COUPLED SYSTEMS OF MECHANICS

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Contents

1. Introduction

We shall develop various classes of initial-boundary-value problems for systems of partial differential equations as models for classical problems of continuum mechanics. These include the deformation of general anisotropic structures of elastic, visco-elastic or plastic materials, the flow of fluids and diffusion through porous media, and combinations of these together with appropriate interface conditions for diffusion through deformable media. The homogeneous and isotropic case will be included, as well as both compressible and incompressible systems.

2. Fundamentals of Mechanics

In describing the kinematics of a continuum, it is important to distinguish a *point* in space from a *particle* occupying that location. A *deformation* is a function that represents a change in position of particles from an initial to a final configuration. The related notion of *flow* is used to describe the entire time-dependent family of intermediate positions. Each point in $\mathbb{R}^3$ has *spatial coordinates* $\mathbf{x} = (x_1, x_2, x_3)$. A *particle* in a region $\Omega \subset \mathbb{R}^3$ is identified with its original location $\mathbf{X} \in \Omega$, and its new position is denoted by the deformation $\mathbf{x} = \mathbf{x}(\mathbf{X}) \in \tilde{\Omega} \subset \mathbb{R}^3$. The original position $\mathbf{X}$ is the *material coordinate* of the particle, and the new position $\mathbf{x}$ is its spatial coordinate. The deformation $\mathbf{X} \mapsto \mathbf{x} = \mathbf{x}(\mathbf{X})$ is assumed to be continuously differentiable from Ω onto $\tilde{\Omega}$ and invertible, and the *deformation gradient*

$$\frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \left(\frac{\partial x_i}{\partial X_j} \right)$$

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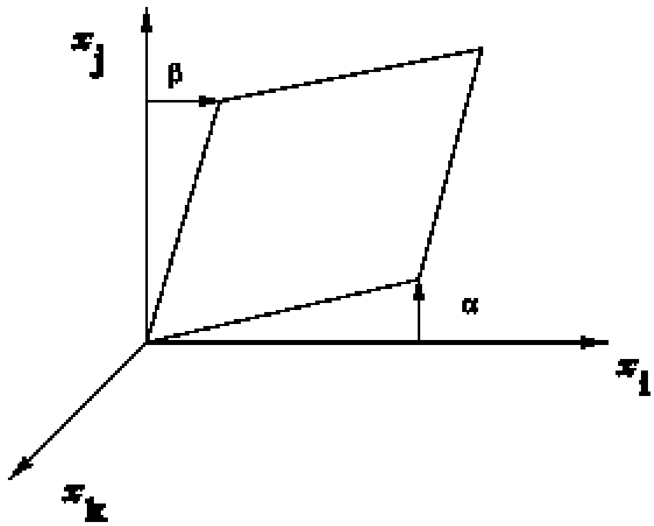

FIGURE 1. Shear strain

is continuous and invertible at each point. The same holds for the corresponding inverse $\mathbf{x} \mapsto \mathbf{X} = \mathbf{X}(\mathbf{x})$. Assume also that the determinant satisfies $det(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}) > 0$, so the deformation is orientation preserving. Any function on Ω can be expressed in terms of either spatial coordinates $\mathbf{x}$ or material coordinates $\mathbf{X}$. The first is the Euler representation and the latter is the *material* or Lagrangian representation.

2.1. **Displacement and Strain.** We consider a deformation of the body Ω for which each point $\mathbf{X} \in \Omega$ is moved to another point $\mathbf{x}(\mathbf{X}) = \mathbf{X} + \mathbf{u}(\mathbf{X})$. The vector field $\mathbf{u}(\mathbf{X})$ is the *displacement* function. The derivative of this map can be written as the sum of its symmetric and skew-symmetric parts as

$$\partial_j u_i \equiv \frac{\partial u_i}{\partial X_j} = \varepsilon_{ij}(\mathbf{u}) + \omega_{ij}(\mathbf{u})\,,$$

$$\varepsilon_{ij}(\mathbf{u}) \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right), \qquad \omega_{ij}(\mathbf{u}) \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right).$$

We investigate the geometric meaning of these expressions. Figure 1 indicates the X_k-projection of the deformed state of a rectangular element with sides originally parallel to the axes. The *shear strain* in the $\mathbf{X}_i\mathbf{X}_j$-plane is measured by the average of the two angles, α, β, made with the X_i axis and the X_j axis, respectively. These can be approximated by

$$\alpha \approx \tan(\alpha) = \frac{\partial u_j}{\partial X_i}\,, \qquad \beta \approx \tan(\beta) = \frac{\partial u_i}{\partial X_j}\,.$$

Thus the shear strain in the three respective planes is given by $\varepsilon_{ij}(\mathbf{u})$ for $i \neq j$. The X_i-*elongation* is the limiting relative change in the length in the X_i direction due to the deformation as given by

$$\lim_{h\to 0} \frac{[u_i(\mathbf{X} + h\mathbf{e}_i) + (X_i + h)] - [u_i(\mathbf{X}) + X_i] - h}{h} = \frac{\partial u_i}{\partial X_i}\,.$$

This accounts for the diagonal terms in $\varepsilon_{ij}(\mathbf{u})$. In summary, the symmetric part of the displacement gradient is the *strain* $\varepsilon_{ij}(\mathbf{u})$, whose off-diagonal terms are shear strain and diagonal terms are elongations. Similarly it follows that the skew-symmetric part $\omega_{ij}(\mathbf{u})$ is the *rotation*. The displacement field $\mathbf{u}(\mathbf{X})$ is a *rigid motion* if $\varepsilon_{ij}(\mathbf{u}) = 0$, and then it is of the form $\mathbf{u}(\mathbf{X}) = B\,\mathbf{X} + c$, where B is a skew-symmetric matrix.

