0 \to \infty$, we obtain formally the Dirichlet boundary condition, while for $k_0 \to 0$ we get the Neumann condition. Thus the *effective tension* k_0 interpolates between the first two types.

4. Another type of boundary condition arises if there is a *concentrated mass* at the end point. Then $u(b,t)$ is the displacement of this mass, and we have the *dynamic* boundary condition

$$\rho_0 u_{tt}(b,t) + \sigma(b,t) = f_b(t), \quad t > 0\,,$$

which is the boundary condition of *fourth type* for the elastic case.

2. The Eigenfunction Expansion, II

We shall apply the method of *separation of variables* to the initial-boundary-value problem for longitudinal vibrations with Dirichlet boundary conditions.

Example 1. *Suppose the rod $(0,\ell)$ is perfectly elastic and set $\alpha^2 = \frac{k}{\rho}$. Assume there are no internal forces,* i.e., *$f(x,t) = 0$, and that both ends of the rod are fixed. The initial displacement and velocity are given by $u_0(x)$ and $v_0(x)$, respectively. The initial-boundary-value problem for this scenario is*

$$u_{tt}(x,t) = \alpha^2 u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0, \tag{3a}$$

$$u(0,t) = 0, \;\; u(\ell,t) = 0, \qquad t > 0, \tag{3b}$$

$$u(x,0) = u_0(x), \;\; u_t(x,0) = v_0(x), \quad 0 < x < \ell. \tag{3c}$$

We look for non-null solutions of the form $u(x,t) = X(x)T(t)$ and find as before that

$$\frac{T''(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}, \qquad 0 < x < \ell,\; t > 0\,.$$

Each side must be equal a common constant, denoted by $-\lambda$, and this leads to the pair of ordinary differential equations

$$\begin{aligned} T''(t) \;&+\; \lambda\,\alpha^2\,T(t) = 0, \quad t > 0, \\ X''(x) \;&+\; \lambda\,X(x) = 0, \quad 0 < x < \ell. \end{aligned}$$

The boundary conditions given in (3b) imply that $X(0) = X(\ell) = 0$.

Note that if $X(\cdot)$ and $T(\cdot)$ are solutions of these respective equations, then it follows directly that their product is a solution (3a). We have already found the non-null solutions of the *boundary-value problem*

$$\begin{cases} X''(x) + \lambda X(x) = 0, & 0 < x < \ell, \\ X(0) = 0, \ X(\ell) = 0. \end{cases} \tag{4}$$

The solutions are the *normalized eigenfunctions*

$$X_n(x) = \sqrt{\frac{2}{\ell}} \sin(\frac{n\pi}{\ell} x)$$

corresponding to the eigenvalues $\lambda_n = (n\pi/\ell)^2$. If we combine these with the corresponding time-dependent solutions $\cos(\alpha\sqrt{\lambda_n} t)$ and $\sin(\alpha\sqrt{\lambda_n} t)$ of the first differential equation and take linear combinations, we obtain a large class of solutions of the partial differential equation (3a) and boundary conditions (3b) in the form of a series

$$u(x,t) = \sum_{n=1}^{\infty} \big(A_n \cos(\alpha\sqrt{\lambda_n} t) + B_n \sin(\alpha\sqrt{\lambda_n} t)\big) X_n(x),$$

where the sequences $\{A_n\}$ and $\{B_n\}$ are to be determined. From the initial conditions (3c), it follows that these coefficients must satisfy

$$\sum_{n=1}^{\infty} A_n X_n(x) = u_0(x), \ \sum_{n=1}^{\infty} B_n \alpha \sqrt{\lambda_n} X_n(x) = v_0(x), \quad 0 < x < \ell,$$

so we obtain

$$A_m = (u_0(\cdot), X_m(\cdot)), \ B_m = \frac{(v_0(\cdot), X_m(\cdot))}{\alpha\sqrt{\lambda_m}}, \quad m \geq 1\,.$$

