MTH 614: Exercise set #2

Problems are due (electronically or in my mailbox in Kidder) on Dec 6 (Thursday) at 4pm = 1600.

1. In the classical theory of Fourier series, the orthonormal basis for $L^2(0,\ell)$ consists of cosine and sine functions. Denote these by $\{v_j(x),\ j = 1, 2, ...$ Each $f \in L^2(0,\ell)$ is represented by $f = \sum_{j=i}^{\infty}(f, v_j)_{L^2}\, v_j$. The n-th partial sum is $f_n = \sum_{j=i}^{n}(f, v_j)_{L^2}\, v_j$, and one can actually sum the series to obtain the representation $f_n(x) = \int_0^\ell f(y)D_n(y-x)\,dy$. The integrand is the Dirichlet kernel $D_n(x)$.

Denote the space of continuous ℓ-periodic functions by $C_p[0,\ell]$. For $f \in C_p[0,\ell]$ define $T_n f = f_n(0) = \int_0^\ell f(y)D_n(y)\,dy$ to get a sequence $\{T_n\}$ in the dual space $C_p[0,\ell]'$. From an explicit computation one finds that $\|T_n\| \to \infty$. Show there exists a continuous periodic function whose Fourier series is not convergent at $x = 0$.

2. Let H be a hyperplane in a normed linear space X. That is, $H = \{x \in X : f(x) = \alpha\}$ where $f \in X^*$ and $\alpha \in \mathbb{R}$. Show that H is either closed or dense in X.

3. Let $\mathcal{B} \in \mathcal{L}(\mathbf{X}, \mathbf{Y}')$ where $\mathbf{X}$ and $\mathbf{Y}$ are Banach spaces. Show the following are equivalent:

- $\mathcal{B} : \mathbf{X} \to \mathbf{Y}'$ has closed range,
- $\mathcal{B}(\mathbf{X}) = (\operatorname{Ker}\mathcal{B}')^o$,
- there is a $\beta > 0$ for which $\sup_{y\in\mathbf{Y}} \frac{\mathcal{B}x(y)}{\|y\|_{\mathbf{Y}}} \geq \beta \inf_{z\in\operatorname{Ker}\mathcal{B}} \|x+z\|_{\mathbf{X}},\ x \in \mathbf{X}$.

4. a) Let $V = H^1(0,\ell)$ and $a(\cdot) \in L^\infty(0,\ell)$ with $a(x) \geq c > 0$. Show that

$$u \in V : \quad \int_0^\ell a(x)\partial u(x)\partial v(x)\,dx = \int_0^\ell F(x)v(x)\,dx + \alpha v(\ell) \text{ for all } v \in V.$$

characterizes a boundary-value problem.

b) Let $\mathcal{A} : V \to V'$ denote the corresponding operator. What is the kernel of $\mathcal{A}$? For the functional

$$f(v) = \int_G F(x)v(x)\,dx + \alpha v(\ell),$$

when is f in $\operatorname{Rg}\mathcal{A}$?