The mechanical meaning of "strain" refers to effects on neighboring points that result from their relative distance being changed by the deformation. Pick $\mathbf{X} \in \Omega$ and a small increment $\xi \in \mathbb{R}^3$. Consider the nearby point $\mathbf{X} + \xi$ and corresponding change in the increment ξ. The points $\mathbf{X}$ and $\mathbf{X}+\xi$ go to $\mathbf{X}+\mathbf{u}(\mathbf{X})$ and $\mathbf{X}+\xi+\mathbf{u}(\mathbf{X}+\xi)$, respectively. Thus the change in ξ due to the displacement $\mathbf{u}$ is given by

$$\Delta\xi = \mathbf{X} + \xi + \mathbf{u}(\mathbf{X} + \xi) - (\mathbf{X} + \mathbf{u}(\mathbf{X})) - \xi = \mathbf{u}(\mathbf{X} + \xi) - \mathbf{u}(\mathbf{X})\,.$$

Using Taylor's expansion on this we get (approximately)

$$\Delta\xi_i \approx \partial_j u_i(\mathbf{X})\,\xi_j\,.$$

This deformed state is characterized by changes in the distances between the points and the angle θ of rotation of ξ to $\Delta\xi$, and these are computed (approximately) by ignoring products of the small displacement gradients to get

$$\begin{aligned}
|\xi + \Delta\xi|^2 - |\xi|^2 &= \sum_{i,j=1}^{3} (\xi_i + \partial_j u_i(\mathbf{X})\,\xi_j)^2 - \sum_{i=1}^{3} (\xi_i)^2 \\
&= \sum_{i,j=1}^{3} \left(2\partial_j u_i(\mathbf{X})\,\xi_i\,\xi_j + \partial_j u_i(\mathbf{X})^2 \xi_i^2\right) \\
&\approx 2\sum_{i,j=1}^{3} \partial_j u_i(\mathbf{X})\,\xi_i\,\xi_j = 2\sum_{i,j=1}^{3} \varepsilon_{ij}\,\xi_i\,\xi_j\,, \\
(\Delta\xi, \xi) &= \sum_{i,j=1}^{3} \varepsilon_{ij}\,\xi_i\,\xi_j = \|\Delta\xi\|\|\xi\|\cos(\theta)\,.
\end{aligned}$$

Likewise, the relative change in volume or *dilation*, i.e., the change in volume per unit volume, is approximated by the divergence,

$$\frac{(\xi_1 + \partial_1 u_1(\mathbf{X})\xi_1)(\xi_2 + \partial_2 u_2(\mathbf{X})\xi_2)(\xi_3 + \partial_3 u_3(\mathbf{X})\xi_3) - \xi_1\xi_2\xi_3}{\xi_1\xi_2\xi_3} \approx \sum \varepsilon_{ii}(\mathbf{u}) = \boldsymbol{\nabla} \cdot \mathbf{u}.$$

These calculations show that if the displacement gradients are all zero, then a small neighborhood of $\mathbf{X}$ will remain in the same state and thus differs from the original configuration by a rigid displacement. More generally, when the displacement gradients are small with respect to unity, the deformed state is characterized by the strain components. This is the basic assumption of *small deformation theory*, which we assume below. If both the displacement gradients and the displacements $\mathbf{u}(\mathbf{X})$ themselves are small, then we identify the spatial and material coordinates, $\mathbf{x} \approx \mathbf{X}$, and corresponding gradients, $\frac{\partial u_i}{\partial X_j} \approx \frac{\partial u_i}{\partial x_j}$. This is the *small strain theory*.

2.2. **Transport and Momentum.** Just as deformation is a function that represents a change in position from an initial to a final configuration, *flow* is used to describe a time-dependent family of intermediate positions. A particle in a region Ω is identified with its original location $\mathbf{X} \in \Omega$ at $t = 0$, and its position at a later time $t > 0$ is denoted by the deformation $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$. This curve in $\mathbb{R}^3$ is the *path* of the particle $\mathbf{X}$.

The distinction between spatial coordinates $\mathbf{x}$ and material coordinates $\mathbf{X}$ is even more important for flows. For any function $F(\mathbf{x}, t)$, $\frac{\partial F}{\partial t}$ is the usual partial derivative with $\mathbf{x}$ fixed. We denote by $\frac{DF}{Dt} = \frac{\partial F(\mathbf{X},t)}{\partial t}$ the *material derivative* in which $\mathbf{X}$ is held fixed. For example, the *velocity* of the particle $\mathbf{X}$ at time t is the material derivative

$$\mathbf{v}(t) = \frac{d\mathbf{x}(t)}{dt} = \frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial t}. \tag{2.1}$$

Thus, for any spatial function $F(\mathbf{x}, t)$, the material derivative is given by

$$\frac{DF}{Dt} = \frac{\partial F(\mathbf{x}(\mathbf{X}, t), t)}{\partial t} = \frac{\partial F(\mathbf{x}, t)}{\partial x_i}\frac{\partial x_i(\mathbf{X}, t)}{\partial t} + \frac{\partial F(\mathbf{x}, t)}{\partial t} = \frac{\partial F(\mathbf{x}, t)}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} F. \tag{2.2}$$

It is also known as the *convective derivative.*

We obtained the velocity at each point $\mathbf{x}$ from the particle paths in (2.1). Conversely, we can recover the particle paths by solving the initial-value problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t),\ t > 0,\ \mathbf{x}(0) = \mathbf{X}, \tag{2.3}$$

for $\mathbf{x} = \mathbf{x}(t)$. The *acceleration* of the particle is given by

$$\mathbf{a} = \frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}\mathbf{v}. \tag{2.4}$$

