LECTURES ON
FLOW IN POROUS MEDIA

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1. Models of Porous Media

A porous medium G in $\mathbb{R}^m$ is filled with a fluid, either liquid or gas, and this fluid diffuses from locations of higher to those of lower pressure. We begin by modeling this situation. Let $p(x,t)$ denote the *pressure* of fluid at the point $x \in G$ and time $t > 0$, and denote the corresponding *density* by $\rho(x,t)$. The quantity of fluid in an element of volume V is $\int_V c(x)\rho(x,t)\,dx$, and this defines the *porosity* $c(x)$ of the medium at the point x. This is the volume fraction of the medium that is accessible to the fluid. The *flux* is the vector flow rate $\vec{J}(x,t)$, so the rate at which fluid moves across a surface element S with normal $\vec{\nu}$ is given by $\int_S \vec{J}(x,t)\cdot\vec{\nu}\,dS$. Then the *conservation of fluid* takes the integral form

$$\frac{\partial}{\partial t}\int_{G_0} c(x)\rho(x,t)\,dx + \int_{\partial G_0} \vec{J}\cdot\vec{\nu}\,dS = \int_{G_0} f(x,t)\,dx\ , \qquad G_0 \subset G\ ,$$

in which $f(x,t)$ denotes any volume distributed *source density*. When the flux and density are differentiable, we can write this in the differential form

$$\frac{\partial}{\partial t}c(x)\rho(x,t) + \vec{\nabla}\cdot\vec{J}(x,t) = f(x,t)\ , \qquad x \in G\ .$$

The statement that the flux depends on the pressure gradient is *Darcy's law*, and it takes the form

$$\vec{J}(x,t) = -\frac{k(x)}{\mu}\rho\vec{\nabla}p(x,t)\ .$$

This defines the *permeability* $k(x)$ of the porous medium. The value of k is a measure of the ease with which the fluid flows through the medium, and μ is the *viscosity* of the fluid. That is, $\frac{1}{k}$ is the *resistance* of the medium to flow, and μ is a corresponding property of the fluid. Finally, the type of fluid considered is described by the *equation of state*,

$$\rho = s(p)\ .$$

The function $s(\cdot)$ which relates the pressure and density is monotone, in fact, it is usually chosen to be strictly increasing. However in problems with *partial saturation*, it is necessary to let $s(\cdot)$ be a *graph*, since at $p = 0$ fluid may only partially fill the pores and thereby give an effective density within the interval $[0, \rho(0)]$ related to the fraction of fluid present. By substituting the appropriate quantities above we obtain

$$\frac{\partial}{\partial t} c(x) s(p(x,t)) - \vec{\nabla} \cdot \frac{k(x)}{\mu} \vec{\nabla} S(p(x,t)) = f(x,t) \ , \qquad x \in G \ , \ t > 0 \ ,$$

where we denote by $S(\cdot)$ the antiderivative of $s(\cdot)$. That is, we set $S(p) = \int_0^p s(r)\, dr$. If we introduce the variable $u = S(p)$ we obtain the *generalized porous medium equation*

$$\frac{\partial}{\partial t} c(x) a(u(x,t)) - \vec{\nabla} \cdot \frac{k(x)}{\mu} \vec{\nabla} u(x,t) = f(x,t) \ , \qquad x \in G \ , \ t > 0 \ , \tag{1.1}$$

with $a(\cdot) \equiv s(\cdot) \circ S^{-1}(\cdot)$.

The classical case is obtained by choosing an equation of state that is specific to the flow of gas. This corresponds to the choice of the function

$$s(p) = \rho_0 p^\alpha$$

in the equation of state where ρ_0 and α are positive constants with $\alpha \leq 1$. If the medium is homogeneous, so the porosity and permeability are independent of $x \in G$, then this leads to the *classical porous medium equation*

$$\frac{\partial}{\partial t} c\rho - \frac{k}{\mu(\alpha+1)\rho_0^{m-1}} \Delta \rho^m = f \ , \tag{1.2}$$

where $m = 1 + \frac{1}{\alpha}$. Other situations in which (1.2) arises include boundary layer theory, where $m = 2$, population models, interstellar diffusion of galactic civilizations, and certain problems of plasma physics. The nonlinearity satisfies $m > 1$ in all but the last of these examples, and there $0 < m < 1$. The situation with $m > 1$ is called the *slow diffusion* case, and $0 < m < 1$ is the *fast diffusion* case. In the former case disturbances have a finite speed of propagation, unlike the linear case $m = 1$, and any solution which initially has compact support will continue to have compact support for all later times.

The simplest situation is that of a *slightly compressible* fluid. Here the equation of state has the form $s(p) = \exp c_0\, p$ where $c_0 > 0$ is the *compressibility* of the fluid. Then (1.1) simplifies to the linear parabolic equation

$$\frac{\partial}{\partial t} c\rho - \frac{k c_0}{\mu} \Delta \rho = f \ . \tag{1.3}$$

This is formally the same as the heat equation, i.e., it is (1.2) with $m = 1$. The corresponding initial-boundary-value problems for (1.3) will be discussed in Section 3.

If we assume that the Darcy Law has the nonlinear form

$$\vec{J} = -\frac{k(\rho\vec{\nabla}p)}{\mu}\rho\vec{\nabla}p$$

in which the permeability depends on the flux, then in terms of the variable $u = S(p)$ we obtain the *quasilinear porous medium equation*

$$\frac{\partial}{\partial t}c(x)a(u(x,t)) - \vec{\nabla}\cdot\frac{k(\vec{\nabla}u(x,t))}{\mu}\vec{\nabla}u(x,t) = f(x,t)\ , \qquad x \in G\ ,\ t > 0\ , \tag{1.4}$$

with $a(\cdot) \equiv s(\cdot)\circ S^{-1}(\cdot)$ as before. Finally, if we specialize this to the case of a slightly compressible fluid, we obtain

$$\frac{\partial}{\partial t}c(x)c_0u(x,t) - \vec{\nabla}\cdot\frac{k(\vec{\nabla}u(x,t))}{\mu}\vec{\nabla}u(x,t) = f(x,t)\ , \qquad x \in G\ ,\ t > 0\ . \tag{1.5}$$

The flow of fluid through a fissured porous medium leads to some related systems. A fissured medium consists of a matrix of porous and permeable blocks of pores or *cells* which are isolated from each other by a highly developed system of *fissures* through which the bulk of the fluid transport occurs. Due to this separation of the cells by the fissure system, there is a negligible amount of transport directly from cell to cell. Another feature of fissured media is that the volume occupied by the fissures is considerably smaller than that occupied by the matrix of cells. Thus, most transport occurs in the fissures and the bulk of the storage takes place in the cells. The system is by nature very much unsymmetric in the structure and function of its components. Specifically, the fissure system provides the primary transport component and the cell system accounts for all storage. Thus, the exchange between the fissure system and the matrix of cells is an important component of the model.

More generally, the *parallel flow model* is a classical description of diffusion in heterogeneous media. The idea is to introduce at each point in space a density, pressure or concentration for each component, each being obtained by averaging in the respective medium over a generic neighborhood sufficiently large to contain a representative sample of each component. The rate of exchange between the components must be expressed in terms of these quantities, and the resulting expressions become distributed source and sink terms for the diffusion equations in the individual components. Thus, one obtains a *system* of diffusion equations, one for each component. The classical linear double porosity model for the flow of slightly compressible fluid in a general heterogeneous medium consisting of two components is

$$\begin{aligned} &\frac{\partial}{\partial t}\,c_1u_1 - \frac{k_1c_0}{\mu}\Delta u_1 + \frac{1}{\alpha}(u_1 - u_2) = f\ , \\ &\frac{\partial}{\partial t}\,c_2u_2 - \frac{k_2c_0}{\mu}\Delta u_2 + \frac{1}{\alpha}(u_2 - u_1) = g\ . \end{aligned} \tag{1.6}$$

The first equation describes the flow in fissures — regions of small relative volume but large permeability. The second describes the flow in the cell system — regions

of large porosity or volume, but largely isolated from one another. Both of these equations are to be understood macroscopically; that is, these equations are obtained by averaging over neighborhoods containing a large number of cells (called "blocks" of pores) and fissures. Although the components of (1.6) are structured symmetrically, fissured media characteristics are modeled by very small coefficients c_1 and k_2.

For the fissured medium model, the coefficient c_1 is almost zero because the relative volume of the fissures is small, and $k_2 = 0$ because there is no direct flow from one block of pores to another, i.e., each cell is isolated from the adjacent cells by the fissure system. The last term on the left of each equation represents the exchange of fluid between the cells and the fissures. The parameter α represents the resistance of the medium to this exchange. When $\alpha = \infty$, no exchange flow is possible. An alternative interpretation is that $1/\alpha$ represents the degree of fissuring in the medium. When the degree of fissuring is infinite, the exchange flow encounters no resistance and $u_0 = u_1$. The external sources of fluid represented by f and g are located in the fissures and in the cells, respectively.

Consider the analogous quasilinear situation in which the permeability is flux-dependent. Then we obtain the system

$$\begin{aligned} &\frac{\partial}{\partial t} c_1(x) a(u_1(x,t)) - \vec{\nabla} \cdot \frac{k_1(\vec{\nabla} u_1(x,t))}{\mu} \vec{\nabla} u_1(x,t) + \tau(u_1 - u_2) = f(x,t) \ , \\ &\frac{\partial}{\partial t} c_2(x) a(u_2(x,t)) - \vec{\nabla} \cdot \frac{k_2(\vec{\nabla} u_2(x,t))}{\mu} \vec{\nabla} u_2(x,t) - \tau(u_1 - u_2) = g(x,t) \ , \end{aligned} \tag{1.7}$$

in terms of the variables $u_1 = S(p_1)$ and $u_2 = S(p_2)$. The fluid exchange rate between the cells and fissures is assumed to be determined by a monotone function $\tau(\cdot)$ of the pressure difference and the density, so it is given by $\tau(\tilde{\rho}(p_1 - p_2))$. Here we choose $\tilde{\rho}$ to be the *average* density on the pressure interval $[p_1, p_2]$, so we have

$$\tilde{\rho}(p_2 - p_1) = \int_{p_1}^{p_2} s(r)\, dr = u_2 - u_1 \ .$$

In this way we have obtained an exchange term which is a function of the difference of the components instead of a difference of functions of the two components. If we specialize this to the case of a slightly compressible fluid we obtain the system

$$\begin{aligned} &\frac{\partial}{\partial t} c_1(x) c_0(u_1(x,t)) - \vec{\nabla} \cdot \frac{k_1(\vec{\nabla} u_1(x,t))}{\mu} \vec{\nabla} u_1(x,t) + \tau(u_1 - u_2) = f(x,t) \ , \\ &\frac{\partial}{\partial t} c_2(x) c_0(u_2(x,t)) - \vec{\nabla} \cdot \frac{k_2(\vec{\nabla} u_2(x,t))}{\mu} \vec{\nabla} u_2(x,t) - \tau(u_1 - u_2) = g(x,t) \ . \end{aligned} \tag{1.8}$$

These provide examples of parabolic partial differential equations and systems with multiple nonlinearities as well as a variety of *types* of nonlinearity that can arise. See Section 7.

We shall show that by making an appropriate choice of the operator A and a space of functions, the *initial-value problem*

$$\frac{du(t)}{dt} + Au(t) = 0 \ , \qquad 0 < t \ , \quad u(0) = u_0$$

corresponds to an appropriate *initial-boundary-value problem.* In particular, we shall show that many of our flow problems can be realized as such initial-value problems in function spaces. For a preview of the types of hypotheses on the operator A that are appropriate for the solvability of the initial-value problem, we approximate it by the backward difference equation

$$\frac{u(t) - u(t-\varepsilon)}{\varepsilon} + Au(t) = 0 \ , \qquad 0 < t \ , \quad u(0) = u_0$$

from which the solution can be obtained from the equivalent identity

$$u(t) = (I + \varepsilon A)^{-1} u(t - \varepsilon).$$

This suggests that we should consider those *unbounded operators* A for which $(I + \varepsilon A)^{-1}$ is *stable* for each $\varepsilon > 0$. This leads below to the basic notion of an *m-accretive* operator.

2. Linear Stationary Problems

We begin with a review of certain topics in Hilbert space and the formulation of boundary value problems. Here we illustrate the ideas with special cases, and one can see [1] or the first Chapter of [2] for more complete treatments of the material of this section and the next.

