PACIFIC JOURNAL OF MATHEMATICS
Vol. 34, No. 3, 1970

LOCAL REGULARITY OF SOLUTIONS OF SOBOLEV-GALPERN PARTIAL DIFFERENTIAL EQUATIONS

R. E. SHOWALTER

Let M and L be elliptic differential operators of orders $2m$ and $2l$, respectively, with $m \leqq l$. The existence and uniqueness of a solution to the abstract mixed initial and boundary value problem

$$Mu'(t) + Lu(t) = 0, \qquad u(0) = u_0$$

was established for u_0 given in the domain of the infinitesimal generator of a strongly-continuous semi-group. The purpose of this paper is to show that this semi-group is holomorphic and then obtain differentiability results for the solution and convergence of this solution to the initial function u_0 as $t \downarrow 0$.

Let G be a bounded open domain of R^n whose boundary ∂G is an $(n-1)$-dimensional manifold with G lying on one side of ∂G. $H^k = H^k(G)$ is the Hilbert space (of equivalence classes) of functions whose distributional derivatives through order k belong to $L^2(G)$ with the usual inner-product and norm,

$$(f, g)_k = \sum \left\{\int_G D^\alpha f \, \overline{D^\alpha g} \, dx \colon |\alpha| \leqq k\right\}$$

and

$$\|f\|_k = \sqrt{(f, f)_k} \,.$$

$H_0^k = H_0^k(G)$ is the closure in H^k of $C_0^\infty(G)$, the space of infinitely differentiable functions with compact support in G.

We specify the problem by means of the bilinear forms

$$B_M(\phi, \psi) = \sum \{(m^{\rho\sigma} D^\sigma \phi, D^\rho \psi)_0 \colon |\rho|, |\sigma| \leqq m\}$$

and

$$B_L(\phi, \psi) = \sum \{(l^{\rho\sigma} D^\sigma \phi, D^\rho \psi)_0 \colon |\rho|, |\sigma| \leqq l\} \,,$$

defined initially for ϕ and ψ in $C_0^\infty(G)$. Furthermore, we require the following:

P_1: The coefficients $m^{\rho\sigma}$, $l^{\rho\sigma}$ are bounded and measurable.

P_2: Re $B_M(\phi, \phi) \geqq k_m \|\phi\|_m^2$, $k_m > 0$
Re $B_L(\phi, \phi) \geqq k_l \|\phi\|_l^2$, $k_l > 0$
for all ϕ in $C_0^\infty(G)$.

P_3: M is symmetric; that is $m^{\rho\sigma} = \overline{m^{\rho\sigma}}$ for all ρ, σ, (hence $B_M(\phi, \phi)$ is real for all ϕ in C_0^∞).

From the assumptions P_1 and P_2 and the general theory of elliptic operators, [1, 6, 7, 11, 12, 13], there are two operators, M_0 and L_0, which are topological isomorphisms of H_0^m onto $H^{-m} = (H_0^m)'$ and H_0^l onto $H^{-l} = (H_0^l)'$ (where "$'$" denotes the continuous linear dual), and these are determined by the respective identities

$$B_M(\phi, \psi) = \langle M_0\phi, \bar{\psi}\rangle$$

and

$$B_L(\phi, \psi) = \langle L_0\phi, \bar{\psi}\rangle$$

on H_0^m and H_0^l, respectively, where "$\langle\, ,\rangle$" denotes $\mathscr{D} - \mathscr{D}'$ duality, $\mathscr{D}'$ being the space of distributions over G.

Since $l \geqq m$ we have a topological inclusion $H_0^l \subsetneq H_0^m$, hence, by duality, $H^{-m} \subsetneq H^{-l}$. Thus the mapping $L_0^{-1}M_0$ is continuous from H_0^m into H_0^l and is a topological isomorphism only if $l = m$. Letting $D = L_0^{-1}M_0(H_0^m) \equiv L_0^{-1}(H^{-m})$, we have an unbounded operator $A = M_0^{-1}L_0$ on H_0^m with domain D dense in H_0^l. In [16] we showed that A is the infinitesimal generator of an equicontinuous semi-group of bounded operators [6, 9, 11] on H_0^m, denoted by $\{S(t): t \geqq 0\}$. We shall prove that this semi-group is holomorphic.