This does not necessarily vanish if the flow is steady, i.e., $\mathbf{v}(\mathbf{x}, t) = \mathbf{v}(\mathbf{x})$, but then

$$\mathbf{a} = \mathbf{v} \cdot \boldsymbol{\nabla}\mathbf{v}.$$

The determinant of the deformation gradient is the *Jacobian*, $J = det(\frac{\partial \mathbf{x}}{\partial \mathbf{X}})$. It appears as the scaling of volume under the change of coordinates, $dx = J\,dX$, and it is called the *expansion.* The time-derivative of each term of the deformation gradient is given by

$$\frac{\partial}{\partial t}\frac{\partial x_i}{\partial X_j} = \frac{\partial}{\partial X_j}\frac{\partial x_i}{\partial t} = \frac{\partial v_i}{\partial X_j} = \frac{\partial v_i}{\partial x_k}\frac{\partial x_k}{\partial X_j}\,.$$

The derivative of any 3×3 determinant is the sum of three terms, each being obtained by differentiating one of the rows. The first term in $\frac{dJ}{dt}$ is

$$\begin{vmatrix} \frac{\partial v_1}{\partial x_k}\frac{\partial x_k}{\partial X_1} & \frac{\partial v_1}{\partial x_k}\frac{\partial x_k}{\partial X_2} & \frac{\partial v_1}{\partial x_k}\frac{\partial x_k}{\partial X_3} \\ \boldsymbol{\nabla}_{\mathbf{X}} x_2 & & \\ \boldsymbol{\nabla}_{\mathbf{X}} x_3 & & \end{vmatrix} = \frac{\partial v_1}{\partial x_1}J + \frac{\partial v_1}{\partial x_2}0 + \frac{\partial v_1}{\partial x_3}0 = \frac{\partial v_1}{\partial x_1}J\,,$$

and the sum of the three terms is then

$$\frac{dJ}{dt} = \boldsymbol{\nabla} \cdot \mathbf{v}J.$$

That is, the divergence of the velocity is the relative rate of change of the expansion,

$$\boldsymbol{\nabla} \cdot \mathbf{v}(t) = \tfrac{d}{dt}\ln J(t). \tag{2.5}$$

This is the *Euler expansion formula.*

Let $F(\mathbf{x},t)$ be a function, and at each $t \geq 0$ let $B(t) = \{\mathbf{x}(\mathbf{X},t) : \mathbf{X} \in B\}$ be the position of a body $B = B(0)$ of particles. We shall calculate the rate of change of

$$F(t) = \int_{B(t)} F(\mathbf{x},t)\,dx.$$

By a change of variable back to the material coordinates, we have from (2.5)

$$\begin{aligned}\frac{d}{dt}\int_{B(t)} F(\mathbf{x},t)\,dx &= \frac{d}{dt}\int_B F(\mathbf{x}(\mathbf{X},t),t)\,J d\mathbf{X}\\ &= \int_B \left(\frac{D}{Dt}F(\mathbf{x}(\mathbf{X},t),t)\,J + F(\mathbf{x}(\mathbf{X},t),t)\frac{D}{Dt}J\right)d\mathbf{X}\\ &= \int_B \left(\frac{D}{Dt}F(\mathbf{x}(\mathbf{X},t),t) + F(\mathbf{x}(\mathbf{X},t),t)\boldsymbol{\nabla}\cdot\mathbf{v}\right)\,J d\mathbf{X}\\ &= \int_{B(t)} \left(\frac{D}{Dt}F(\mathbf{x},t) + F(\mathbf{x},t)\boldsymbol{\nabla}\cdot\mathbf{v}\right)dx\,.\end{aligned}$$

Thus, we obtain the *Reynolds transport theorem* in either form

$$\begin{aligned}\frac{d}{dt}\int_{B(t)} F(\mathbf{x},t)\,dx &= \int_{B(t)} \left(\frac{\partial}{\partial t}F(\mathbf{x},t) + \boldsymbol{\nabla}\cdot(F(\mathbf{x},t)\mathbf{v})\right)dx\\ &= \int_{B(t)} \frac{\partial}{\partial t}F(\mathbf{x},t)dx + \int_{\partial B(t)} F(\mathbf{x},t)\mathbf{v}\cdot\mathbf{n}dS\,.\end{aligned} \tag{2.6}$$

The rate of change of the integral of F along a moving volume is the integral of the rate of change of F plus the integral of *flux* of F over the boundary of that volume.

Let the region Ω be deformed with displacement $\mathbf{u}(t) = \mathbf{x}(t) - \mathbf{X}$ as above. The velocity is $\mathbf{v}(t) = \frac{\partial \mathbf{u}}{\partial t}$, and the *momentum* of the particles with density ρ in the moving body $B(t)$ with velocity $\mathbf{v}(\mathbf{x},t)$ is given by $\int_{B(t)} \rho\mathbf{v}(x,t)\,dx$. It's rate of change has i-th component in any one of the forms

$$\begin{aligned}\frac{d}{dt}\int_{B(t)} \rho v_i(\mathbf{x},t)\,dx &= \int_{B(t)} \left(\frac{D}{Dt}\rho v_i(\mathbf{x},t) + \rho v_i(\mathbf{x},t)\boldsymbol{\nabla}\cdot\mathbf{v}\right)dx\\ &= \int_{B(t)} \left(\frac{\partial}{\partial t}\rho v_i(\mathbf{x},t) + \boldsymbol{\nabla}\cdot(\rho v_i(\mathbf{x},t)\mathbf{v})\right)dx\\ &= \int_{B(t)} \frac{\partial}{\partial t}\rho v_i(\mathbf{x},t)dx + \int_{\partial B(t)} \rho v_i(\mathbf{x},t)\mathbf{v}\cdot\mathbf{n}dS.\end{aligned} \tag{2.7}$$

2.3. **Force and Stress.** Forces acting on a body are divided into two categories, *point forces* and *traction forces*. Point forces depend on the *location* at which they are applied, and they are usually distributed with respect to mass or volume. A force density $\mathbf{f}(x)$ distributed over Ω denotes the force applied at the point x. Thus, the total body force applied to a region $B \subset \Omega$ is given by the volume integral $\int_B \mathbf{f}(x)\,dx$. Gravity and magnetism are examples of this type. Traction forces act on *surface elements* at the point, so they depend not only on the location but also on the orientation of the surface element on which they act, i.e., its orientation. Fluid pressure is the simplest example, since it acts normal to the surface to which it is applied. If the fluid is viscous and in motion, then its action also has a force component tangential to the surface.