In summary, the solution of the initial-boundary-value problem (3) is given by the series

$$u(x,t) = \sum_{n=1}^{\infty} \big(\cos(\alpha\sqrt{\lambda_n} t)(u_0(\cdot), X_n(\cdot)) + \sin(\alpha\sqrt{\lambda_n} t)\frac{(v_0(\cdot), X_n(\cdot))}{\alpha\sqrt{\lambda_n}}\big) X_n(x).$$

Denote the second term in the preceding formula by

$$[S(t)v_0](x) = \sum_{n=1}^{\infty} \big(\sin(\alpha\sqrt{\lambda_n} t)\frac{(v_0(\cdot), X_n(\cdot))}{\alpha\sqrt{\lambda_n}}\big) X_n(x).$$

This defines the operator $S(t)$ on the space of functions on $[0,\ell]$. We can use this operator to represent the solution by

$$u(\cdot,t) = S'(t)u_0 + S(t)v_0\,. \tag{5}$$

Example 2. *Suppose the rod $(0,\ell)$ is perfectly elastic and set $\alpha^2 = \frac{k}{\rho}$. Assume that both ends of the rod are fixed, the initial displacement and velocity are both null, and that there* *are distributed forces $f(x,t)$ along its length. The initial-boundary-value problem for this case is*

$$u_{tt}(x,t) = \alpha^2 u_{xx}(x,t) + f(x,t), \quad 0 < x < \ell, \quad t > 0, \tag{6a}$$

$$u(0,t) = 0, \; u(\ell,t) = 0, \qquad t > 0, \tag{6b}$$

$$u(x,0) = 0, \; u_t(x,0) = 0, \quad 0 < x < \ell. \tag{6c}$$

For this problem with a non-homogeneous partial differential equation (6a), we look for the solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x). \tag{7}$$

If we assume that $f(x,t)$ has the eigenfunction expansion

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) X_n(x)\,,$$

then the coefficients are given by

$$f_n(t) \equiv \int_0^{\ell} f(s,t) X_n(s)\, ds\,,$$

and substituting this expansion into equation (6a) yields

$$\sum_{n=1}^{\infty} \left[\ddot{u}_n(t) + \lambda_n \alpha^2 u_n(t)\right] X_n(x) = \sum_{n=1}^{\infty} f_n(t) X_n(x)\,.$$

By equating the coefficients of the series given in this last equation, we are led to the sequence of initial-value problems

$$\ddot{u}_n(t) + \lambda_n \alpha^2 u_n(t) = f_n(t), \qquad t > 0 \tag{8a}$$

$$u_n(0) = 0, \; \dot{u}_n(0) = 0\,. \tag{8b}$$

The solution to (8) is

$$u_n(t) = \int_0^t \frac{\ell}{n\pi\alpha} \sin(\frac{n\pi\alpha}{\ell}(t-\tau)) f_n(\tau)\, d\tau$$

Now, if we use this in (7), we find that the solution to our non-homogeneous initial-boundary-value problem (6) is

$$u(x,t) = \int_0^t \sum_{n=1}^{\infty} \frac{\ell}{n\pi\alpha} \sin(\frac{n\pi\alpha}{\ell}(t-\tau)) f_n(\tau) X_n(x)\, d\tau. \tag{9}$$

Note that we can use the operator $S(t)$ to represent this formula as

$$u(\cdot,t) = \int_0^t S(t-\tau) f(\tau)\, d\tau. \tag{10}$$

As before, each $S(t)$ is an integral operator of the form

$$\begin{aligned}
[S(t)v_0](x) &= \sum_{n=1}^{\infty} \frac{\ell}{n\pi\alpha} \sin(\frac{n\pi\alpha}{\ell}t) \int_0^{\ell} (v_0(s)\ \sin(\frac{n\pi}{\ell}s))\, ds \frac{2}{\ell} \sin(\frac{n\pi}{\ell}x) \\
&= \int_0^{\ell} \frac{2}{\ell} (\sum_{n=1}^{\infty} \frac{\ell}{n\pi\alpha} \sin(\frac{n\pi\alpha}{\ell}t) \sin(\frac{n\pi}{\ell}s) \sin(\frac{n\pi}{\ell}x)\big) v_0(s)\, ds \\
&= \int_0^{\ell} H(x,s,t) v_0(s)\, ds
\end{aligned}$$