Let V be a linear space over the reals $\mathbb{R}$, and let the function $x, y \mapsto (x, y)$ from $V \times V$ to $\mathbb{R}$ be a *scalar product.* That is, $(x, x) > 0$ for non-zero $x \in V$, $(x, y) = (y, x)$ for $x, y \in V$, and for each $y \in V$ the function $x \mapsto (x, y)$ is linear from V to $\mathbb{R}$. For each pair $x, y \in V$ it follows that

$$\big|(x, y)\big|^2 \leq (x, x)(y, y) \ . \tag{2.1}$$

To see this, we note that

$$0 \leq (tx + y, tx + y) = t^2(x, x) + 2t(x, y) + (y, y) \ , \qquad t \in \mathbb{R} \ ,$$

and so the discriminant of the quadratic must be non-positive. From (2.1) it follows that $\|x\| \equiv (x, x)^{1/2}$, $x \in V$, defines a *norm* on V : $\|x\| \geq 0$, $\|tx\| = |t| \ \|x\|$, and $\|x + y\| \leq \|x\| + \|y\|$ for $x, y \in V$ and $t \in \mathbb{R}$. Thus every scalar product induces a norm and corresponding *metric* $d(x, y) = \|x - y\|$. A sequence $\{x_n\}$ *converges to* x in V if $\lim_{n\to\infty} \|x_n - x\| = 0$. This is denoted by $\lim_{n\to\infty} x_n = x$. A convergent sequence is always *Cauchy*: $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$. The space V with norm $\|\cdot\|$ is *complete* if each Cauchy sequence is convergent in V. A complete normed linear space is a *Banach space*, and a complete scalar product space is a *Hilbert space.*

Some familiar examples of Hilbert spaces include Euclidean space $\mathbb{R}^m = \{\vec{x} = (x_1, x_2, \dots, x_m) : x_j \in \mathbb{R}\}$ with $(\vec{x}, \vec{y}) = \sum_{j=1}^m x_j y_j$, the sequence space $\ell^2 = \{\vec{x} = \{x_1, x_2, x_3, \dots\} : \sum_{j=1}^\infty |x_j|^2 < \infty\}$ with $(\vec{x}, \vec{y}) = \sum_{j=1}^\infty x_j y_j$, and the Lebesgue space $L^2(\Omega) = \{\text{equivalence classes of measurable functions } f : \Omega \to \mathbb{R} : \int_\Omega |f|^2 \, d\mu < \infty\}$ with $(f, g) = \int_\Omega f(w) g(w) \, d\mu$, where (Ω, μ) is a measure space.

Another example is the Sobolev space $H^1(a,b)$ given by

$$H^1(a,b) = \{u \in L^2(a,b) : \partial u \in L^2(a,b)\}$$

where ∂u denotes the distribution derivative of the function u. Thus each $u \in H^1(a,b)$ is absolutely continuous with

$$u(x) - u(y) = \int_y^x \partial u \ , \qquad a \le x \ , y \le b \ .$$

This gives the Hölder continuity estimate

$$|u(x) - u(y)| \le |x-y|^{1/2} \|\partial u\|_{L^2(a,b)} \ , \quad u \in H^1(a,b), a \le x, y \le b \ . \tag{2.2}$$

If also we have $u(a) = 0$ then there follow

$$|u(x)| \le (b-a)^{1/2} \|\partial u\|_{L^2(a,b)} \ , \qquad a \le x \le b \ , \tag{2.3}$$

$$\|u\|_{L^2(a,b)} \le \big((b-a)/\sqrt{2}\big) \|\partial u\|_{L^2(a,b)} \ , \tag{2.4}$$

and such estimates also hold for those $u \in H^1(a,b)$ with $u(b) = 0$. Let $\lambda(x) = (x-a)(b-a)^{-1}$ and $u \in H^1(a,b)$. Then $\lambda u \in H^1(a,b)$ and $\partial(\lambda u) = \lambda \partial u + (b-a)^{-1} u$, so $\|\partial(\lambda u)\|_{L^2} \le \|\partial u\|_{L^2} + (b-a)^{-1}\|u\|_{L^2}$. The same holds for $\partial((1-\lambda)u)$ so by writing $u = \lambda u + (1-\lambda)u$ we obtain

$$\begin{aligned} \max\{|u(x)| : a \le x \le b\} &\le 2(b-a)^{1/2} \|\partial u\|_{L^2} \\ &\quad + 2(b-a)^{-1/2} \|u\|_{L^2} \ , \quad u \in H^1(a,b) \ . \end{aligned} \tag{2.5}$$

Such simple estimates will be very useful, and analogous ones can be easily obtained in appropriate subspaces. To verify that this space is complete, let $\{u_n\}$ be a Cauchy sequence, so that both $\{u_n\}$ and $\{\partial u_n\}$ are Cauchy sequences in $L^2(a,b)$. Since $L^2(a,b)$ is complete there are $u, v \in L^2(a,b)$ for which $\lim u_n = u$ and $\lim \partial u_n = v$ in $L^2(a,b)$. For each $\varphi \in C_0^\infty(a,b)$ we have

$$-\int_a^b u_n \cdot \partial\varphi = \int_a^b \partial u_n \varphi \ , \qquad n \ge 1 \ ,$$

so letting $n \to \infty$ shows $v = \partial u$. Thus $u \in H^1(a,b)$ and $\lim u_n = u$ in $H^1(a,b)$.

More generally, we define for each integer $k \ge 1$ the *Sobolev space*

$$H^k(a,b) = \{u \in L^2(a,b) : \partial^j u \in L^2(a,b) \ , \ 1 \le j \le k\} \ .$$

Let V_1 and V_2 be normed linear spaces with corresponding norms $\|\cdot\|_1$, $\|\cdot\|_2$. A function $T : V_1 \to V_2$ is *continuous at* $x \in V_1$ if $\{T(x_n)\}$ converges to $T(x)$ in V_2 whenever $\{x_n\}$ converges to x in V_1. It is *continuous* if it is continuous at every x. For example, the norm is continuous from V_1 into $\mathbb{R}$. If T is linear, we shall also denote its value at x by Tx instead of $T(x)$.

Proposition 2.1. *If $T : V_1 \to V_2$ is linear, the following are equivalent:*

(a) *T is continuous at 0,*
(b) *T is continuous at every $x \in V_1$,*
(c) *there is a constant $K \geq 0$ such that $\|Tx\|_2 \leq K\|x\|_1$ for all $x \in V_1$.*

Denote by $\mathcal{L}(V_1, V_2)$ the set of all continuous linear functions from V_1 to V_2; these are called the *bounded linear functions* because of (c). Additional structure on this set is given as follows.

Proposition 2.2. *For each $T \in \mathcal{L}(V_1, V_2)$ we have*

$$\begin{aligned}\|T\| &\equiv \sup\{\|Tx\|_2 : x \in V_1, \|x\|_1 \leq 1\} = \sup\{\|Tx\|_2 : \|x\|_1 = 1\} \\ &= \inf\{K > 0 : \|Tx\|_2 \leq K\|x\|_1 \ , \qquad x \in V_1\} \ ,\end{aligned}$$

and this gives a norm on $\mathcal{L}(V_1, V_2)$. If V_2 is complete, then $\mathcal{L}(V_1, V_2)$ is complete.

As a consequence it follows that the *dual* $V' \equiv \mathcal{L}(V, \mathbb{R})$ of any normed linear space V is complete with the dual norm

$$\|f\|_{V'} \equiv \sup\{|f(x)| : x \in V, \|x\|_V \leq 1\} \tag{2.6}$$

for $f \in V'$.

Hereafter we let V denote a Hilbert space with norm $\|\cdot\|$, scalar product $(\cdot, \cdot)$, and dual space V'. A subset K of V is called *closed* if each $x_n \in K$ and $\lim x_n = x$ imply $x \in K$. The subset K is *convex* if $x, y \in K$ and $0 \leq t \leq 1$ imply $tx + (1-t)y \in K$. The following *minimization principle* is fundamental. Let $a : V \times V \to \mathbb{R}$ be bilinear (linear in each variable separately), continuous, symmetric ($a(x, y) = a(y, x), x, y \in V$) and *$V$-elliptic*: there is a $c_0 > 0$ such that

$$a(x, x) \geq c_0\|x\|^2 \ , \qquad x \in V \ . \tag{2.7}$$

Theorem 2.1. *Let $a(\cdot, \cdot)$ be a bilinear, symmetric, continuous and V-elliptic form on the Hilbert space V, let K be a closed, convex and non-empty subset of V, and let $f \in V'$. Set $\varphi(x) = (1/2)a(x, x) - f(x)$, $x \in V$. Then there is a unique*

$$x \in K : \varphi(x) \leq \varphi(y) \ , \qquad y \in K \ . \tag{2.8}$$

The solution of (2.8) *is characterized by*

$$x \in K : a(x, y - x) \geq f(y - x) \ , \qquad y \in K \ . \tag{2.9}$$

If, in addition, K is a subspace of V, then (2.9) *is equivalent to*

$$x \in K : a(x, y) = f(y) \ , \qquad y \in K \ . \tag{2.10}$$

Corollary 2.1. *For each closed subspace K of V and each $x_0 \in V$ there is a unique*

$$x \in K : (x - x_0, y) = 0 \ , \qquad y \in K \ .$$

Two vectors $x, y \in V$ are called *orthogonal* if $(x, y) = 0$, and the *orthogonal complement* of the set S is $S^\perp \equiv \{x \in V : (x, y) = 0 \text{ for } y \in S\}$. Corollary 2.1 says each $x_0 \in V$ can be uniquely written in the form $x_0 = x_1 + x_2$ with $x_1 \in K$ and $x_2 \in K^\perp$ whenever K is a closed subspace. We denote this orthogonal decomposition by $V = K \oplus K^\perp$.

The *Riesz map* $\underset{\sim}{\mathcal{R}}$ of V into V' is defined by $\underset{\sim}{\mathcal{R}}x(y) = (x, y)$ for $x, y \in V$. It is clear that $\|\underset{\sim}{\mathcal{R}}x\|_{V'} = \|x\|_V$; Theorem 2.1 with $K = V$ shows by way of (2.10) that $\underset{\sim}{\mathcal{R}}$ is onto V', so $\underset{\sim}{\mathcal{R}}$ is an isometric isomorphism of the Hilbert space V onto its dual V'. Specifically, for each $f \in V'$ there is a unique $x = \underset{\sim}{\mathcal{R}}^{-1}(f) \in V$.

Corollary 2.2. *For each $f \in V'$ there is a unique*

$$x \in V : (x, y) = f(y) \ , \qquad y \in V \ .$$

Even when $a(\cdot, \cdot)$ is not symmetric we can still solve the linear problem (2.10), although it no longer is related to a minimization problem.

Theorem 2.2 (Lax-Milgram). *Let $a(\cdot, \cdot)$ be bilinear, continuous and V-coercive: there is a $c_0 > 0$ such that*

$$|a(x, x)| \geq c_0 \|x\|^2 \ , \qquad x \in V \ .$$

Then for each $f \in V'$ there is a unique

$$x \in V : a(x, y) = f(y) \ , \qquad y \in V \ . \tag{2.11}$$

Proof. For each $x \in V$ the function "$y \mapsto a(x, y)$" belongs to V', so by Corollary 2.2 there is a unique $\alpha(x) \in V : (\alpha(x), y) = a(x, y)$, $y \in V$. This defines $\alpha \in \mathcal{L}(V, V)$, and we similarly construct $\beta \in \mathcal{L}(V, V)$ with $(x, \beta(y)) = a(x, y)$ for $x, y \in V$. Since (2.11) is equivalent to $\alpha(x) = \underset{\sim}{\mathcal{R}}^{-1}(f)$, it suffices to show α is invertible. First, α is one-to-one, since

$$c_0 \|x\|^2 \leq |a(x, x)| = \big|\big(\alpha(x), x\big)\big| \leq \|\alpha(x)\| \ \|x\| \ ,$$

and so $\alpha(x) = 0$ implies $x = 0$. Also, $c_0 \|x\| \leq \|\alpha(x)\|$ for all $x \in V$. Second, we show the range of α, $Rg(\alpha)$, is closed. If $\lim_{n \to \infty} z_n = z$ and $z_n = \alpha(x_n)$, then $c_0 \|x_n - x_m\| \leq \|z_n - z_m\|$ so $\{x_n\}$ is Cauchy, hence, convergent to some $x \in V$. But α is continuous, so $\alpha(x) = z \in Rg(\alpha)$. Finally, since $K \equiv Rg(\alpha)$ is a closed subspace, hence $V = Rg(\alpha) \oplus Rg(\alpha)^\perp$, we need only show $Rg(\alpha)^\perp = \{0\}$. But if $y \in Rg(\alpha)^\perp$ then for every $x \in V$, $0 = (\alpha(x), y) = (x, \beta(y))$, so $\beta(y) = 0$. As above, β is one-to-one, so $y = 0$. Thus $Rg(\alpha) = V$. $\square$

Let H be a Hilbert space, D a subspace (algebraic) of H and let $A : D \to H$ be linear. Such a map we call an *unbounded operator* on H with *domain* D. The *graph* of A is the subspace

$$G(A) = \{[x, Ax] : x \in D\}$$

of the product $H \times H$. Note that $H \times H$ is also a Hilbert space with componentwise addition and scalar multiplication, and its scalar product is

$$\big([x_1, x_2], [y_1, y_2]\big)_{H\times H} = (x_1, y_1)_H + (x_2, y_2)_H \ .$$

The operator A is called *closed* if $G(A)$ is a closed subspace of $H \times H$. That is, A is closed if whenever $x_n \in D$, $x_n \to x$ and $Ax_n \to y$ in H imply that $x \in D$ and $Ax = y$. This is a much weaker condition than continuity of A, since convergence of $\{Ax_n\}$ is an assumption, not a conclusion.