A deformation or change of shape of a body from its equilibrium position changes the array of forces which represent the local action on each small part of the body of the complement of that part by way of its boundary. These internal forces are characterized by the *stress* functional, and this concept can be visualized as follows. Let x be a point in the body and S be a small element of surface located at x. The stress gives the traction force distribution acting on one side of S which represents the effect of the body from the opposite side of S, so it is a function of x and also of the *orientation* of S, i.e., the unit *normal* $\mathbf{n}$ to S. Denote the force of the body on the positive side of S by $\Sigma(S)$ and consider the ratio $\frac{\Sigma(S)}{|S|}$. The limit of this ratio as the measure $|S| \to 0$ depends on the point x and the vector $\mathbf{n}$, and it defines the *stress*,

$$\sigma(x, \mathbf{n}) \equiv \lim_{|S|\to 0} \left(\frac{\Sigma(S)}{|S|} \right) .$$

If the deformation is time dependent, so also is the resulting stress field, $\sigma(x, t, \mathbf{n})$.

Stress is characterized by a symmetric matrix $\sigma_{ij}(x)$ for which the i^{th} component of the traction force on S at x is given by $\sigma(x, \mathbf{n})_i = \sigma_{ij}(x)\, n_j$. The total of all traction forces acting on $B \subset \Omega$ is given by the surface integral $\int_{\partial B} \sigma_{ij}\, n_j\, dS$. Here $\mathbf{n} = (n_1, n_2, n_3)$ is the unit outward normal vector on the boundary ∂B. Finally, we should note that stress is defined to be positive with *tension*, i.e., negative with *compression*. By contrast, *pressure* is a special form of stress which acts only in the *normal* direction to a surface, and it is taken to be positive in compression, so it has the form $\sigma(x, \mathbf{n})_i = -p(x)\, n_i$, that is, $\sigma_{ij}(x) = -\delta_{ij}\, p(x)$ with a scalar-valued function, $p(x)$.

Let the region Ω be deformed with displacement $\mathbf{u}(t)$ as before. The velocity is $\mathbf{v}(t) = \mathbf{u}'(t)$, and the momentum of the particles with density ρ in the moving body $B(t)$ with velocity $\mathbf{v}(\mathbf{x}, t)$ is given by $\int_{B(t)} \rho \mathbf{v}(x, t)\, dx$. The forces acting on the body $B(t)$ consist of the traction forces applied by the complement of $B(t)$ across its boundary $\int_{\partial B(t)} \sigma_{ij}(x, t)\, n_j\, dS$ and the volume-distributed exterior forces $\int_{B(t)} \mathbf{f}(x, t)\, dx$. Thus we obtain the equation of *balance of momentum*

$$\frac{d}{dt} \int_{B(t)} \rho \mathbf{v}(\mathbf{x}, t)\, dx = \int_{\partial B(t)} \sigma(\cdot, t, \mathbf{n})\, dS + \int_{B(t)} \mathbf{f}(x, t)\, dx$$

for each such $B(t)$, and with (2.7) this gives the system of partial differential equations

$$\rho \frac{\partial v_i(\mathbf{x}, t)}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{v}(x, t)\, \rho\, v_i(\mathbf{x}, t)) - \partial_j \sigma_{ij}(\mathbf{x}, t) = f_i(\mathbf{x}, t)\,, \qquad 1 \le i \le 3\,, \tag{2.8}$$

where $\mathbf{v}(\mathbf{x}, t) = (v_1(\mathbf{x}, t), v_2(\mathbf{x}, t), v_3(\mathbf{x}, t))$ is the (small) *displacement rate* at the position $x \in \Omega$, and components of $\mathbf{f}(\mathbf{x}, t)$ are indicated similarly. The boundary conditions on $\partial\Omega$ will involve the displacement or the *tractions* $\sigma_{ij}(\mathbf{x}, t)n_j$ on $\partial\Omega$. For example, we could impose

$$u_i = 0 \text{ on } \Gamma_0\,, \quad \sigma_{ij}(\mathbf{x}, t) n_j = g_i \text{ on } \Gamma_1\,, \quad 1 \le i \le 3,$$

where Γ_0 and Γ_1 are complementary parts of the boundary $\partial\Omega$. The traction forces can be further resolved

$$\sigma_{\mathbf{n}} \equiv \sigma_{ij} n_i n_j \,,$$
$$\sigma_{\mathbf{t}} \equiv \{\sigma_{i\mathbf{t}}\} = \sigma - \sigma_{\mathbf{n}}\,\mathbf{n}\,,$$
$$\sigma_{i\mathbf{t}} \equiv \sigma_{ij} n_j - \sigma_{\mathbf{n}} n_i \,,$$

as indicated into their normal and tangential components.

3. Constitutive Laws

3.1. **Linear Elasticity.** The necessarily symmetric stress σ represents the internal forces on surface elements resulting from deformations. The material is characterized by a *stress-strain* law which relates the stress to the strain ε_{ij}. *Linearly elastic* materials are those characterized by the *generalized Hooke's law*

$$\sigma_{ij}(\mathbf{u}) = a_{ijkl}\,\varepsilon_{kl}(\mathbf{u})\,.$$

The positive and symmetric *elasticity* a_{ijkl} provides a model for general anisotropic materials. It satisfies the symmetry conditions

$$a_{ijkl} = a_{klij}, \quad a_{ijkl} = a_{jikl} = a_{ijlk}\,.$$

Note that the sign of the elasticity is consistent with the convention that stress components are positive in tension. The body is said to be *homogeneus* if the elasticity is constant, and it is called *non-homogeneus* if the elasticity varies with the point x. The quadratic form $\frac{1}{2}a_{ijkl}\xi_{ij}\xi_{kl}$ is the *elastic energy density*, and it determines the *strain energy*

$$\mathcal{E}(\mathbf{u}) \equiv \frac{1}{2}\int_\Omega (a_{ijkl}\,\partial_j u_i\,\partial_k u_l\,)\;dx\,.$$

We shall consider only those elasticities which are *coercive*:

$$c\,|\xi|^2 \le a_{ijkl}\xi_{ij}\xi_{kl} \le K\,|\xi|^2\,, \quad \xi \in \mathbb{R}^3$$

for some $c > 0$.