We have already shown that the nonnegative real axis belongs to the resolvent set of A and, in fact,

$$(1) \qquad |R(\lambda, A)|_M = |(\lambda - A)^{-1}|_M \leqq (\mathrm{Re}\,(\lambda))^{-1}$$

for all real $\lambda \geqq 0$, where the norm $|\cdot|_M$ defined by

$$|\phi|_M = \sqrt{B_M(\phi, \phi)}$$

on H_0^m is equivalent to $\|\cdot\|_m$ by P_1 and P_2. Actually the whole right half of the complex plane belongs to the resolvent set of A, and (1) is true there. This can be shown by noting that for $\lambda = \sigma + i\tau$ we have

$$B_M((A - \lambda)\phi, \phi) = B_M((A - \sigma)\phi, \phi) - i\tau B_M(\phi, \phi)$$

and hence

$$\mathrm{Re}\, B_M((A - \lambda)\phi, \phi) = \mathrm{Re}\, B_M((A - \sigma)\phi, \phi)$$

in the argument leading to (1) for λ real. See [16] for details.

2. Our goal is to improve the estimate (1) to show that the family $\{\lambda R(\lambda, A)\}$ is uniformly bounded in $\mathscr{L}(H_0^m)$ for $\mathrm{Re}\,(\lambda) > 0$. First let ϕ be in D; then

$$B_M((\lambda - A)\phi, \phi) = (\sigma + i\tau)B_M(\phi, \phi) + B_L(\phi, \phi)\ .$$

Since M is symmetric it follows that $B_M(\phi, \phi)$ is real, so we obtain

$$\text{(2)} \qquad \operatorname{Re} B_M((\lambda - A)\phi, \phi) = \sigma B_M(\phi, \phi) + \operatorname{Re} B_L(\phi, \phi) \geqq k_l \,||\phi||_l^2 ,$$

since $\sigma > 0$. Similarly, from

$$\operatorname{Im} B_M((\lambda - A)\phi, \phi) = \tau B_M(\phi, \phi) + \operatorname{Im} B_L(\phi, \phi)$$

we obtain the estimate

$$\text{(3)} \qquad |\operatorname{Im} B_M((\lambda - A)\phi, \phi)| \geqq |\tau|\, |\phi|_M^2 - K_l \,||\phi||_l^2 .$$

From (2) and (3) we conclude that either

$$\text{(4)} \qquad |\operatorname{Im} B_M((\lambda - A)\phi, \phi)| \geqq \frac{|\tau|}{2} |\phi|_M^2$$

or

$$\text{(5)} \qquad |\operatorname{Re} B_M(\lambda - A)\phi, \phi)| \geqq \frac{k_l}{2K_l} |\tau|\, |\phi|_M^2 ,$$

for if (4) is not true then by (3)

$$|\tau|\, |\phi|_M^2 - K_l \,||\phi||_l^2 \leqq \frac{|\tau|}{2} |\phi|_M^2 ,$$

hence

$$\frac{|\tau|}{2} |\phi|_M^2 \leqq K_l \,||\phi||_l^2 ,$$

which with (2) implies (5). From (4) and (5) we obtain the estimate

$$\text{(6)} \qquad |B_M((\lambda - A)\phi, \phi)| \geqq \frac{k_l}{2K_l} |\tau|\, |\phi|_M^2$$

for all ϕ in D, and this in turn yields

$$\text{(7)} \qquad |R(\lambda, A)|_M \leqq \frac{2K_l}{k_l} \frac{1}{|\tau|} ,$$

whenever $\operatorname{Re}(\lambda) > 0$. The calculation is as follows:

$$\frac{k_l}{2K_l} |\tau|\, |\phi|_M^2 \leqq |B_M((\lambda - A)\phi, \phi)| \leqq |(\lambda - A)\phi|_M \,|\phi|_M$$

implies

$$|(\lambda - A)\phi|_M \geqq |\tau| \frac{k_l}{2K_l} |\phi|_M$$

for all ϕ in D, the domain of A, so (7) follows. The estimates (1) and (7) imply that

$$|\lambda R(\lambda, A)|_M \leqq \frac{|\tau|}{\sigma} + 1$$

when $\sigma > 0$ and, respectively, that

$$|\lambda R(\lambda, A)|_M \leqq \frac{2K_l}{k_l}\left(\frac{\sigma}{|\tau|} + 1\right)$$

whenever $|\tau| \neq 0$, where $\lambda = \sigma + i\tau$. By considering the two cases, $|\tau| \geqq \sigma$ and $|\tau| < \sigma$, we obtain, finally,

$$|\lambda R(\lambda, A)|_M \leqq \frac{4K_l}{k_l} \tag{8}$$

for all λ in the right half of the complex plane. The estimate (8) yields the following result.