Definition. An (unbounded) operator $A : D \to H$ is *accretive* if

$$(Ax, x)_H \geq 0 \ , \qquad x \in D \ ,$$

and it is *m-accretive* if, in addition, $A + I$ maps D onto H, i.e., $Rg(A + I) = H$.

Proposition 2.3. *The following are equivalent:*

(a) *$A : D \to H$ is accretive and there exists a $\mu > 0$ such that $Rg(\mu I + A) = H$,*
(b) *A is m-accretive, and*
(c) *A is accretive, D is dense in H, and $Rg(\lambda I + A) = H$ for every $\lambda > 0$.*

Such operators will occur frequently and play an important role in our work below. We give some examples in $H = L^2(a, b)$.

Example 2.a. Set $D = \{v \in H^1(a, b) : v(a) = cv(b)\}$ and $A = \partial$. Then A is closed, and A is accretive only if $|c| \leq 1$, and then it is m-accretive with

$$(I + A)u = f \qquad \text{if and only if}$$
$$u(x) = \int_a^b G(x, s) f(s)\, ds \ , \qquad a \leq x \leq b \ ,$$

where

$$G(x, s) = \frac{e^{-(x-s)}}{e^{b-a} - c} \begin{cases} e^{b-a} \ , & a \leq s < x \ , \\ c \ , & x < s \leq b \ . \end{cases}$$

The integrand $G(\cdot, \cdot)$ is the Green's function for $A + I$, and it is characterized for each $s \in (a, b)$ as the solution of

$$G(\cdot, s) \in D \ , \qquad (I + A)G(\cdot, s) = \delta_s \ ,$$

where δ_s is the Dirac functional at s.

Example 2.b. Set $D = \{v \in H_0^1(a, b) : \partial^2 v \in L^2(a, b)\}$ and $A = -\partial^2$. Then one can show directly that A is closed. It is easy to check that A is accretive and from Theorem 2.1 that A is m-accretive. This operator corresponds to the *Dirichlet problem.*

Actually Example 2.B illustrates a general situation that occurs frequently. Let V be a Hilbert space which is dense in another Hilbert space H, and assume the identity $V \to H$ is continuous. Let $a(\cdot,\cdot)$ be a continuous bilinear form on V. Then we define D to be the set of all $u \in V$ such that the function $v \mapsto a(u,v)$ is continuous on V *with the H-norm*. For each such $u \in D$ there is then a unique $Au \in H$ such that

$$a(u,v) = (Au,v)_H \ , \qquad u \in D \ , v \in V \ ,$$

and this defines a linear operator $A : D \to H$. The special case of

$$H = L^2(a,b) \ , \qquad V = H_0^1(a,b) \ , \qquad a(u,v) = (\partial u, \partial v)_H$$

gives our last example.

Consider the *adjoint form* on V given by $b(u,v) = a(v,u)$, $u,v \in V$. This leads likewise to an operator $B : D(B) \to H$ given by

$$a(u,v) = (u,Bv)_H \ , \qquad u \in V, v \in D(B) \ .$$

Then we obtain the following.

Proposition 2.4. *Assume there is a $\lambda \in \mathbb{R}$ and a $c > 0$ such that*

$$a(v,v) + \lambda \|v\|_H^2 \geq c\|v\|_V^2 \ , \qquad v \in V \ .$$

Then D is dense in H, the operator $A + \lambda I : D(A) \to H$ is one-to-one and onto and its inverse is continuous, and A is closed.

Proof. If $F \in H$ then $v \mapsto (F,v)_H$ is continuous and linear on V, so by Theorem 2.2 there is a unique

$$u \in V : a(u,v) + \lambda(u,v)_H = (F,v)_H \ , \qquad v \in V \ .$$

Thus $u \in D$ and $(A+\lambda)u = F$, so $A + \lambda$ maps D one-to-one onto H. Similarly, $B + \lambda$ maps $D(B)$ one-to-one onto H. If $w \in D^\perp$ in H then there is a $v \in D(B)$ with $(B+\lambda)v = w$, and so

$$0 = (u,w)_H = \big(u,(B+\lambda)v\big)_H = \big((A+\lambda)u,v\big)_H \ , \qquad u \in D \ .$$

Since $A + \lambda I$ is onto, $v = 0$ and $w = 0$, so $D^\perp = \{0\}$. This shows D is dense. We can deduce that A is closed from the fact that $(A+\lambda)^{-1}$ is continuous. $\square$

Corollary 2.3. *Assume that $a(\cdot,\cdot)$ is non-negative, i.e.,*

$$a(v,v) \geq 0 \ , \qquad v \in V \ ,$$

and that there is a $c > 0$ for which

$$a(v,v) + \|v\|_H^2 \geq c\|v\|_V^2 \ , \qquad v \in V \ .$$

Then A is m-accretive.

Here is another example to illustrate the situation, the *Neumann problem*.

Example 2.c. Set $H = L^2(a,b)$, $V = H^1(a,b)$ and define

$$a(u,v) = \int_a^b \partial u \partial v \ , \qquad u, v \in V \ .$$

As above this determines an unbounded operator A on $L^2(a,b) : Au = F \in L^2(a,b)$ is equivalent to

$$u \in V : a(u,v) = (F,v)_{L^2} \ , \qquad v \in V \ ,$$

and this weak Neumann problem is equivalent to

$$u \in V \ , \qquad -\partial^2 u = F \ , \qquad \partial u(a) = \partial u(b) = 0 \ .$$

Thus, we find

$$D = \left\{ u \in V : \partial^2 u \in L^2(a,b) \ , \partial u(a) = \partial u(b) = 0 \right\}$$

and $A = -\partial^2 : D \to L^2(a,b)$. From Corollary 2.3 it follows that A is m-accretive, and that $A + \lambda I$ is a bijection of D onto $L^2(a,b)$ for every $\lambda > 0$. The situation is different at $\lambda = 0$. Specifically, it is clear that $Au = F$ is possible only if $\int_a^b F = 0$, i.e., F is orthogonal in L^2 to the constant functions. Conversely, one can show that this condition on F is sufficient for the existence of a solution. (For example, solve the mixed problem

$$-\partial^2 u = F \ , \qquad u(a) = \partial u(b) = 0 \ ,$$

and then note that $\int_a^b F = \partial u(a)$.) In summary, we find the range of A and kernel of A are given by

$$\begin{aligned} \mathrm{Ker}(A) &= \{ \text{ constant functions } \} \ , \\ \mathrm{Rg}\ (A) &= \mathrm{Ker}(A)^{\perp} \ . \end{aligned}$$

We have seen that (2.10) arises as the weak formulation of certain boundary value problems. Specifically, when $V = H_0^1$ or H^1, (2.10) is the Dirichlet or Neumann problem, respectively. This weak formulation is the special case of a *variational inequality* (2.9) which is the characterization of the solution of the minimization problem (2.8).

It is easy to see how the unbounded operator A with domain D in H constructed as above from the continuous bilinear form $a(\cdot,\cdot)$ on V is related to the continuous $\mathcal{A} \in \mathcal{L}(V, V')$ which is equivalent to $a(\cdot,\cdot)$. In fact, the graph of A is the restriction of the graph of $\mathcal{A}$ to $V \times H$. That is, note that $H' \hookrightarrow V'$ by "restriction to V" of functionals on H, so $D = \{u \in V : \mathcal{A}u \in H'\}$ and then $Au \in H$ is just that $\mathcal{A}u \in H'$ which corresponds through the identification of H with H' by its Riesz map. Thus, with this identification $H = H'$ in the proof of Proposition 2.4, it is clear that $\mathcal{A} + \lambda I$ is an isomorphism of V onto V' and $A + \lambda I$ is just its (necessarily onto) restriction to $H \subset V'$. (More generally, if $\mathcal{R}$ is the Riesz map of H onto H', then $A = \mathcal{R}^{-1}\mathcal{A}$.) Finally, note that A is accretive on H exactly when the linear operator $\mathcal{A}$ satisfies

$$\mathcal{A}v(v) \geq 0 \ , \qquad v \in V \ .$$

This property of $\mathcal{A}$ is called *monotone*. Not every m-accretive A corresponds to a monotone $\mathcal{A}$ as above; those which do are a special class.

Definitions. Let V, H be Hilbert spaces with $H \cong H'$ and let $\mathcal{A} \in \mathcal{L}(V, V')$ be *monotone*:

$$\mathcal{A}v(v) \geq 0 \ , \qquad v \in V \ .$$

The corresponding unbounded operator on H, $A = \mathcal{A}|_{V \times H}$, is then accretive and we shall call it *regular accretive* when it is so determined by a triple $\{\mathcal{A}, V, H\}$. Assume further that for every $\varepsilon > 0$, $\mathcal{A} + \varepsilon I$ is V-elliptic. (This implies that $\mathcal{A}$ is monotone.) Then $\mathrm{Rg}(\mathcal{A} + \varepsilon I) = V'$ for each $\varepsilon > 0$ and so $\mathrm{Rg}(A + \varepsilon I) = H$, hence, A is m-accretive, and we shall call it *regular m-accretive.*

3. The Cauchy Problem

Let H be a Hilbert space, D a subspace, and $A : D \to H$ an unbounded linear operator. The *Cauchy Problem* for the evolution equation

$$u'(t) + Au(t) = 0 \ , \qquad t > 0 \ , \tag{3.1}$$

is to find a *solution* $u \in C([0, \infty), H) \cap C^1((0, \infty), H)$ such that $u(t) \in D(A)$ for $t > 0$ and $u(0) = u_0$, where $u_0 \in H$ is prescribed. The continuity or differentiability of the vector-valued function $u : [0, \infty) \to H$ is defined exactly as in the real-valued case $H = \mathbb{R}$, but with absolute-value replaced by the H norm.

Suppose that for each $u_0 \in D$ there is a unique solution $u(\cdot)$ of the Cauchy Problem; then define $S(t)u_0 = u(t)$ for $t \geq 0$, $u_0 \in D$. Since A is linear it follows each $S(t) : D \to D$ is linear for $t \geq 0$. Furthermore, since the translate $u(t + \tau)$ is a solution of (3.1) for each $\tau \geq 0$, we find from the uniqueness that

$$S(t + \tau)u_0 = S(t)S(\tau)u_0 \ , \qquad t, \tau \geq 0 \ ; S(0) = I \ .$$

We also have

$$1/2 \frac{d}{dt} \|u(t)\|^2 = -\big(Au(t), u(t)\big)_H \ ,$$

so if A is accretive then $\|u(t)\|$ is decreasing for $t \geq 0$, hence, $\|S(t)u_0\| \leq \|u_0\|$ and thus each $S(t)$ is a contraction on D. If D is dense in H each $S(t)$ has a unique extension to a contraction on H and we obtain the following.