For the special case of an *isotropic medium*, the elasticity is given by

$$a_{ijkl} \equiv \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

with the positive Lamé constants λ and μ. These represent the *dilation* and *shear* moduli of elasticity, respectively. The first accounts for *compression* and the second for *distortion* of the medium. From the computations

$$\delta_{ij}\delta_{kl}\varepsilon_{kl} = \delta_{ij}\varepsilon_{kk}$$
$$\delta_{ik}\delta_{jl}\varepsilon_{kl} = \varepsilon_{ij}$$
$$\delta_{il}\delta_{jk}\varepsilon_{kl} = \varepsilon_{ij}$$

we obtain the stress

$$\sigma_{ij}(\mathbf{u}) = a_{ijkl}\varepsilon_{kl}(\mathbf{u}) = \lambda\delta_{ij}\varepsilon_{kk}(\mathbf{u}) + 2\mu\varepsilon_{ij}(\mathbf{u})\,.$$

The elastic energy density is given by

$$\frac{1}{2}a_{ijkl}\xi_{ij}\xi_{kl} = \frac{\lambda}{2}(\xi_{ii})^2 + \mu\xi_{ij}\xi_{ij}\,,$$

and the corresponding strain energy functional is

$$\mathcal{E}(\mathbf{u}) \equiv \int_\Omega \left(\frac{\lambda}{2}(\partial_i u_i)^2 + \mu \partial_j u_i \partial_j u_i \right) \, dx \, .$$

The Lamé constants λ and μ are convenient for the theory, but there are other constants that have a more direct mechanical interpretation. We introduce them by three *experiments*, which are illustrated below in Figure 2.. First, assume the material is loaded vertically so $\sigma_3 = (0, 0, \sigma_{33})$. Then we have

$$0 = \lambda \varepsilon_{kk} + 2\mu \varepsilon_{11} \, , \quad 0 = \lambda \varepsilon_{kk} + 2\mu \varepsilon_{22} \, , \quad \sigma_{33} = \lambda \varepsilon_{kk} + 2\mu \varepsilon_{33} \, .$$

By summing these three and then using the last, we obtain

$$\sigma_{33} = (3\lambda + 2\mu)\varepsilon_{kk} = E \varepsilon_{33}$$

where the constant E is *Young's modulus*,

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \, .$$

This represents the resistance of the material under uniaxial loading. Also, by solving for $\lambda \varepsilon_{kk}$ in the first two equations and then using the last, we get

$$\varepsilon_{11} = \varepsilon_{22} = -\nu \varepsilon_{33}$$

where

$$\nu = \frac{\lambda}{2(\lambda + \mu)}$$

is the *Poisson's ratio*. This is a measure of the 'bulging' of the material, a transfer of its deformation to the direction perpendicular to the uniaxial load. These permit the expression of strain in terms of stress through the inverse form of Hooke's law as

$$\begin{aligned}
\varepsilon_{11} &= \frac{1}{E}\left(\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})\right) \, , \quad & \varepsilon_{12} &= \frac{1}{2\mu}\sigma_{12} \\
\varepsilon_{22} &= \frac{1}{E}\left(\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})\right) \, , \quad & \varepsilon_{13} &= \frac{1}{2\mu}\sigma_{13} \\
\varepsilon_{33} &= \frac{1}{E}\left(\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})\right) \, , \quad & \varepsilon_{23} &= \frac{1}{2\mu}\sigma_{23}
\end{aligned}$$

In the second experiment, the material is loaded by a constant (decrease of) pressure, that is,

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = P \, .$$

Then we have from Hooke's law

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \frac{1}{E}\left(P(1 - 2\nu)\right)$$

so the expansion is given by

$$\varepsilon_{kk} = \frac{P}{K}$$

where

$$K = \frac{E}{3(1 - 2\nu)} = \lambda + \frac{2}{3}\mu$$

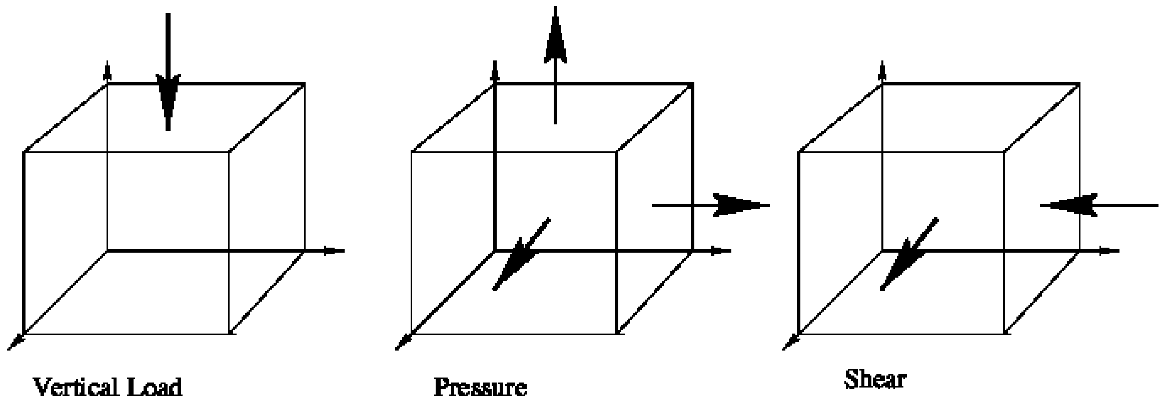

FIGURE 2. Experiments

is the *bulk modulus*, a measure of the pressure required for a decrease of volume. Finally, assume that the material is loaded by a 'pure shear' for which

