Final Exercises

Let $c \geq 0$, $\alpha \geq 0$, $f_0 \in \Re$ and $F \in L^2(0,\ell)$ be given. Define the bilinear form

$$a(u,v) = \int_0^\ell (\partial u(x)\, \partial v(x) + c\, u(x)\, v(x))\, dx + \alpha u(\ell)\, v(\ell)$$

and the linear functional $f(v) = \int_0^\ell F(x)v(x)\, dx + f_0 v(\ell)$ on the space $V = \{v \in H^1(0,\ell) :\ v(0) = 0\}$.

1. Show that f is continuous and that $a(\cdot,\cdot)$ is continuous and V-elliptic.
2. Characterize the solution of

$$u \in V :\ a(u,v) = f(v) \text{ for all } v \in V$$

as the solution of a boundary-value problem.
3. Repeat #2 for the space

$$V = \{v \in H^1(0,\ell) :\ v(0) = v(\ell)\}.$$

4. Show that $a(\cdot,\cdot)$ is $H^1(0,\ell)$-elliptic if either $c > 0$ or $\alpha > 0$.

Show that it is *not* $H^1(0,\ell)$-elliptic if $c = 0$ and $\alpha = 0$.