Definition. $\{S(t) : t \geq 0\}$ is a *linear contraction semigroup* (or LCS) if $S(t) : H \to H$ is a linear contraction for each $t \geq 0$,

$$S(t + \tau) = S(t)S(\tau) \ \text{ for } \ t, \tau \geq 0 \ , \ S(0) = I \ ,$$

$$\text{and} \qquad S(\cdot)x \in C\big([0, \infty), H\big) \ \text{ for each } \ x \in H \ .$$

The *generator* of the LCS $\{S(t) : t \geq 0\}$ is the operator B defined by $Bx \equiv D^+(S(0)x)$ for each x belonging to

$$D(B) = \{x \in H : \lim_{h \to 0^+} h^{-1}\big(S(h)x - x\big) \equiv D^+\big(S(0)x\big) \ \text{ exists}\} \ .$$

Our preceding remarks verify most of the following.

Proposition 3.1. *Let $A \in L(D,H)$ be closed and accretive, D dense in H, and assume for every $u_0 \in D$ there exists a solution $u \in C^1([0,\infty),H)$ of (3.1) on $t \geq 0$ with $u(0) = u_0$. Construct $\{S(t) : t \geq 0\}$ as above, so $u(t) = S(t)u_0$, $t \geq 0$. Then $\{S(t) : t \geq 0\}$ is a LCS on H whose generator is an extension of $-A$.*

Proof. Since A is accretive, each Cauchy problem has at most one solution, so the construction of $\{S(t) : t \geq 0\}$ is done as above. For each $u_0 \in D$ we have

$$S(t)u_0 - u_0 = \int_0^t u' = -\int_0^t Au(s)\,ds\ , \qquad t > 0\ ,$$

and the integrand is continuous on $[0,\infty)$, so $D^+(S(0)u_0) = -Au_0$. $\square$

Our objective is to find sufficient conditions on an operator A in order that the Cauchy problem for (3.1) will have a solution. These are contained in the following characterization of the operators which are generators of linear contraction semigroups.

Theorem 3.1 (Hille-Yosida). *A necessary and sufficient condition for B to be the generator of a linear contraction semigroup is that $D(B)$ is dense and $\lambda(\lambda - B)^{-1}$ is a contraction on H for every $\lambda > 0$.*

Its proof contains the following:

Corollary 3.1. *A necessary and sufficient condition for $-A : D(A) \to H$ to be the generator of a linear contraction semigroup is that A be m-accretive. If $u_0 \in D(B)$ and $u(t) \equiv S(t)u_0$, $t \geq 0$, then*
$u \in C^1([0,\infty),H)$ satisfies $u'(t) + Au(t) = 0$, $t \geq 0$, and $u(0) = u_0$.

Example 3.a. Let $H = L^2(0,1)$ and $A = \partial$ on $D(A) = \{u \in H^1(0,1) : u(0) = cu(1)\}$ with $|c| \leq 1$. We showed in Section 2 that A is m-accretive, so by Corollary 3.1 we see the initial-boundary-value problem

$$\partial_t u(x,t) + \partial_x u(x,t) = 0\ , \qquad 0 < x < 1\ ,\ t \geq 0\ , \tag{3.2.a}$$

$$u(0,t) = cu(1,t) \tag{3.2.b}$$

$$u(x,0) = u_0(x) \tag{3.2.c}$$

has a unique solution for each $u_0 \in D(A)$. This is a *linear transport equation* for a purely convective flow. An explicit representation for this solution can be easily found. Since any solution of (3.2.a) is of the form $u(x,t) = F(x-t)$, it follows that

$$u(x,t) = u_0(x-t)\ , \qquad 0 \leq t \leq x \leq 1\ ,$$

and then (3.2.b) implies

$$u(x,t) = cu_0(1+x-t)\ , \qquad x \leq t \leq x+1\ .$$

By an easy induction we obtain

$$u(x,t) = c^n u_0(n+x-t)\ , \qquad n-1+x \leq t \leq n+x\ ,\ n \geq 1\ .$$

This representation of the solution gives some additional information. First, the Cauchy problem can be solved only if $u_0 \in D(A)$, because $u(\cdot,t) \in D(A)$ implies $u(\cdot,t)$ is (absolutely) continuous and this is possible only if u_0 satisfies the boundary condition (3.2.b). Second, the solution satisfies $u(\cdot,t) \in H^1(0,1)$ for every $t \geq 0$ but will *not* belong to $H^2(0,1)$ unless $\partial u_0 \in D(A)$. That is, we do not in general have $u(\cdot,t) \in H^2(0,1)$, no matter how smooth the initial function u_0 may be. Finally, the representation above defines a solution of (3.2) on $-\infty < t < \infty$ by allowing n to be *any* integer. Thus, the problem can be solved backwards in time as well as forward. This is related to the fact that $-A$ generates a *group* of operators.

Example 3.b. For our second example we take $H = L^2(a,b)$, $V = H_0^1(a,b)$ and define $a(u,v) = (\partial u, \partial v)_H$ for $u, v \in V$. Corollary 2.3 shows that the operator

$$A = -\partial^2 \ , \ D(A) = H_0^1(a,b) \cap H^2(a,b)$$

is m-accretive. Thus by Corollary 3.1 we obtain existence and uniqueness for the initial-boundary-value problem

$$\partial_t u(x,t) = \partial_x^2 u(x,t) \ , \qquad a < x < b \ , \ t > 0 \ , \tag{3.3.a}$$

$$u(a,t) = u(b,t) = 0 \ , \tag{3.3.b}$$

$$u(x,0) = u_0(x) \ , \tag{3.3.c}$$

for each $u_0 \in D(A)$. This is the linear diffusion equation (1.3).

Due to the *parabolic* equation in this problem, this last example illustrates some *regularizing effects* that occur with evolutions governed by regular m-accretive operators. Consider the case of such an operator A which arises from a triple $\{a(\cdot,\cdot), V, H\}$ for which $a(\cdot,\cdot)$ is symmetric. Let $\{S(t) : t \geq 0\}$ be the semigroup generated by $-A$, $u_0 \in D(A)$, and $u(t) = S(t)u_0$. Thus, (3.1) holds and $u(0) = u_0$. We seek estimates which imply "regularity" of the solution $u(t)$. First, from $(Au(t), u(t))_H = -\frac{1}{2}\frac{d}{dt}\|(u(t)\|_H^2$ we obtain

$$\int_0^T a(u(t),u(t))\,dt = \frac{1}{2}(\|u_0\|_H^2 - \|u(T)\|_H^2) \ . \tag{3.4}$$

For each $h > 0$ and $t > \tau > 0$, $u(t+h) - u(t) = S(t-\tau)(u(\tau+h) - u(\tau))$, so $S(t-\tau)$ being a contraction shows $\|u'(t)\|_H \leq \|u'(\tau)\|$, hence, $\|u'(\cdot)\|_H$ is non-increasing. We have

$$t\|u'(t)\|_H^2 = -\frac{t}{2}\frac{d}{dt}a(u(t),u(t)) = -\frac{d}{dt}\left(\frac{t}{2}a(u(t),u(t))\right) + \frac{1}{2}a(u(t),u(t))$$

since $a(\cdot,\cdot)$ is symmetric, and this yields

$$\int_0^T t\|u'(t)\|_H^2\,dt + \frac{T}{2}a(u(T),u(T)) = \frac{1}{2}\int_0^T a(u(t),u(t))\,dt \ . \tag{3.5}$$

Using the non-increase of $\|u'(\cdot)\|_H$ and (3.4), we obtain

$$\|u'(T)\|_H^2 \frac{T^2}{2} \leq \frac{1}{4}\|u_0\|_H^2 \ ,$$

and this leads to the following *parabolic regularizing property* of the LCS.

Theorem 3.2. *If A is regular m-accretive and the associated bilinear form is symmetric, then the generated LCS satisfies the following:*
$S(t)$ maps H into $D(A)$, and $\|tAS(t)\|_{\mathcal{L}(H)} \leq \frac{1}{\sqrt{2}}$, $\qquad t > 0$.

Proof. Let $w \in H$ and $w_n \in D(A)$ for $n \geq 1$ with $w_n \to w$. We have $S(t)w_n \to S(t)w$ and $\|AS(t)(w_m - w_n)\|_H \leq \|w_m - w_n\|_H/\sqrt{2}t$, so $\{AS(t)w_n\}$ converges in H. But A is closed, and so the desired result follows. □

Corollary 3.2. *For every $t > 0$ and integer $p \geq 1$*

$$S(t) \text{ maps } H \text{ into } D(A^p) \text{ , and } \|A^p S(t)\|_{\mathcal{L}(H)} \leq \left(\frac{p}{t\sqrt{2}}\right)^p \text{ .}$$

This gives the *spatial regularity* of the solution $u(t)$ of the Cauchy problem.

Corollary 3.3. *For every $u_0 \in H$ there is a unique solution $u \in C([0,\infty), H) \cap C^\infty((0,\infty), H)$ of (3.1) with $u(0) = u_0$, and it satisfies $u(t) \in D(A^p)$ for every $t > 0$ and $p \geq 1$.*

For any m-accretive operator, $A + I$ is a bijection of $D(A)$ onto H, and if we define a norm on $D(A)$ by

$$\|x\|_D \equiv (\|x\|_H^2 + \|Ax\|_H^2)^{1/2} \text{ ,} \qquad x \in D(A) \text{ ,}$$

it follows that $D(A)$ is a Hilbert space isomorphic to H. Similarly, $D(A^p)$ is a Hilbert space with scalar-product

$$(x, y)_{D^p} \equiv (x, y)_H + (A^p x, A^p y)_H \text{ ,} \qquad x, y \in D(A^p) \text{ ,}$$

and $(A + I)^p$ is an isomorphism of $D(A^p)$ onto H. For the special case of a self-adjoint regular accretive operator as above we can deduce from the identity

$$A^p \frac{u(t+h) - u(t)}{h} = \frac{S(t - \varepsilon + h) - S(t - \varepsilon)}{h} A^p S(\varepsilon) u_0 \text{ ,} \qquad 0 < \varepsilon < t \text{ , } h > 0 \text{ ,}$$

that $\lim_{h \to 0^+} \left(\frac{u(t+h)-u(t)}{h}\right) = u'(t)$ in the space $D(A^p)$. When A is a differential operator this shows $u'(t)$ agrees with the partial derivative with respect to time and a corresponding *temporal regularity* of the solution of (3.1). The preceding shows that (3.3) has a *smooth* solution for each $u_0 \in L^2(a, b)$.

Similar results hold also for the corresponding non-homogeneous equation.

Theorem 3.3. *Let the operator A be m-accretive on the Hilbert space H. Then for every $u_0 \in D(A)$ and $f \in C^1([0,\infty), H)$ there is a unique solution $u \in C^1([0,\infty), H)$ of the initial-value problem*

$$\begin{aligned} u'(t) + Au(t) &= f(t) \text{ ,} \qquad t > 0 \text{ ,} \\ u(0) &= u_0 \text{ .} \end{aligned} \tag{3.6}$$

If additionally A is symmetric, then for each $u_0 \in H$ and Hölder continuous $f \in C^\alpha([0,\infty), H)$, $0 < \alpha < 1$, there is a unique solution $u \in C([0,\infty), H) \cap C^1((0,\infty), H)$ of (3.6).