PROPOSITION [22]. *The semi-group $\{S(t): t \geqq 0\}$ has a holomorphic extension into a sector of the complex plane. Furthermore, $S(t)$ maps H_0^m into D whenever $t > 0$, so $S(t)$ is infinitely differentiable and $S^{(p)}(t) = A^p S(t)$ for any integer $p \geqq 1$.*

The significance of this result for our problem is that, for each $t > 0$, $S(t)$ maps H_0^m into the domain of A^p for an arbitrary integer $p \geqq 1$.

3. The differentiability of the semi-group yields differentiability of the solution to the problem being considered; the latter is obtained by means of the following.

Let H_{loc}^k denote those (equivalence classes of) functions on G which are locally in H^k; that is,

$$H_{\text{loc}}^k = \{f: f \in H^k(K) \text{ for each compact subset } K \text{ of } G\} .$$

The following result on the local regularity of solutions of elliptic equations is well known.

THEOREM [1, 4, 5, 7, 12, 13, 14]. *Let p be an integer $\geqq -l$ for which $l^{\rho\sigma}$ is $\max\{1, |\rho| + p\}$ times continuously differentiable in G whenever $|\rho|$ and $|\sigma|$ are $\leqq l$. If u belongs to H_0^l, and if $L_0 u$ is in H_{loc}^p, then u belongs to H_{loc}^{2l+p}. That is, L_0 is a topological isomorphism of $H_0^l \cap H_{\text{loc}}^{2l+p}$ onto $H^{-l} \cap H_{\text{loc}}^p$.*

Let k be a nonnegative integer and assume that we have

$P(k)$: $m^{\rho\sigma}$ and $l^{\rho\sigma}$ are $\max\{1, |\rho| - m + k\}$ times continuously differentiable in G.

From the above theorem it follows that M_0 is a bijection of $H_0^m \cap H_{\mathrm{loc}}^{m+k}$ onto $H^{-m} \cap H_{\mathrm{loc}}^{k-m}$. Also L_0^{-1} is a bijection of $H^{-l} \cap H_{\mathrm{loc}}^{k-m}$ onto $H_0^l \cap H_{\mathrm{loc}}^{2l-m+k}$. Since $H^{-m} \subset H^{-l}$, it follows that $A^{-1} = -L_0^{-1}M_0$ maps $H_0^m \cap H_{\mathrm{loc}}^{m+k}$ into $H_0^l \cap H_{\mathrm{loc}}^{2l-m+k}$.

COROLLARY. *$P(2(p-1)(l-m))$ implies that the domain of A^p is contained in $H_0^l \cap H_{\mathrm{loc}}^{m+2p(l-m)}$ for $p \geqq 1$.*

From §2 we know that $u(t)$ is in the domain of A^p for all $t > 0$ and $p > 1$. The corollary thus yields the following results.

THEOREM. *Assume P_1, P_2 and P_3 of §2. Let the coefficients in M and L satisfy $P(2(p-1)(l-m))$ for some integer $p \geqq 1$. Then $u(t) = S(t)u_0$ belongs to $H_0^l \cap H_{\mathrm{loc}}^{m+2p(l-m)}$ for each $t > 0$, where u_0 is any element of H_0^m.*

If p is sufficiently large we obtain pointwise-solutions by Sobolev's Lemma [17]:

If m is an integer $> (n/2)$, then H_{loc}^m is imbedded in $C^j(G)$, $j = m - [n/2] - 1$, and the injection is continuous when the range space is given the topology of uniform convergence in all derivatives of order $\leqq j$ on compact of subsets of G.

COROLLARY. *Assume the hypotheses of the above theorem hold with $m + 2p(l-m) - [n/2] - 1 = j \geqq 0$. Then, for $t > 0$, $u(t)$ has j continuous derivatives in G and, for each point x in G, the function $t \rightarrow u(x, t)$ is infinitely differentiable.*

Proof. Choose t' such that $t > t' > 0$. Since $u(t') = S(t')u_0$ belongs to $D(A^p)$, the semi-group property yields

$$\delta^{-1}[u(t+\delta) - u(t)] = A^{-p}\delta^{-1}[S(t+\delta-t') - S(t-t')]A^p u(t')$$

for δ sufficiently small. Since $A^p u(t')$ belongs to $D = D(A)$, the function to the right of A^{-p} has a limit in H_0^m as $\delta \rightarrow 0$, so the function $\delta^{-1}[u(t+\delta) - u(t)]$ has a limit in $H^{m+2p(l-m)}(K)$, where K is any compact subset of G. By Sobolev's Lemma, the function

$$\delta \rightarrow \delta^{-1}[u(x, t+\delta) - u(x, t)]$$

has a limit as $\delta \rightarrow 0$, so $u(x, t)$ is differentiable. A repetition of this argument shows that $u(x, t)$ is infinitely differentiable in t without any further assumptions on the coefficients.