$$\sigma_{11} = -\sigma_{22}\,, \quad \sigma_{33} = 0\,.$$

Then we obtain as before

$$\varepsilon_{11} = \frac{\sigma_{11}}{2G}\,, \quad \varepsilon_{22} = \frac{\sigma_{22}}{2G}\,, \quad \varepsilon_{33} = 0\,,$$

where $G = \frac{E}{2(1+\nu)} = \mu$ is the *shear modulus.*

The *stationary elasticity system* is given by the equations of equilibrium

$$-\partial_j a_{ijkl}\,\varepsilon_{kl}(\mathbf{u}) = f_i \text{ in } \Omega\,, \quad 1 \le i \le 3,$$
$$u_i = 0 \text{ on } \Gamma_0\,, \quad a_{ijkl}\,\varepsilon_{kl}(\mathbf{u})n_j = g_i \text{ on } \Gamma_1\,,$$

where $\mathbf{u}(x) = (u_1(x), u_2(x), u_3(x))$ is the (small) *displacement* from the position $x \in \Omega$, and $\varepsilon_{kl}(\mathbf{u}) \equiv \frac{1}{2}(\partial_k u_l + \partial_l u_k)$ is the (linearized) *strain*, which provides a measure of the local deformation of the body. We shall seek a weak solution $\mathbf{u}$ in the *complex* Sobolev product space

$$\mathbf{V}_1 \equiv \{\mathbf{v} \in H^1(\Omega)^3 : \ \mathbf{v} = 0 \text{ on } \Gamma_0\}\,.$$

Multiply the system equations by the complex conjugates of the respective components of a $\mathbf{v} \in H^1(\Omega)^3$ and use Greens theorem to integrate by parts and obtain

$$\int_\Omega a_{ijkl}\varepsilon_{kl}(\mathbf{u})\overline{\partial_j v_i}\,dx = \int_\Omega f_i\overline{v_i}\,dx + \int_\Gamma a_{ijkl}\varepsilon_{kl}(\mathbf{u})n_j\overline{v_i}\,ds\,.$$

From the symmetry $a_{ijkl} = a_{jikl}$ and boundary condition $v_i = 0$ on Γ_0 we obtain the weak formulation of the problem,

$$\mathbf{u} \in \mathbf{V}_1 : \quad \int_\Omega a_{ijkl}\varepsilon_{kl}(\mathbf{u})\overline{\varepsilon_{ij}(\mathbf{v})}\,dx = \int_\Omega f_i\overline{v_i}\,dx + \int_{\Gamma_1} g_i\overline{v_i}\,ds \quad \forall \mathbf{v} \in \mathbf{V}_1\,.$$

This is of the form

$$\mathbf{u} \in \mathbf{V}_1 : \quad e(\mathbf{u},\mathbf{v}) = \mathbf{f}_0(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_1$$

with the appropriate definitions of the sesquilinear form $e(\cdot,\cdot)$ and the conjugate-linear functional $\mathbf{f}_0(\cdot)$ on the Hilbert space $\mathbf{V}_1$. Hereafter we denote the resulting *elasticity operator* above by $\mathcal{E}(\mathbf{u}) = \mathbf{f}_0$. That is, $\mathcal{E} : \mathbf{V}_1 \to \mathbf{V}_1'$ is the linear operator determined by the sesquilinear form $e(\cdot,\cdot)$ on $\mathbf{V}_1$, $\mathcal{E}(\mathbf{u})(\mathbf{v}) = e(\mathbf{u},\mathbf{v})$, $\mathbf{v} \in \mathbf{V}_1$, so it is the Gateaux derivative of the strain energy, $\mathcal{E} = \mathcal{E}'$. It follows from *Korn's inequality* that this form is coercive, and so $\mathcal{E}$ is an isomorphism. See Ciarlet for regularity results on $\mathcal{E}$.

We return to the elasticity system posed in terms of the Lamé constants. The weak form of the *isotropic* elasticity system is given by

$$\mathbf{u} \in \mathbf{V}_1 : \quad \int_\Omega (\lambda(\boldsymbol{\nabla}\cdot\mathbf{u})\,\overline{(\boldsymbol{\nabla}\cdot\mathbf{v})} + 2\mu\,\varepsilon_{ij}(\mathbf{u})\,\overline{\varepsilon_{ij}(\mathbf{v})})\,dx = \int_\Omega \mathbf{f}\cdot\overline{\mathbf{v}}\,dx + \int_{\Gamma_1} \mathbf{g}\cdot\overline{\mathbf{v}}\,ds \quad \forall \mathbf{v} \in \mathbf{V}_1 .$$

If we write out in components the corresponding partial differential equations, we have

$$-\partial_j(\lambda\delta_{ij}\boldsymbol{\nabla}\cdot\mathbf{u} + \mu(\partial_i u_j + \partial_j u_i)) = f_i$$

or

$$-(\lambda+\mu)\partial_i(\boldsymbol{\nabla}\cdot\mathbf{u}) - \mu\Delta u_i = f_i .$$

The boundary conditions are

$$u_i = 0 \text{ on } \Gamma_0 , \quad \lambda(\boldsymbol{\nabla}\cdot\mathbf{u})n_i + 2\mu\,\varepsilon_{ij}(\mathbf{u})n_j = g_i \text{ on } \Gamma_1 , \quad 1 \le i \le 3 .$$