Example 3.c. For our final example we take the product spaces $H = L^2(a,b) \times L^2(a,b)$, $V = H_0^1(a,b) \times H_0^1(a,b)$ and define the bilinear form

$$a(\tilde{u},\tilde{v}) = (k_1 \partial u_1, \partial v_1)_{L^2(a,b)} + (k_2 \partial u_2, \partial v_2)_{L^2(a,b)} + \frac{1}{\alpha}(u_1 - u_2, v_1 - v_2)_{L^2(a,b)}$$

for $\tilde{u} = [u_1, u_2], \tilde{v} = [v_1, v_2] \in V$. For the scalar-product on H we take

$$(\tilde{u},\tilde{v})_H = (c_1 u_1, v_1)_{L^2(a,b)} + (c_2 u_2, v_2)_{L^2(a,b)} \ .$$

Here we assume that $k_1,\ k_2,\ c_1,\ c_2$ are positive numbers. Corollary 2.3 shows that we obtain an m-accretive operator A in H, for which the resolvent equation

$$\tilde{u} \in V : (\tilde{u},\tilde{v})_H + a(\tilde{u},\tilde{v}) = \tilde{f}(\tilde{v}) \,, \quad \tilde{v} \in V \,,$$

corresponds to

$$\begin{aligned} c_1 u_1 - k_1 \partial^2 u_1 + \frac{1}{\alpha}(u_1 - u_2) &= f_1 \ , \\ c_2 u_2 - k_2 \partial^2 u_2 + \frac{1}{\alpha}(u_2 - u_1) &= f_2 \ , \\ u_1(a) = u_1(b) = u_2(a) &= u_2(b) = 0 \ , \end{aligned}$$

on the domain $D(A) = \{\tilde{u} \in V \cap (H^2(a,b) \times H^2(a,b))\}$. Then from Theorem 3.3 we obtain existence and uniqueness of a solution of the initial-boundary-value problem

$$\begin{aligned} &c_1 \frac{\partial}{\partial t}\, u_1(x,t) - k_1 \partial^2 u_1(x,t) + \frac{1}{\alpha}(u_1(x,t) - u_2(x,t)) = f_1(x,t) \ , \\ &c_2 \frac{\partial}{\partial t}\, u_2(x,t) - k_2 \partial^2 u_2(x,t) + \frac{1}{\alpha}(u_2(x,t) - u_1(x,t)) = f_2(x,t) \ , \\ &u_1(a,t) = u_1(b,t) = u_2(a,t) = u_2(b,t) = 0 \ , \\ &u_1(x,0) = u_1(x) \ , u_2(x,0) = u_2(x) \ , \qquad a < x < b \ , \ t > 0 \ , \end{aligned}$$

for each $\tilde{u}_0 = [u_1, u_2] \in H$ and $\tilde{f} = [f_1(t), f_2(t)] \in C^\alpha([0,\infty), H)$. This system is the parallel flow model (1.6). We can easily modify the above to allow k_1 or k_2 to be zero, and by a different construction even to permit c_1 or c_2 to be zero.

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4. Distributed Microstructure Models

As a first approximation to flow in a region G which consists of a composition of two finely interspersed materials, one can consider *averaged solutions*, one for each material and both defined at every point $x \in G$. This leads to a pair of partial differential equations, one identified with each of the two components, and a coupling term that describes the flow across the interface between these components. The values at each point x of the two dependent variables in this system (the solutions) have been obtained by averaging in the respective media over a generic neighborhood, which is located at $x \in G$ and is sufficiently large to contain a representative sample of each component. Since the two components are treated symmetrically in the resulting system of two parabolic partial differential equations, such a double porosity model is said to be of *parallel flow* type. Although appropriate for many situations, this symmetric treatment of the two components can be a real limitation. For a fissured medium, for example, such a representation is particularly restrictive, since the porous and permeable cells within the structure have flow properties radically different from those of the surrounding highly developed system of fissures. Moreover the geometry of the individual cells and the corresponding interface are lost in the averaging process leading to such models. For layered media similar remarks apply, and these could be supplemented by nonisotropic considerations.

The classical example of a *parallel flow model* for single phase flow in a composite medium is the parabolic system

$$\begin{aligned} \frac{\partial}{\partial t}(au_1) - \vec{\nabla} \cdot (A\vec{\nabla}u_1) + \frac{1}{\delta}(u_1 - u_2) &= f_1 \,, \\ \frac{\partial}{\partial t}(bu_2) - \vec{\nabla} \cdot (B\vec{\nabla}u_2) + \frac{1}{\delta}(u_2 - u_1) &= f_2 \,, \end{aligned} \tag{4.1}$$

discussed in [8] for which u_1 represents the density of fluid in the first material and u_2 the density in the second. The coefficients $a(x)$ and $A(x)$ are porosity and permeability of the first material, respectively, while $b(x)$ and $B(x)$ are corresponding properties of the second material. The first equation quantifies the rate of flow in the first component of the composite, and the second equation quantifies the corresponding flow rate in the second. Both of these equations are to be understood macroscopically; that is, they were obtained by averaging over a generic neighborhood sufficiently large to contain contributions from each component. The third term in each equation is an attempt to quantify the exchange of fluid between the two components. See [52] for a corresponding system which describes heat conduction in such a composite medium.

A *fissured medium* consists of a matrix of porous and permeable material cells through which is intertwined a highly developed system of fissures. The bulk of the flow occurs in the highly permeable fissure system, and most of the storage of fluid is in the matrix of cells which accounts for almost all of the total volume. One approach to constructing a model of such a medium is to regard the fissure system as the first component and the cell matrix as the second component of a general composite by adjusting the coefficients in (4.1) appropriately. These fissured media characteristics are modeled by choosing very small values for the coefficients $a(x)$ and $B(x)$ in (4.1). Since one component is essentially responsible

for storage and the other for transport, the distributed exchange of fluid between the two components is of fundamental importance. The parameter δ represents the resistance of the medium to this exchange. (When $\delta = \infty$, no exchange flow is possible, and the system is completely decoupled.) An alternative interpretation is that $1/\delta$ represents the degree of fissuring in the medium. (When the degree of fissuring is infinite, the exchange flow encounters no resistance and $u_1 = u_2$.)

In order to specialize the system (4.1) to a totally fissured medium in which the individual cells are isolated from each other, one sets $B = 0$, because there is no direct flow through the matrix of cells; only an indirect exchange occurs by way of the fissures. Thus, the condition $B = 0$ corresponds to a *totally fissured* medium in which each cell of the matrix is isolated from adjacent cells by the fissure system. The resulting system of parabolic-ordinary differential equations

$$\frac{\partial}{\partial t}(au_1) - \vec{\nabla} \cdot (A\vec{\nabla}u_1) + \frac{1}{\delta}(u_1 - u_2) = f\,, \tag{4.2.a}$$

$$\frac{\partial}{\partial t}(bu_2) + \frac{1}{\delta}(u_2 - u_1) = 0\,, \tag{4.2.b}$$

is called the *first-order kinetic* model, since the cell storage is regarded as an added kinetic storage perturbation of the global fissure system.

The equation (4.2.b) models the delay that is inherent in the flow between the fissures and blocks. It is precisely this delay that led to the introduction of such models by Barenblatt, Zheltov, and Kochina [8] and Warren and Root [62] three decades ago in order to better match observed reservoir behavior. See [2], [18], [15], [39], [26], [13], [22], [38], [57] for additional applications and mathematical developments of such models.

If we further specialize this model by setting $a = 0$ in order to realize that the relative volume of the fissures is zero, we obtain the *pseudoparabolic* partial differential equation

$$\frac{\partial}{\partial t} b\Big(u_1 - \delta\ \vec{\nabla} \cdot (A\vec{\nabla}u_1)\Big) - \vec{\nabla} \cdot (A\vec{\nabla}u_1) = f + \delta\frac{\partial f}{\partial t}. \tag{4.3}$$

See [12] for the development of such equations. Their solutions are determined by a *group* of operators on appropriate spaces, and their dynamics is regularity-*preserving*.

See [53] for a discussion and development of *partially fissured* media, those in which there are substantial flow paths directly joining the cells.

Distributed Microstructure Models. In general, essential limitations of the parallel flow models are the suppression of the geometry of the cells and their corresponding interfaces on which the coupling occurs and the lack of any distinction between the space and time scales of the two components of the medium. The introduction of *distributed microstructure models* for diffusion in porous media represents an attempt to recognize the geometry and the multiple scales in these problems in order to better quantify the exchange of fluid across the intricate interface between the components. We are given a domain G which represents the global region of the model. At each point $x \in G$ there is specified a cell G_x, a magnified or scaled

representation of the microstructure that is present near x. One partial differential equation is specified to describe the global flow in the region G, and a separate partial differential equation is specified in each cell G_x to describe the flow internal to that cell. Any coupling between these equations will occur on the boundary of G_x, denoted by Γ_x. It is the collection $\{\Gamma_x : x \in G\}$ which provides the interface on which this exchange takes place. Now we use this concept to model the example of single phase flow in a fissured medium.

The global flow in the fissure system is described in the macro-scale x by

$$\frac{\partial}{\partial t}\big(a(x)u(x,t)\big) - \vec{\nabla} \cdot A(x)\vec{\nabla}u + q(x,t) = f(x,t) \ , \qquad x \in G \ , \tag{4.4.a}$$

where $q(x,t)$ is the exchange term representing the flow into the cell G_x. The flow within each local cell G_x is described by

$$\frac{\partial}{\partial t}\big(b(x,y)U(x,y,t)\big) - \vec{\nabla}_y \cdot B(x,y)\vec{\nabla}_y U = F(x,y,t) \ , \qquad y \in G_x \ . \tag{4.4.b}$$

The subscript y on the gradient indicates that the gradient is with respect to the *local* variable y. A gradient operator without any subscript will mean that the gradient is taken with respect to the *global* variable x. Because of the smallness of the cells, the fissure pressure is assumed to be well approximated by the "constant" value $u(x,t)$ at every point of the cell boundary, so the effect of the fissures on the cell pressure is given by the interface condition

$$B(x,s)\vec{\nabla}_y U(x,s,t) \cdot \nu + \frac{1}{\delta}(U(x,s,t) - u(x,t)) = 0 \ , \qquad s \in \Gamma_x \ , \tag{4.4.c}$$

where ν is the unit outward normal on Γ_x. (When $\delta = 0$, this becomes (and converges to) the *matched* boundary condition, $u(x,t) = U(x,s,t)$ for $s \in \Gamma_x$.) Finally, the amount of fluid flux across the interface scaled by the cell size determines the remaining term in (4.4.a) by

$$q(x,t) = \frac{1}{|G_x|}\int_{\Gamma_x} B(x,s)\vec{\nabla}_y U \cdot \nu \, ds \ , \tag{4.4.d}$$

where $|G_x|$ denotes the Lebesgue measure of G_x, and this contributes to the *cell storage*. The system (4.4) comprises a double porosity model of *distributed microstructure* type for a totally fissured medium. It needs only to be supplemented by appropriate boundary conditions for the global pressure $u(x,t)$ and initial conditions for $u(x,0)$ and $U(x,y,0)$ in order to comprise a well-posed problem. See [40], [51], [50], [19], [61], [9], [25], [6], [7], [3], [5], [19], [20], [23], [24], [28], [29], [30], [31], [32], [33], [34], [36], [35], [43], [59], [58], [60], [56], [55], [54] for applications and mathematical theory for (4.4) and various related problems. A typical development of well-posedness results is given below in Section 5.

Finally, we remark that the system (4.4) can be rewritten as a single equation of functional-differential type. By applying Gauss' theorem to (4.4.b) we obtain from (4.4.d)

$$\frac{\partial}{\partial t}\int_{G_x} bU \, dy = \int_{\Gamma_x} B\frac{\partial U}{\partial \nu} \, ds + \int_{G_x} F \, dy \ .$$

Then by using the Green's function for the problem (4.4.b) to represent the solution $U(x,y,t)$ as an integral over Γ_x of $u(x,t)$, we substitute this into (4.4.a) to get the implicit *convolution evolution equation*

$$\frac{\partial}{\partial t}\left\{a(x)u(x,t)+\int_0^t k(x,t-\tau)u(x,\tau)\,d\tau\right\}-\vec{\nabla}\cdot A(x)\vec{\nabla}u=f(x,t)\ . \tag{4.4}$$

The convolution term represents a storage effect with memory. See [37] for a direct treatment of this equation and particularly [45], [48], [49] where this equation forms the basis for an independent theoretical and numerical analysis. Also see [46], [47] for related work.

A related model for *partially fissured medium* was introduced in [14] to describe the highly anisotropic situation in layered media and developed in [16] for more general situations. See [4] for an earlier discrete version and numerical work, and see [21] for such models which were derived by homogenization (see below) from corresponding ε-models.