for which the kernel

$$H(x,s,t) = \frac{2}{\ell} (\sum_{n=1}^{\infty} \frac{\ell}{n\pi\alpha} \sin(\frac{n\pi\alpha}{\ell}t) \sin(\frac{n\pi}{\ell}s) \sin(\frac{n\pi}{\ell}x)\big)$$

is the *Green's function* for the problem.

Example 3. *Suppose the left end of the elastic rod is free while the right end has an elastic constraint given by $u(\ell,t) + u_x(\ell,t) = 0$. The initial-boundary-value problem for this situation is*

$$u_{tt}(x,t) = \alpha^2 u_{xx}(x,t), \qquad 0 < x < \ell, \qquad t > 0, \tag{11a}$$

$$u_x(0,t) = 0, \quad u(\ell,t) + u_x(\ell,t) = 0, \qquad t > 0, \tag{11b}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), \qquad 0 < x < \ell. \tag{11c}$$

We seek a solution in the form

$$u(x,t) = X(x)T(t),$$

where the boundary conditions imply that $X_x(0) = 0$ and $X(\ell) + X_x(\ell) = 0$. The method of separation of variables leads us to the boundary-value problem

$$\begin{aligned}
X''(x) + \lambda X(x) = 0, &\qquad 0 < x < \ell, \\
X'(0) = 0,\ X(\ell) + X'(\ell) = 0.&
\end{aligned}$$

We obtain a sequence of eigenvalues λ_n and corresponding eigenfunctions given by

$$X_n(x) = \big(\frac{2}{\ell}\big)^{\frac{1}{2}} \cos\,(\lambda_n^{\frac{1}{2}} x). \tag{12}$$

Note that for large λ_n we have

$$\lambda_n \approx (\frac{n\pi}{\ell})^2,$$

so the eigenvalues are asymptotically close to those of the preceding example. Combining these results with the time-dependent solutions and using the orthogonality of the eigenfunctions, we find solutions of the initial-boundary-value problem (11) in the form

$$u(x,t) = \sum_{n=1}^{\infty} \big(\cos(\alpha\sqrt{\lambda_n}t)(u_0(\cdot), X_n(\cdot)) + \sin(\alpha\sqrt{\lambda_n}t) \frac{(v_0(\cdot), X_n(\cdot))}{\alpha\sqrt{\lambda_n}}\big) X_n(x),$$

with the eigenfunctions given by (12). Once again, this can be represented in the form (5) for an appropriate family of operators $\{S(t) : t \geq 0\}$.

Exercise 1. *Suppose the rod* $(0, \ell)$ *is elastic and that we account for the inertia of lateral extension. Set* $\alpha^2 = \frac{k}{\rho}$ *and* $\beta^2 = r^2 P$*, where* P *is Poisson's ratio and* r *is the average radius of a cross section as above. Assume there are no internal forces,* i.e., $f(x,t) = 0$*, and that both ends of the rod are fixed. The initial displacement and velocity are given by* $u_0(x)$ *and* $v_0(x)$*, respectively. The initial-boundary-value problem is*

$$u_{tt}(x,t) = \alpha^2 u_{xx}(x,t) + \beta^2 u_{xxtt}(x,t), \quad 0 < x < \ell, \quad t > 0, \tag{13a}$$

$$u(0,t) = 0, \ u(\ell,t) = 0, \qquad t > 0, \tag{13b}$$

$$u(x,0) = u_0(x), \ u_t(x,0) = v_0(x), \quad 0 < x < \ell. \tag{13c}$$

Find the solution by separation of variables. Find the family of operators $\{S(t) : t \geq 0\}$ *for which this can be represented in the form* (5)*.*

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