This boundary-value problem in vector form is given by

$$-(\lambda+\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) - \mu\Delta\mathbf{u} = \mathbf{f} \text{ in } \Omega ,$$
$$\mathbf{u} = 0 \text{ on } \Gamma_0 , \quad \sigma(\mathbf{u},\mathbf{n}) = \mathbf{g} \text{ on } \Gamma_1 ,$$

where the *boundary traction* is given by

$$\sigma(\mathbf{u},\mathbf{n}) = \lambda(\boldsymbol{\nabla}\cdot\mathbf{u})\mathbf{n} + 2\mu\,\varepsilon(\mathbf{u},\mathbf{n}) , \qquad \varepsilon(\mathbf{u},\mathbf{n})_i \equiv \varepsilon_{ij}(\mathbf{u})n_j, \quad 1 \le i \le 3 .$$

The *operator form* of this boundary-value problem is

$$\mathbf{u} \in \mathbf{V}_1 : \quad \mathcal{E}(\mathbf{u}) = \mathbf{f}_0 \text{ in } \mathbf{V}_1' .$$

For the *fully dynamic* case of a general linearly elastic body with small displacement rates, the *momentum equation* (2.8) takes the form of the initial-boundary value problem for the hyperbolic system

$$\rho\frac{\partial^2 u_i(x,t)}{\partial t^2} - \partial_j a_{ijkl}\,\varepsilon_{kl}(\mathbf{u}(x,t)) = f_i(x,t) \text{ in } \Omega , \quad 1 \le i \le 3 ,$$
$$u_i = 0 \text{ on } \Gamma_0 , \quad a_{ijkl}\,\varepsilon_{kl}(\mathbf{u})n_j = g_i \text{ on } \Gamma_1 .$$

In the *isotropic case* this elasticity system is given by

$$\rho\frac{\partial^2 u_i(x,t)}{\partial t^2} - \partial_j(\lambda\delta_{ij}\boldsymbol{\nabla}\cdot\mathbf{u} + \mu(\partial_i u_j + \partial_j u_i)) = f_i(x,t) \text{ in } \Omega , \quad 1 \le i \le 3 ,$$
$$u_i = 0 \text{ on } \Gamma_0 , \quad \lambda(\boldsymbol{\nabla}\cdot\mathbf{u})n_i + 2\mu\,\varepsilon_{ij}(\mathbf{u})n_j = g_i \text{ on } \Gamma_1 ,$$

and in vector form we have the wave equation

$$\rho\ddot{\mathbf{u}} - (\lambda+\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) - \mu\Delta\mathbf{u} = \mathbf{f} \text{ in } \Omega ,$$
$$\mathbf{u} = 0 \text{ on } \Gamma_0 , \quad \lambda(\boldsymbol{\nabla}\cdot\mathbf{u})\mathbf{n} + 2\mu\,\varepsilon(\mathbf{u},\mathbf{n}) = \mathbf{g} \text{ on } \Gamma_1 .$$

The *operator form* of this evolution equation is

$$\rho\ddot{\mathbf{u}} + \mathcal{E}(\mathbf{u}) = \mathbf{f}_0.$$

Note that the compression component of the stress satisfies $\sigma_{kk}(\mathbf{u}) = (3\lambda + 2\mu)\varepsilon_{kk}(\mathbf{u})$. The limiting case of $3\lambda + 2\mu \to \infty$ leads to the situation in which $\varepsilon_{kk}(\mathbf{u}) = 0$, that is, the material is *incompressible*. Thus, when the dilational modulus of elasticity $\lambda \to +\infty$, the system of partial differential equations becomes a pair of complementary equations

$$\rho\ddot{\mathbf{u}} - \mu\Delta\mathbf{u} + \boldsymbol{\nabla}p = \mathbf{f}\,, \quad \boldsymbol{\nabla}\cdot\mathbf{u} = 0 \text{ in } \Omega\,,$$
$$\mathbf{u} = 0 \text{ on } \Gamma_0\,, \quad -p\,\mathbf{n} + 2\mu\,\varepsilon(\mathbf{u},\mathbf{n}) = \mathbf{g} \text{ on } \Gamma_1\,,$$

in which the limiting variable $p = -\lim_{\lambda\to\infty}\lambda\boldsymbol{\nabla}\cdot\mathbf{u}$ is an unknown function which we identify formally as a *pressure*.

Remark 3.1. *The* incompressible elastic solids *are those which are capable of sustaining substantial deformations without a change in volume. For such a material the stress is* not *determined solely by the deformation! To the stress of an incompressible solid may be added any multiple of the type of stress associated with a pure volume change or dilation without modifying the deformation of the body, i.e., a pressure. Thus, the addition of a pressure p to an incompressible solid body will change the stress to*

$$\sigma_{ij}(\mathbf{u}) = -p\delta_{ij} + 2\mu\varepsilon_{ij}(\mathbf{u})$$

but it will not affect the strain.

3.2. **Visco-elasticity.** Consider a medium in which there are additional internal forces generated by the *strain rate*, $\frac{\partial\varepsilon_{ij}(\mathbf{u})}{\partial t} = \varepsilon_{ij}(\mathbf{v})$, where $\mathbf{v} \equiv \frac{\partial\mathbf{u}}{\partial t}$ is the *displacement velocity.* Thus, we have a *dissipation functional* of the form

$$\mathcal{F}(\mathbf{v}) \equiv \frac{1}{2}\int_\Omega \left(b_{ijkl}\,\partial_j v_i\,\partial_k v_l\right)\,dx\,,$$

and the corresponding stress is given by the derivatives

$$\sigma_{ij} \equiv \frac{\partial\mathcal{E}}{\partial\mathbf{u}} + \frac{\partial\mathcal{F}}{\partial\mathbf{v}} = a_{ijkl}\partial_k u_l + b_{ijkl}\partial_k v_l\,,$$

with the positive and symmetric *viscosity* b_{ijkl}. The momentum equations take the form

$$\rho\ddot{u}_i = \partial_j(b_{ijkl}\partial_k\dot{u}_l) + \partial_j(a_{ijkl}\partial_k u_l) + f_i\,.$$