5. A Variational Formulation

We illustrate the mathematical formulation of microstructure models as evolution equations in Hilbert space. This provides a means of establishing directly that they are well-posed problems, and it identifies the natural *energy* and *state* spaces for these dynamical problems. Let G be an open, bounded domain in $\mathbb{R}^3$ and for every $x\in G$, let G_x be a bounded region contained in $\mathbb{R}^3$. Identify the product space $\prod_{x\in G}G_x\equiv\mathbb{Q}$ as a subset of $\mathbb{R}^6$; we require that $\mathbb{Q}$ be a *measurable* subset of $\mathbb{R}^6$, hence, each of the cells $G_x\equiv\{y:(x,y)\in\mathbb{Q}\}$ is a measurable subset of $\mathbb{R}^3$. Here we will formulate the Cauchy-Dirichlet problem for the *linear parabolic system*

(5.1.a)
$$\begin{aligned}\frac{\partial}{\partial t}\big(a(x)u(x,t)\big)-\vec{\nabla}\cdot A(x)\vec{\nabla}u(x,t)+\int_{\Gamma_x}B(x,s)\vec{\nabla}_yU(x,s,t)\cdot\vec{\nu}\,ds\\ =f(x,t)\ ,\qquad x\in G\ ,\end{aligned}$$

(5.1.b)
$$\begin{aligned}\frac{\partial}{\partial t}\big(b(x,y)U(x,y,t)\big)-\vec{\nabla}_y\cdot B(x,y)\vec{\nabla}_yU(x,y,t)\\ =F(x,y,t)\ ,\qquad x\in G\ ,\ y\in G_x\ ,\end{aligned}$$

(5.1.c)
$$B(x,s)\vec{\nabla}_yU(x,s)\cdot\vec{\nu}+\frac{1}{\delta}\big(U(x,s,t)-u(x)\big)=0\ ,\qquad x\in G\ ,\ s=\in\Gamma_x\ ,$$

as an evolution equation in an appropriate Hilbert space. This is just the system (4.4) in which the measure ds on Γ_x is used to absorb the extra factor of $|G_x|$. We shall assume $a\in L^\infty(G)$, $b\in L^\infty(\mathbb{Q})$, A and B are uniformly positive definite and bounded measurable matrix functions, $\vec{\nu}$ is the unit outward normal on Γ_x and $\delta>0$. We will further assume that each boundary Γ_x is piecewise C^1 and that the measures $|\Gamma_x|$ and $|G_x|$ are uniformly bounded in x. We shall use the Lebesgue space $L^2(\mathbb{Q})=L^2(G,L^2(G_x))$ with the norm

$$\|U\|_{L^2(\mathbb{Q})}=\left(\int_G\int_{G_x}|U(x,y)|^2\,dy\,dx\right)^{1/2}\ ,$$

and the Sobolev spaces $H_0^1(G)$ and

$$L^2(G, H^1(G_x)) = \left\{ U \in L^2\big(G, L^2(G_x)\big) : \vec{\nabla}_y U \in L^2\big(G, L^2(G_x)\big) \right\} .$$

See [1] for information on Sobolev spaces. For the norm on $L^2(G, H^1(G_x))$ we employ the notation

$$|U|_2 = \left(\int_G \int_{G_x} |\vec{\nabla}_y U(x,y)|^2 \, dy \, dx \right)^{1/2} \quad ,$$

so $\|U\|_2^2 = |U|_2^2 + \|U\|_{L^2(\mathbb{Q})}^2$. Denote by $V = H_0^1(G) \times L^2(G, H^1(G_x))$ the indicated product space with norm

$$\|[u, U]\|_V = \|u\|_{H_0^1(G)} + \|U\|_2 \ .$$

Let γ_x be the usual trace map of $H^1(G_x)$ into $L^2(\Gamma_x)$, and define $\mathcal{B} = L^2(G, L^2(\Gamma_x))$ and the distributed trace $\gamma : L^2(G, H^1(G_x)) \to \mathcal{B}$ by $\gamma U(x,s) = (\gamma_x U(x))(s)$. We will require that the trace maps γ_x be uniformly bounded, so γ is continuous from $L^2(G, H^1(G_x))$ into $\mathcal{B}$. Define $\lambda : H_0^1(G) \to \mathcal{B}$ by: $\lambda u(x,s) = u(x) 1_s$, $x \in G$, $s \in \Gamma_x$, where $u(x)1_s$ is the constant function on Γ_x with value $u(x)$. We will employ the notation $\tilde{u} = [u, U]$.

Define the Hilbert space $H = L^2(G) \times L^2(G, L^2(G_x))$ with the inner product

$$(\tilde{u}, \tilde{\varphi})_H = \int_G a(x) u(x) \varphi(x) \, dx + \int_G \int_{G_x} b(x,y) U(x,y) \Phi(x,y) \, dy \, dx \ ,$$
$$\text{for } \ \tilde{u} = [u, U] \ , \ \tilde{\varphi} = [\varphi, \Phi] \in H \ .$$

Define $V_h = \{\tilde{u} \in V : \gamma U = \lambda u \text{ in } \mathcal{B}\}$. Since γ and λ are continuous, V_h is a closed subspace of V. Also define $V_0 \equiv \{U \in L^2(G, H^1(G_x)) : \gamma U = 0\}$. It can be shown that $H_0^1(G) \times V_0$ and V_h are dense in H.

We shall write the system (5.1) as an evolution equation over the spaces described above. To obtain the variational form for the system, choose $[\varphi, \Phi] \in V$, multiply (5.1.a) by φ and integrate over G. Multiply (5.1.b) by Φ and integrate over both G_x and G. Add these equations and apply Green's Theorem to obtain

$$\begin{aligned}
&\int_G \left\{ \frac{\partial}{\partial t}\big(a(x)u(x,t)\big)\varphi(x) + \int_{G_x} \frac{\partial}{\partial t}\big(b(x,y)U(x,y,t)\big)\Phi(x,y)\, dy \right\} dx \\
&\quad + \int_G \left\{ A(x)\nabla u(x,t) \cdot \vec{\nabla}\varphi(x) + \int_{\Gamma_x} B(x,s)\vec{\nabla}_y U(x,s,t) \cdot \vec{\nu}\varphi(x)\, ds \right. \\
&\quad + \int_{G_x} B(x,y)\vec{\nabla}_y U(x,y,t) \cdot \vec{\nabla}_y \Phi(x,y)\, dy \\
&\quad \left. - \int_{\Gamma_x} B(x,y)\vec{\nabla}_y U(x,y,t) \cdot \vec{\nu}\gamma_x \Phi(x,s)\, ds \right\} dx \\
&\qquad = \int_G f(x)\varphi(x)\, dx + \int_G \int_{G_x} F(x,y)\Phi(x,y)\, dy\, dx \ .
\end{aligned}$$

Combining the boundary integrals and substituting for $B(x,s)\vec{\nabla}_y U(x,y,t)\cdot\vec{\nu}$ yields

$$
\begin{aligned}
(5.2)\qquad \big[u(t),U(t)\big]\in V:\ &\int_G \frac{\partial}{\partial t}a(x)u(x,t)\varphi(x)\,dx\\
&+\int_G\int_{G_x}\frac{\partial}{\partial t}b(x,y)U(x,y,t)\Phi(x,y)\,dy\,dx\\
&+\int_G A(x)\vec{\nabla}u(x,t)\cdot\vec{\nabla}\varphi(x)\,dx\\
&+\int_G\int_{G_x}B(x,y)\vec{\nabla}_y U(x,y,t)\cdot\vec{\nabla}_y\Phi(x,y)\,dy\,dx\\
&+\int_G\int_{\Gamma_x}\frac{1}{\delta}\big(\gamma U(x,s,t)-\lambda u(x,s,t)\big)\big(\gamma\Phi(x,s,t)-\lambda\varphi(x,s,t)\big)\,ds\,dx\\
&=\int_G f(x)\varphi(x)\,dx+\int_G\int_{G_x}F(x,y)\Phi(x,y)\,dy\,dx\ ,\quad [\varphi,\Phi]\in V\ .
\end{aligned}
$$

A special case of the above is obtained when (5.1.c) is replaced by

$$
(5.1.c)'\qquad \gamma U(x,s,t)=\lambda u(x,s,t)\qquad x\in G\ ,\ s\in\Gamma_x\ ,\ t>0\ .
$$

This is the formal result obtained by allowing $\delta\to 0^+$, so that (5.1.c)$'$ is forced to hold, and it corresponds to

$$
\begin{aligned}
(5.2)'\qquad \big[u(t),U(t)\big]\in V_h:\ &\int_G \frac{\partial}{\partial t}a(x)u(x,t)\varphi(x)\,dx\\
&+\int_G\int_{G_x}\frac{\partial}{\partial t}b(x,y)U(x,y)\Phi(x,y)\,dy\,dx\\
&+\int_G A(x)\vec{\nabla}u(x,t)\cdot\vec{\nabla}\varphi(x)\,dx\\
&+\int_G\int_{G_x}B(x,y)\vec{\nabla}_x U(x,y,t)\cdot\vec{\nabla}_y\Phi(x,y)\,dy\,dx\\
&=\int_G f(x)\varphi(x)\,dx+\int_G\int_{G_x}F(x,y)\Phi(x,y)\,dy\,dx\ ,\quad [\varphi,\Phi]\in V_h\ .
\end{aligned}
$$

The problem (5.1) will be called the *regularized model*, and (5.1)$'$, i.e., (5.1.a), (5.1.b) and (5.1.c)$'$ will be called the *matched model*. Conversely, starting from (5.2) it is not difficult to recover (5.1).

Define a bilinear form on V by

$$
\begin{aligned}
a(\tilde{u},\tilde{\varphi})=&\int_G A(x)\vec{\nabla}u\cdot\vec{\nabla}\varphi\,dx+\int_G\int_{G_x}B(x,y)\vec{\nabla}_y U\cdot\vec{\nabla}_y\Phi\,dy\,dx\\
&+\frac{1}{\delta}\int_G\int_{\Gamma_x}(\gamma U-\lambda u)(\gamma\Phi-\lambda\varphi)\,ds\,dx\ ,\quad \tilde{u},\tilde{\varphi}\in V\ .
\end{aligned}
$$

Let $\tilde{f} \in H$ be given in the form

$$\tilde{f}(\tilde{\varphi}) = \int_G f\varphi \, dx + \int_G \int_{G_x} F\Phi \, dy \, dx \ , \qquad \tilde{\varphi} \in V \ .$$

In the situation described above, (5.5) and (5.5)′ have unique solutions.

Theorem. *Given the spaces and operators as above, suppose that* $\tilde{u}_0 = [u_0, U_0] \in H$ *and* $\tilde{f} = [f, F] \in C^\alpha([0,\infty), H)$, $0 < \alpha < 1$. *Then for every* $\delta > 0$ *there is a unique* $\tilde{u}_\delta \in C([0,\infty), H) \cap C^1((0,\infty), H)$ *which satisfies* (5.2) *for each* $t > 0$ *and* $\tilde{u}_\delta(0) = \tilde{u}_0$. *Likewise, there is a unique* $\tilde{u} \in C([0,\infty), H) \cap C^1((0,\infty), H)$ *which satisfies* (5.2)′ *for each* $t > 0$ *and* $\tilde{u}(0) = \tilde{u}_0$.

The proof is a direct application of Theorem 3.3. Furthermore, one can show that $\tilde{u}_\delta$ converges to $\tilde{u}$ as $\delta \to 0$.

We have shown that (5.2) and (5.2)′ have unique solutions and that the two models which they represent are related. In particular, allowing $\delta \to 0^+$ formally transforms the regularized model into the matched model.

Finally note that the models and results here could be generalized or extended in several ways. In (5.1.c) we might choose $\frac{1}{\delta}$ to be something other than a constant. If, for example, $\frac{1}{\delta}$ is assumed to be a monotone graph which is also a subgradient operator, an approach similar to that in [17] might be used to show existence of a solution. As stated earlier, Dirichlet boundary conditions on ∂G are not necessary, so some generalization is also possible in that respect. Finally, if additional assumptions about the differentiability of A and B and the smoothness of Γ_x and ∂G were made, then it might be possible to say more about the regularity of u and U.

6. Homogenization

So far we have given a direct but only heuristic justification of the microstructure models. In order to employ them to simulate real phenomena, one must obtain realistic values for the coefficients, e.g., by matching with data. Here we briefly recall the derivation by homogenization of the distributed microstructure model of a totally fissured medium following [4],[6]. This provides simultaneously a justification of the model and a means to compute the *effective* coefficients in the microstructure model from known coefficients in the *exact* case.

We begin with the exact microscopic model of single phase flow in a fissured domain G, a bounded open subset of R^3. We assume that the geometry and the physical parameters of the problem have ε–periodic character. This implies that the solution to the problem exhibits periodic behavior. It has also some macroscopic (non–periodic) behavior which is seen on the scale of the whole region, G. The basic problem is to investigate the asymptotics of the solution as $\varepsilon \to 0$ in a family of properly scaled problems posed on domains G^ε formed by a lattice of copies of cells εY, where the unit reference cell is the cube $Y = (0,1)^3$. We use ε as a superscript or subscript on coefficients or variables to denote objects periodic with respect to εY; we omit this notation when $\varepsilon = 1$.

