MATH 512, Winter 2009 **Exercise 4**

1. For Lebesgue measure on $X = Y = [a, b]$, show that each *open set* in $X \times Y$ is measurable.

2. Let $g(\cdot)$ be integrable on the measure space $(X,\ \mu_1)$ and $h(\cdot)$ be integrable on the measure space $(Y,\ \mu_2)$. Define $f(x, y) = g(x)h(y)$ for $(x, y) \in X \times Y$. Show that f is integrable on $(X \times Y,\ \mu)$ with $\mu = \mu_1 \times \mu_2$ and that

$$\int_{X\times Y} f\, d\mu = \int_X g\, d\mu_1 \int_Y h\, d\mu_2\,.$$

(Do not assume $\sigma-$finiteness of the measure spaces.)

3. Let $X = Y$ be positive integers with the counting measure, and define the function on $X \times Y$ by

$$f(x, y) = \begin{cases} 2 - 2^{-x} & \text{if } x = y, \\ -2 + 2^{-x} & \text{if } x = y + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Evaluate each of the following:

(a) $\int_X f(\cdot, y)\, d\mu_1$

(b) $\int_Y f(x, \cdot)\, d\mu_2$

(c) $\int_X \int_Y f\, d\mu_2\, d\mu_1$

(d) $\int_Y \int_X f\, d\mu_1\, d\mu_2$

(e) $\int_{X\times Y} f\, d\mu$

4. Show that the existence of the integral

$$\int_X \Big(\int_Y |f(x, y)|\, d\mu_2\Big)\, d\mu_1$$

implies the existence of

$$\int_{X\times Y} |f(x, y)|\, d(\mu_1 \times \mu_2).$$

(This is Problem 6 on page 361 of Kolmogorov - Fomin. See the Hint given there.)