For the special case of an *isotropic medium*, the viscosity is given by

$$b_{ijkl} \equiv \lambda_1\delta_{ij}\delta_{kl} + \mu_1(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

with the positive Lamé constants λ_1 and μ_1 which denote the *dilational viscosity* and *shear viscosity*, respectively. Then the vector form of the momentum equation is given by the strongly-damped wave equation

$$\rho\ddot{\mathbf{u}} - (\lambda_1 + \mu_1)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\dot{\mathbf{u}}) - \mu_1\Delta\dot{\mathbf{u}} - (\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) - \mu\Delta\mathbf{u} = \mathbf{f} \text{ in } \Omega$$

and the boundary conditions are

$$\mathbf{u} = 0 \text{ on } \Gamma_0\,, \quad \lambda_1(\boldsymbol{\nabla}\cdot\dot{\mathbf{u}})\mathbf{n} + 2\mu_1\,\varepsilon(\dot{\mathbf{u}},\mathbf{n}) + \lambda(\boldsymbol{\nabla}\cdot\mathbf{u})\mathbf{n} + 2\mu\,\varepsilon(\mathbf{u},\mathbf{n}) = \mathbf{g} \text{ on } \Gamma_1\,.$$

The *operator form* of this evolution equation is given by

$$\rho\ddot{\mathbf{u}} + \mathcal{F}(\dot{\mathbf{u}}) + \mathcal{E}(\mathbf{u}) = \mathbf{f}_0\,,$$

where $\mathcal{F}$ denotes the operator representing the *viscous friction*. In order to obtain the case of an *incompressible* material, we let $\lambda \to \infty$ to obtain the system of complementary equations and boundary conditions

$$\rho\ddot{\mathbf{u}} - \mu_1\Delta\dot{\mathbf{u}} - \mu\Delta\mathbf{u} + \boldsymbol{\nabla}p = \mathbf{f}\,, \quad \boldsymbol{\nabla}\cdot\mathbf{u} = 0 \text{ in } \Omega\,,$$
$$\mathbf{u} = 0 \text{ on } \Gamma_0\,, \quad -p\,\mathbf{n} + 2\mu_1\,\varepsilon(\dot{\mathbf{u}},\mathbf{n}) + 2\mu\,\varepsilon(\mathbf{u},\mathbf{n}) = \mathbf{g} \text{ on } \Gamma_1\,,$$

where p is a formal pressure variable.

Remark 3.2. Orientation *One can formally obtain a* viscous fluid *by eliminating the elastic forces in the preceding discussion. That is, the fluid is regarded as an isotropic medium for which internal forces arise* only *from the* strain rate, *i.e., from the* motion. *In terms of the velocity, $\mathbf{v} = \dot{\mathbf{u}}$, this leads to a strongly-parabolic system of the form*

$$\rho\dot{\mathbf{v}} - (\lambda_1 + \mu_1)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \mu_1\Delta\mathbf{v} = \mathbf{f} \ \textit{in}\ \Omega\,,$$
$$\mathbf{v} = 0 \ \textit{on}\ \Gamma_0\,, \quad \lambda_1(\boldsymbol{\nabla}\cdot\mathbf{v})\mathbf{n} + 2\mu_1\,\varepsilon(\mathbf{v},\mathbf{n}) = \mathbf{g} \ \textit{on}\ \Gamma_1\,,$$

and in the incompressible *case we obtain the* Stokes system

$$\rho\dot{\mathbf{v}} - \mu_1\Delta\mathbf{v} + \boldsymbol{\nabla}p = \mathbf{f}\,, \quad \boldsymbol{\nabla}\cdot\mathbf{v} = 0 \ \textit{in}\ \Omega\,,$$
$$\mathbf{v} = 0 \ \textit{on}\ \Gamma_0\,, \quad -p\,\mathbf{n} + 2\mu_1\,\varepsilon(\mathbf{v},\mathbf{n}) = \mathbf{g} \ \textit{on}\ \Gamma_1\,,$$

with p being the (unknown) pressure. However, in the first compressible case, this is inconsistent with the assumption that the density is a prescribed autonomous function. A non-zero mass flux across the boundary of a small region will change with time the fluid density within that region. Thus the density becomes an additional unknown variable and requires a corresponding additional equation to complete the system.

Remark 3.3. *The* material derivative *of velocity has been approximated here by the acceleration. For the calculation of the acceleration of a fluid element, the displacement of that element along with the points must be considered. The momentum of the small subdomain $B \subset \Omega$ travelling with the fluid is $\int_B \rho\mathbf{v}(x + \mathbf{u}(x,t), t)\,dx$, and its derivative is given by the Chain rule as*

$$\int_B \rho\left(\frac{\partial\mathbf{v}(x+\mathbf{u}(x,t),t)}{\partial t} + \partial_j\mathbf{v}(x+\mathbf{u}(x,t),t)\,v_j(x+\mathbf{u}(x,t),t)\right)\,dx\,.$$

Thus, the momentum equation for the fluid includes the additional term $(\mathbf{v}\cdot\nabla)\mathbf{v} = v_j\partial_j\mathbf{v}$, and the corresponding system is the Navier-Stokes system

$$\rho\dot{\mathbf{v}} - \mu_1\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \boldsymbol{\nabla}p = \mathbf{f}\,, \quad \boldsymbol{\nabla}\cdot\mathbf{v} = 0 \ \textit{in}\ \Omega\,,$$
$$\mathbf{v} = 0 \ \textit{on}\ \Gamma_0\,, \quad -p\,\mathbf{n} + 2\mu_1\,\varepsilon(\mathbf{v},\mathbf{n}) = \mathbf{g} \ \textit{on}\ \Gamma_1\,,$$

for a viscous incompressible fluid. Note that the quadratic nonlinearity arises from the geometry *of the motion, and it is* not *based on any independent assumptions.*

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