The reference cell Y defines the double component structure of the fissured domain, and we write $\overline{Y} = \overline{Y_1} \cup \overline{Y_2}$, with Y_1 and Y_2 denoting the fissure and matrix

parts of the cell, respectively. Their respective boundaries are denoted by Γ_1 and Γ_2. The fissure–matrix interface is given by $\Gamma_{12} = \Gamma_1 \cap \Gamma_2$. Let Γ be that part of Γ_{12} which is contained in Y, and let Γ_{11} and Γ_{22} denote the respective intersections of ∂Y with Y_1 and Y_2. In the totally fissured case, we assume $\overline{Y}_2 \subset Y$, so the matrix interface Γ_{22} is empty and $\Gamma = \Gamma_{12}$. We then refer to Y_2 as a *block*. By ν we denote the normal unit vector to Γ which points in the direction *out* of Y_2.

The system of fissures and matrix blocks in G^ε are denoted by G_1^ε and G_2^ε, respectively. The exact (but singular) ε–model consists of a pair of differential equations, one on each of the subdomains G_1^ε and G_2^ε for the density, which will be denoted by u^ε. These equations are coupled by standard interface conditions on Γ_{12}^ε to insure conservation of mass and momentum across the fissure–matrix interface Γ_{12}^ε. An exterior boundary condition and an initial condition must also be specified, but they do not enter into the derivation of the limit model. In order to preserve the magnitude of the flux crossing the interfaces contained within a fixed volume of the medium as $\varepsilon \to 0$, it is necessary to scale the permeability in the blocks by the factor ε^2. Thus, the ε–model of diffusion in a totally fissured medium has the form

$$\varphi_1 \frac{\partial u^\varepsilon}{\partial t} - \nabla \cdot (\lambda_1 \nabla u^\varepsilon(x,t)) = 0, \qquad x \in G_1^\varepsilon, \tag{6.1.a}$$

$$\varphi_2 \frac{\partial u^\varepsilon}{\partial t} - \nabla \cdot (\varepsilon^2 \lambda_2 \nabla u^\varepsilon(x,t)) = 0, \quad x \in G_2^\varepsilon, \tag{6.1.b}$$

$$u^\varepsilon|_{G_1^\varepsilon}(s,t) = u^\varepsilon|_{G_2^\varepsilon}(s,t), \qquad s \in \Gamma_{12}^\varepsilon, \tag{6.1.c}$$

$$\lambda_1 \nabla u^\varepsilon|_{G_1^\varepsilon} \cdot \nu = \varepsilon^2 \lambda_2 \nabla u^\varepsilon|_{G_2^\varepsilon} \cdot \nu, \qquad s \in \Gamma_{12}^\varepsilon. \tag{6.1.d}$$

If u^ε is expanded in powers of ε and the formal analysis of this expansion is carried out, it will be seen that the leading terms for the density $u^\varepsilon|_{G_1^\varepsilon}$ in the fractures and the density $u^\varepsilon|_{G_2^\varepsilon}$ in the matrix blocks will be a pair of functions, $u(x,t)$ and $U(x,y,t)$, $x \in G$, $y \in Y_2$, $t > 0$, respectively, which satisfy the system of equations

$$|Y_1| \frac{\partial u}{\partial t} - \nabla \cdot (\Lambda_1 \nabla u(x,t)) + q(x,t) = 0, \quad x \in G, \tag{6.2.a}$$

$$\varphi_2 \frac{\partial U}{\partial t} - \nabla_y \cdot (\lambda_2 \nabla_y U(x,y,t)) = 0, \qquad y \in Y_2(x),\ x \in G, \tag{6.2.b}$$

$$U(x,s,t) = u(x,t), \qquad s \in \Gamma, \qquad x \in G, \tag{6.2.c}$$

$$q(x,t) = \int_\Gamma \lambda_2 \nabla_y U \cdot \nu \, dS, \qquad x \in G\ . \tag{6.2.d}$$

This is just the *matched* microstructure model (6.4). Here $|Y_1|$ denotes the φ_1–weighted volume of the reference set Y_1. The *effective* permeability tensor Λ_1 is given by

$$(\Lambda_1)_{ij} = \int_{Y_1} \lambda_1 \left(\delta_{i,j} |Y_1| + \frac{\partial \omega_i}{\partial y_j} \right) dy,$$

with the auxiliary functions ω_k, $k = 1,2,3$, being Y–periodic solutions of the *cell problem*

$$\nabla_y^2 \omega_k = 0, \qquad y \in Y_1,$$
$$\nabla_y \omega_k \cdot \nu = -e_k \cdot \nu, \quad y \in \Gamma,$$

where e_k is the unit vector in the direction of the k–axis.

Equation (6.2.a) is the macroscopic equation to be solved in G for the (macroscopic) density u. The distributed source term q accounts for the flux across the boundary Γ of the block Y_2; we denote it here by $Y_2(x)$ to emphasize that a copy of it is identified with each point $x \in G$. Blocks over different points in G are disconnected; thus, no flow can take place directly from one such block to another. It is this feature that limits this model to flow in a totally fissured medium. If the ε^2 scaling of the permeability in the blocks had been omitted, then the limit process would have led to a single diffusion equation with *effective* or averaged coefficient that fails to represent the desired delayed storage effects. Vogt [61] appears to have been the first to have recognized this idea in the development of a model for chromotography.

All of the models and results we have presented here could be generalized or extended in several ways. In fact, most are special cases of what is already available in the literature. For example, the linear elliptic operators in the above examples can be replaced by quasilinear operators of divergence type, such as p-Laplace operators, and one can include semilinear operators such as the porous medium equation. We have not even mentioned results for equations of other types, such as hyperbolic. Furthermore, we have restricted discussion to problems with the simplest geometry, and we have not mentioned those involving, e.g., flow on boundaries with concentrated capacity, or, more generally, manifolds arising from periodic cells that have a non-flat geometry.

Experience suggests that the distributed microstructure models are conceptually easy to work with, they provide accurate models which include the fine scales and geometry appropriate for many problems, and their theory can be developed in a straightforward manner using conventional techniques. The numerical analysis of these systems provides a natural application of parallel methods.

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7. Nonlinear Problems

Porous medium equation. We normalize the generalized *porous medium equation* (1.1) to obtain the *initial-boundary-value problem*

$$\frac{\partial}{\partial t}a(u(x,t)) - \Delta u(x,t) = f(x,t)\ , \quad x \in G\ , \tag{7.1.a}$$

$$u(s,t) = 0\ , \quad s \in \partial G\ , t \in (0,T]\ , \tag{7.1.b}$$

$$a(u(x,0)) = w_0(x)\ , x \in G\ , \tag{7.1.c}$$

in which $u = u(x,t)$ is a function defined on a bounded domain G in Euclidean space $\mathbb{R}^m$, and $T > 0$ denotes the length of the time interval. We will show that the dynamics of problem (7.1) is determined by a nonlinear semigroup of contractions on an appropriate Banach space X. The negative of the generator of this contraction semigroup is constructed as an operator $\mathbb{C}$ which realizes the initial-boundary-value problem (7.1) as a Cauchy problem

$$w'(t) + \mathbb{C}(w(t)) = f(t), \quad 0 < t, \quad w(0) = w_0 \tag{7.2}$$

in X. Then one needs to show that $\mathbb{C}$ is an *m-accretive* operator on this Banach space. We will see that it is useful to extend the above to cover the case in which the operator $\mathbb{C}$ is actually multivalued. (It is the monotonicity of the operator that is important but limiting for the applications.) Thus, we shall extend the above to allow the "=" sign to be replaced by an inclusion symbol $\ni$ in the equations above.

A (possibly multi-valued) operator or relation $\mathbb{C}$ in a Banach space X is a collection of related pairs $[x,y] \in X \times X$ denoted by $y \in \mathbb{C}(x)$; the range $Rg(\mathbb{C})$ consists of all such y. The operator $\mathbb{C}$ is called *accretive* if for all $y_1 \in \mathbb{C}(x_1)$, $y_2 \in \mathbb{C}(x_2)$ and $\varepsilon > 0$

$$\|x_1 - x_2\| \le \|x_1 - x_2 + \varepsilon(y_1 - y_2)\|\ .$$

This is equivalent to requiring that $(I + \varepsilon\mathbb{C})^{-1}$ is a contraction on $Rg(I + \varepsilon\mathbb{C})$ for every $\varepsilon > 0$. If, in addition, $Rg(I + \varepsilon\mathbb{C}) = X$ for some (equivalently, for all) $\varepsilon > 0$, then $\mathbb{C}$ is called *m-accretive*. For such an operator, one can approximate the derivative in the evolution equation by a backward-difference quotient of step size $h > 0$ and the function $f(t)$ by the step function $f^h(t)$ $(= f_k^h$ for $kh \le t < (k+1)h)$ and get a unique solution $\{w_k^h : 1 \le k\}$ of

$$\frac{w_k^h - w_{k-1}^h}{h} + \mathbb{C}(w_k^h) \ni f_k^h\ , \qquad k = 1, 2, \dots\ ,$$

with $w_0^h = w_0$. Since $\mathbb{C}$ is m-accretive, this scheme is uniquely solved recursively to obtain w_k^h and, hence, the piecewise-constant approximate solution $w^h(t)$ $(= w_k^h$ for $kh \le t < (k+1)h)$ of the Cauchy problem. The fundamental result is the following.

Theorem 7.1(Crandall-Liggett). *Assume* $\mathbb{C}$ *is m-accretive,* $w_0 \in \overline{D(\mathbb{C})}$, $f \in L^1([0,T],X)$ *and that* $f^h \to f$ *in* $L^1([0,T],X)$. *Then* $w^h \to w(\cdot)$ *uniformly as* $h \to 0$ *and* $w(\cdot) \in C([0,T],X)$.

Thus, $w(\cdot)$ is an obvious candidate for a solution of the Cauchy problem. It can be uniquely characterized as an *integral solution*. Moreover, if $f_1, f_2 \in L^1([0,T],X)$

and w_1, w_2 are integral solutions of $w_j' + \mathbb{C}(w_j) \ni f_j \ , \quad 0 \le t, \quad j = 1,2$, then

$$\|w_1(t) - w_2(t)\| \le \|w_1(0) - w_2(0)\| + \int_0^t \|f_1(s) - f_2(s)\| \, ds \ , \quad 0 \le t.$$

However, this rather technical characterization does not even require any differentiability of the solution. For an introduction to the abstract Cauchy problem in Banach space and its applications to initial-boundary-value problems for partial differential equations, see [2], [3], [6], [22] and their included references.

We would like to indicate the types of estimates that are involved for the application of these abstract results to problem (7.1), and we will do this for simplicity in the case of a *monotone function*, $a(\cdot)$. We seek (the closure of) an operator $\mathbb{C}$ for which the *resolvent equation*, $(I + \varepsilon\mathbb{C})(a) \ni f$ with $\varepsilon > 0$, takes the form

$$a(u(x)) - \varepsilon \Delta u(x) \ni f(x) \ , \quad x \in G \ , \tag{7.3.a}$$

$$u(s) = 0 \ , \quad s \in \partial G \ . \tag{7.3.b}$$

To get a variational formulation of the operator $\mathbb{C}$, multiply the equation by a smooth function φ on G which vanishes on ∂G, and integrate to obtain

$$\int_G \big(a(u)\varphi + \varepsilon \vec{\nabla} u \cdot \vec{\nabla}\varphi\big) \, dx = \int_G f\varphi \, dx \ .$$

Thus, one seeks $u \in H_0^1(G)$ for which the above holds for each $\varphi \in H_0^1(G)$. In order to obtain estimates on the difference of solutions, suppose that $f_1(x)$, $f_2(x)$ are given and that we have corresponding solutions $u_1(x)$, $u_2(x)$ of (7.3). Subtract the two equations, multiply the difference by a smooth function φ on G which vanishes on ∂G, and integrate to obtain

$$\int_G \big((a(u_1) - a(u_2))\varphi + \varepsilon \vec{\nabla}(u_1 - u_2) \cdot \vec{\nabla}\varphi\big) \, dx = \int_G (f_1 - f_2)\varphi \, dx \ .$$

This leads to the essential a-priori estimates. For example, if we choose $\varphi = \operatorname{sgn}(u_1 - u_2)$ and can obtain simultaneously $\varphi = \operatorname{sgn}(a(u_1) - a(u_2))$, then we (formally) obtain the contraction estimate

$$\|a(u_1) - a(u_2)\|_{L^1(G)} \le \|f_1 - f_2\|_{L^1(G)} \ . \tag{7.4}$$

(To make this precise, we first do this with a smooth approximation $\operatorname{sgn}_\varepsilon(\cdot)$ with $\operatorname{sgn}_\varepsilon(0) = 0$ and let $\varepsilon \to 0$.) This establishes the *accretiveness* of the operator $\mathbb{C}$ on the Banach space $L^1(G)$. One can then verify the range condition to show that $\mathbb{C}$ is m-accretive as desired. The estimate above depends on the monotonicity of the function $a(\cdot)$, and it shows that we must work in the Banach space $X = L^1(G)$. (Specifically, $\operatorname{sgn}(\cdot)$ is the duality map for L^1, and this forces us to work in this particular space.) Moreover, it can be extended to the degenerate case $a(\cdot) \equiv 0$, which reduces (7.1) to an *elliptic* equation, as well as the cases of multivalued relations, $a(\cdot)$ (which we develop below) and gradient nonlinearities of p-Laplacean type as in (1.4).