All of the above results have been obtained for a solution with initial value u_0 in H_0^m. We note further that if u_0 is sufficiently smooth then $u(t) \to u_0$ pointwise. (It is always true that $u(t) \to u_0$ in H_0^m.)

COROLLARY. *Assume the hypotheses of the above corollary and that u_0 belongs to the domain of A^p. Then each $u(t), t \geqq 0$ is a continuous function on G, and for each point x in G, $u(x, t) \to u_0(x) = u(x, 0)$ as $t \to 0$.*

Proof. This follows by an argument similar to the proof of the preceding corollary applied to the equation

$$u(t) - u_0 = A^{-p}(S(t) - I)(A^p u_0) .$$

We note that a sufficient condition for u_0 to be in $D = D(A)$ is that u_0 be in $H_0^l \cap H^{2l-m}$. Also if the initial function and all coefficients in M and L are infinitely differentiable, then the solution is infinitely differentiable.

BIBLIOGRAPHY

1. S. Agmon, *Lectures on elliptic boundary value problems*, Van Nostrand, New York, 1965.
2. G. Barenblat, I. Zheltov, and I. Kochiva, *Basic concepts in the theory of seepage of homogeneous liquids for fissured rocks*, J. Appl. Math. Mech. **24** (1960), 1286-1303.
3. G. Barenblat and G. Chernyi, *On moment relations on surface discontinuity in dissipative media*, J. Appl. Math. Mech. **27** (1963), 1205-1218.
4. F. E. Browder, *On the regularity properties of solutions of elliptic differential equations*, Comm. Pure Appl. Math. **9** (1956), 351-361.
5. ———, *Strongly elliptic systems of differential equations*, Contributions to the Theory of Differential Equations, Ann. Math. Studies, no. 33, Princeton University Press, Princeton, 1954, 15-51.
6. R. Carroll, *Partial differential equations*, Harper and Row, New York, 1969.
7. A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, Englewood Cliffs, N. J., 1965.
8. C. A. Galpern, *Cauchy problem for general systems of linear partial differential equations*, Doklady Akad. Nauk. SSSR **119** (1958), 640-643.
9. E. Hille and R. S. Phillips, *Functional analysis and semigroups*, A. M. S. Colloquium Publications, vol. 31, New York, 1957.
10. A. G. Kostachenko and G. I. Eskin, *Cauchy problem for an equation of Sobolev-Galpern type*, Trudy Moscow Math. Obshch. **10** (1961), 273-284.
11. P. D. Lax and A. Milgram, *Parabolic equations*, Contributions to the theory of Partial Differential Equations, Ann. Math. Studies, no. 33, Princeton University Press, Princeton, 1954, 167-190.
12. J. L. Lions, *Lectures on elliptic partial differential equations*, Tata Institute, Bombay, 1957.
13. ———, *Problems aux limites dans les equations aux derivees partielles*, Univ. Montreal, Montreal, 1962.
14. L. Nirenberg, *Remarks on strongly elliptic partial differential equations*, Comm. Pure Appl. Math. **8** (1955), 648-674.

15. R. E. Showalter and T. W. Ting, *Pseudo-parabolic partial differential equations*, SIAM J. Math. Analysis **1** (1970), 1-25.

16. R. E. Showalter, *Partial differential equations of Sobolev-Galpern type*, Pacific J. Math. **31** (1969), 787-794.

17. S. Sobolev, *On a theorem of functional analysis*, Math. Sbornik **4** (1938), 471-497.

18. ———, *Some new problems in mathematical physics*, Izv. Akad. Nauk. SSSR Ser. Mat. **18** (1954), 3-50.

19. D. W. Taylor, *Research on consolidation of clays*, M. I. T. Press, Cambridge, 1942.

20. T. W. Ting, *Certain non-steady flows of second order fluids*, Arch. Rat. Mech. Anal. **14** (1963), 1-26.

21. ———, *Parabolic and pseudo-parabolic partial differential equations*, J. Math. Soc. Japan **21** (1970), 440-453.

22. K. Yosida, *Holomorphic semigroups in a locally convex linear topological space*, Osaka Math. J. **15** (1963), 51-57.

23. T. Zalenyak, *The behavior for $t \to \infty$ of solutions of a problem of S. L. Sobolev*, Soviet Math. **2** (1961), 956-958.

24. ———, *A problem of S. L. Sobolev*, Soviet Math. **3** (1962), 1756-1759.

Received October 23, 1969.

THE UNIVERSITY OF TEXAS