Porous Media System. We next describe a system consisting of a parabolic equation and an ordinary differential equation which are coupled by terms which depend on the difference of the unknowns. This system takes the form

(7.5.a)
$$\frac{\partial}{\partial t}a(u(x,t)) - \Delta u(x,t) - c(v(x,t) - u(x,t)) \ni f(x,t) \ ,$$

(7.5.b)
$$\frac{\partial}{\partial t}b(v(x,t)) + c(v(x,t) - u(x,t)) \ni g(x,t) \ , \quad x \in G \ , \ t \in (0,T] \ ,$$

(7.5.c)
$$-\frac{\partial}{\partial \nu}u(s,t) \in d(u(s,t)) \ , \quad s \in \partial G \ ,$$

in which $u = u(x,t)$ and $v = v(x,t)$ are functions defined on the domain G. See (1.7) and (4.2). Note that (7.5) contains a generalized *porous medium equation*, and we make no assumptions of strict monotonicity of $a(\cdot)$. In particular, we allow the degenerate case $a(\cdot) \equiv 0$, and this reduces (7.5) to a *pseudoparabolic* equation (4.3) [7].

If each of $a(\cdot)$, $b(\cdot)$, $c(\cdot)$ and $d(\cdot)$ were a monotone (non-decreasing) function, then the inclusion symbols, $\ni$, would be replaced by the corresponding equality symbol. Such systems arise in many contexts, for example, in the diffusion of chemicals through a saturated porous medium in which (7.5.b) models the local storeage or adsorbtion in immobile nondiffusive sites. In that case, u is the concentration of a chemical species in the fluid which occupies the pores and v is the concentration on the surface of the medium. These are commonly called *first order kinetic models*, and they arise in many applications to describe diffusion through an adsorbing medium. These systems can be regarded as a degenerate case of a corresponding parabolic system like (1.7) which contains an additional term $-\Delta v(x,t)$ in (7.5.b). We saw earlier that these arise as *parallel models* of flow through a heterogeneous medium consisting of two components with different diffusivities and an exchange flux driven by the difference in concentration between the two components. In (7.5.b) this diffusion term has been deleted because of the immobility of the concentration in the adsorbtion sites.

For our purposes it will be very useful to permit $a(\cdot), b(\cdot)$, and especially $c(\cdot)$ to be multi-valued. That is, we shall consider the case where these are *maximal monotone graphs* in $\mathbb{R} \times \mathbb{R}$ [5]. In particular, if $b(0) = \mathbb{R}$, hence, b is the inverse of the zero graph, then the system (7.5) reduces to (7.5.a) with $v = 0$. Likewise, if $c(0) = \mathbb{R}$, then $u = v$, and the system reduces to

$$\frac{\partial}{\partial t}(a(u(x,t)) + b(u(x,t))) - \Delta u(x,t) = f(x,t) + g(x,t) \qquad x \in G \ , \ t \in (0,T]$$

together with the boundary condition (7.5.c). But these are merely additive perturbations of the porous medium equation.

The generalization to multi-valued graphs permits a very elegant treatment of a class of parabolic problems with *hysteresis*. These are of the form

(7.6)
$$\frac{\partial}{\partial t}\left(a(u) + \mathcal{H}(u)\right) - \Delta u = f$$

in which $\mathcal{H}$ denotes a *hysteresis* functional, that is, its value depends not only on the current value of the input, u, but also on the *history* of the input in a very nonlinear way. Due to the complex nature of the operators customarily used to represent hysteresis [12], their addition to systems of differential equations leads to substantial technical problems for the development of a good theory. An excellent introduction to hysteresis is the monograph [20], and one should consult the recent survey [18]. The new book [28] is an excellent source for history and recent developments of mathematical models of hysteresis as well as their addition to partial differential equations, especially those of parabolic type. See also [11], [27]. The forthcoming monograph [13] concerns quasilinear wave equations with elasto-plastic hysteretic constitutive laws arising in mechanics.

We develop some estimates for solutions of (7.5). Let $\mathbb{C}$ be (the closure of) an operator for which the *resolvent equation*, $(I + \varepsilon\mathbb{C})([a,b]) \ni [f,g]$ with $\varepsilon > 0$, takes the form

$$\begin{aligned} &a(u(x)) - \varepsilon\Delta u(x) - \varepsilon c(v(x) - u(x)) \ni f(x) \ , \\ &b(v(x)) + \varepsilon c(v(x) - u(x)) \ni g(x) \ , \qquad x \in G \ , \\ &-\frac{\partial}{\partial\nu}u(s) \in d(u(s)) \ , \qquad s \in \Gamma \ , \end{aligned}$$

in the state space $L^1(G) \times L^1(G)$. Multiply the respective equations by smooth functions φ and ψ on G and integrate to obtain

$$\begin{aligned} \int_G \big(a(u)\varphi + \varepsilon\vec{\nabla}u \cdot \vec{\nabla}\varphi\big)\,dx + \int_G \big(b(v)\psi + \varepsilon c(v-u)(\psi-\varphi)\big)\,dx \\ + \varepsilon\int_\Gamma d(u)\varphi\,ds \ = \int_G \big(f\varphi + g\psi\big)\,dx \ . \end{aligned}$$

This shows the variational form and leads to the essential a-priori estimates. For example, if we choose $\varphi = \operatorname{sgn}(u)$, $\psi = \operatorname{sgn}(v)$ and can obtain simultaneously $\varphi = \operatorname{sgn}(a(u))$, $\psi = \operatorname{sgn}(b(v))$, then we (formally) obtain the stability estimate

$$\|a(u)\|_{L^1(G)} + \|b(v)\|_{L^1(G)} \le \|f\|_{L^1(G)} + \|g\|_{L^1(G)} \ .$$

By estimating similarly the *differences* of solutions, we find that the *resolvent* map $[f,g] \mapsto [a(u), b(v)]$ is a contraction, and this yields the required *accretiveness* of the operator $\mathbb{C}$. Under some additional conditions on the monotone graphs $a(\cdot)$, $b(\cdot)$, $c(\cdot)$, and $d(\cdot)$, we find that $\mathbb{C}$ is m-accretive as desired.

Dynamic Boundary Conditions. Next we describe a problem with the same formal structure as (7.5), the degenerate-parabolic initial boundary value problem

$$\frac{\partial}{\partial t}a(u) - \Delta u \ni f \ , \qquad x \in G \ , \tag{7.7.a}$$

$$\frac{\partial}{\partial t}b(v) + \frac{\partial u}{\partial\nu} \ni g \quad \text{and} \tag{7.7.b}$$

$$\frac{\partial u}{\partial\nu} \in c(v-u) \ , \qquad s \in \partial G \tag{7.7.c}$$

with initial values specified at $t = 0$ for $a(u)$ and $b(v)$. At each $t > 0$, $u(t)$ is a function on the bounded domain G in $\mathbb{R}^n$ with smooth boundary ∂G, and $v(t)$ is a function on ∂G. Each of $a(\cdot), b(\cdot), c(\cdot)$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ [5]. Thus, the system (7.7) consists of a generalized porous medium equation in the interior of G subject to a nonlinear dynamic constraint on the boundary.

Because of the generality attained via our use of maximal monotone graphs, this class includes boundary conditions of all the usual types, including Dirichlet, Neumann, Robin, and the fourth type, i.e., the *dynamic* boundary conditions [7], [24]. For example, if $b \equiv 0$ we have an explicit *Neumann boundary condition*, and if $c \equiv 0$ it is homogeneous. If $b(0) = \mathbb{R}$ (i.e., $b^{-1} = 0$), then $v \equiv 0$ and we have a nonlinear Robin constraint, and if $c(0) = \mathbb{R}$ we get $v = u$ on ∂G, and this satisfies a nonlinear *dynamic boundary condition*

$$\frac{\partial}{\partial t} b(u) + \frac{\partial u}{\partial \nu} \ni g. \tag{7.7.b'}$$

If $b(0) = c(0) = \mathbb{R}$ we have the homogeneous *Dirichlet boundary condition*. Additional interest in (7.7) arises primarily from the fact that (7.7.b) together with (7.7.c) can represent *boundary hysteresis*.

We have shown in [10] that the dynamics of the problem (7.7) is given by a nonlinear semigroup of contractions on the Banach space $L^1(G) \times L^1(\partial G)$[11]. The method works as well with the various multivalued relations as indicated. This is merely a reflection of the power of the method as developed in [23]; the method permits the extension to gradient nonlinearities of p-Laplacean type in (7.7.a) as well as corresponding elliptic Laplace-Beltrami operators in (7.7.b) for the manifold ∂G. See [19] for a treatment of the degenerate case $a(\cdot) = 0$ corresponding to a Stefan problem on the boundary ∂G. Adsorption in porous media may be governed by conditions on the surfaces of the solid material that are of hysteresis type. If one assumes that the process is governed by certain thresholds, the adsorption rate shows a hysteresis phenomenon of the kind discussed here. In [9] this idea is applied to homogenization of reactive transport through porous media. See also [8], [23].

Finally we would like to indicate the types of estimates that are involved for the problem (7.7), and we will do this for simplicity in the special case of *functions* $a(\cdot)$, $b(\cdot)$, $c(\cdot)$. The *resolvent equation*, $(I + \varepsilon \mathbb{C})([a, b]) \ni [f, g]$ with $\varepsilon > 0$, takes the form

$$a(u) - \varepsilon \Delta u \ni f \ , \qquad x \in G \ , \tag{7.8.a}$$

$$b(v) + \varepsilon \frac{\partial u}{\partial \nu} \ni g \ , \quad \text{and} \tag{7.8.b}$$

$$\frac{\partial u}{\partial \nu} \in c(v - u) \ , \qquad s \in \partial G \ , \tag{7.8.c}$$

in the state space $L^1(G) \times L^1(\partial G)$. Proceeding as before to show how one obtains the essential estimates that are needed, multiply the respective equations by smooth functions φ on G and ψ on ∂G and integrate to obtain

$$\int_G \big(a(u)\varphi + \varepsilon \vec{\nabla} u \cdot \vec{\nabla} \varphi\big)\, dx + \int_{\partial G} \big(b(v)\psi + \varepsilon c(v-u)(\psi - \varphi)\big)\, ds = \int_G f\varphi\, dx + \int_{\partial G} g\psi\, ds \ . \tag{7.9}$$

This shows the variational form of (7.8) and leads to the essential a-priori estimates. For example, if we choose $\varphi = \mathrm{sgn}(u)$, $\psi = \mathrm{sgn}(v)$ and can obtain simultaneously $\varphi = \mathrm{sgn}(a(u))$, $\psi = \mathrm{sgn}(b(v))$, then we (formally) obtain the stability estimate

$$\|a(u)\|_{L^1(G)} + \|b(v)\|_{L^1(\partial G)} \leq \|f\|_{L^1(G)} + \|g\|_{L^1(\partial G)} \cdot \tag{7.10}$$

By estimating similarly the *differences* of solutions, we find that the *resolvent* map $[f, g] \mapsto [a(u), b(v)]$ is a contraction, and this yields the required *accretiveness* of the operator $\mathbb{C}$. Under some additional conditions on the monotone graphs $a(\cdot)$, $b(\cdot)$, and $c(\cdot)$, we find that $\mathbb{C}$ is m-accretive as desired.

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