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WELL-POSED PROBLEMS FOR SOME NONLINEAR DISPERSIVE WAVES

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CONTENTS

1. Introduction

We introduce two systems of partial differential equations, each of which provides an approximate model for long gravity waves of small amplitude in one dimension. The first system is a variation on the classical Boussinesq model obtained from a standard formal expansion procedure in hydrodynamics. This system permits wave motion in either direction. The derivation of this system is given in Section 2 and we show in Section 6 that certain appropriate initial-boundary value problems for the system are well-posed over sufficiently small time intervals. A second system is presented in Section 3 under the additional assumption that the waves are essentially unidirectional. In Section 7 this second system is solved globally in time with certain initial and boundary conditions. The given examples include periodic waves and an undular bore. The existence-uniqueness-stability results of Sections 6 and 7 are obtained from corresponding results on the abstract Cauchy problem for an evolution equation in Hilbert space. This equation is solved in Section 5 for two types of solutions, each of which appears in the applications of Sections 6 and 7.

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The unidirectional propogation of long waves of small amplitude has classically been modeled on the well-known equation of Korteweg and de Vries ([4], [5], [11]). More recently, a similar equation of Sobolev type has been introduced ([1], [8], [10]) and it has the same formal justification as the K d V equation for the description of long water waves as above. The derivation of both of these equations is given in Section 4 where we describe the assumptions that lead to them. In particular, we show that each of our two systems describes nonlinear waves under conditions more general than those for which either of the two single equations are appropriate.

When using a formal expansion as below, one must be concerned to arrive at an equation or system for which relevant problems are well-posed. For example, the last two equations of Section 3 are formally equivalent, appropriate initial-boundary value problems are well-posed for the first, and the second is elliptic! Similarly, a pair of equations with the same formal justification is given in Section 4 although their corresponding well-posed problems are different.

Approximate Equations for Long Gravity Waves. — We consider gravity waves which occur on the free surface of a two dimensional horizontal layer of inviscid and incompressible fluid of finite depth. The units are chosen so that the gravitational constant, depth and fluid density are all unity. The fluid motion is assumed irrotational and the wave propogation in the x-direction is therefore described by a velocity potential $\varphi(x, y, t)$, where y is the vertical coordinate. If $\eta(x, t)$ denotes the height of the surface above the undisturbed level, then

$$\varphi_{xx}+\varphi_{yy}=0, \qquad 0<y<1+\eta(x, t).$$

The relevant boundary conditions are

$$\varphi_y=0 \quad \text{at} \quad y=0,$$

$$\eta+\varphi_t+(1/2)(\varphi_x^2+\varphi_y^2)=0, \quad \text{and} \quad \eta_t-\varphi_y+\eta_x\varphi_x=0 \quad \text{at} \quad y=1+\eta(x, t).$$

We refer to [11] (Sect. 13.1) for the derivation of these equations.

2. Boussinesq's System

We shall describe a class of flows depending on a small parameter $\varepsilon>0$ which will simultaneously measure the maximal amplitude and minimal wavelength. In particular, we assume $\varphi=\mathcal{O}(\varepsilon^{1/2})$, $\eta=\mathcal{O}(\varepsilon)$, and the dominant wavelength is $\mathcal{O}(1/\varepsilon^{1/2})$. These assumptions suggest a choice of new coordinates $\xi=\varepsilon^{1/2}x$, $\tau=\varepsilon^{1/2}t$ and scaled variables $\varphi(x, y, t)=\varepsilon^{1/2}\psi(\xi, y, \tau)$, $\eta(x, t)=\varepsilon v(\xi, \tau)$ in terms of which the problem is written as

(1) $$\varepsilon\psi_{\xi\xi}+\psi_{yy}=0, \qquad 0<y<1+\varepsilon v(\xi, \tau),$$

(2) $$\psi_y=0 \quad \text{at} \quad y=0,$$

(3) $$v+\psi_\tau+(1/2)(\varepsilon\psi_\xi^2+\psi_y^2)=0,$$

and

(4) $$\varepsilon v_\tau-\psi_y+\varepsilon v_\xi\psi_\xi=0, \qquad y=1+\varepsilon v(\xi, \tau).$$

We consider a formal asymptotic expansion of ψ in powers of ε and retain only those terms of orders zero and one. From (1) and (2) we obtain

$$\psi(\xi, y, \tau) = F(\xi, \tau)-(y^2 \varepsilon/2\,!)\,F_{\xi\xi}+(y^4 \varepsilon^2/4\,!)\,F_{\xi\xi\xi\xi}+\mathcal{O}(\varepsilon^3), \tag{5}$$

where $F(\xi, \tau)$ is the scaled potential at the bottom. Substitution of (5) into (3) and (4) yields

$$v+F_\tau-(\varepsilon/2)\,F_{\xi\xi\tau}+(\varepsilon/2)\,F_\xi^2 = \mathcal{O}(\varepsilon^2), \tag{6}$$

$$v_\tau+(1+\varepsilon v)\,F_{\xi\xi}-(\varepsilon/6)\,F_{\xi\xi\xi\xi}+\varepsilon v_\xi F_\xi = \mathcal{O}(\varepsilon^2), \tag{7}$$

Denote the scaled horizontal velocity at the level y by $u(\xi, \tau) \equiv \psi_\xi(\xi, y, \tau)$ From (5) we obtain

$$u = F_\xi-(y^2 \varepsilon/2)\,F_{\xi\xi\xi}+\mathcal{O}(\varepsilon^2), \tag{8}$$

and it follows from (8) that

$$\begin{aligned} F_\xi &= u+(y^2 \varepsilon/2)\,F_{\xi\xi\xi}+\mathcal{O}(\varepsilon^2) \\ &= u+(y^2 \varepsilon/2)\,u_{\xi\xi}+\mathcal{O}(\varepsilon^2). \end{aligned}$$

Differentiation of (6) with respect to ξ gives

$$v_\xi+u_\tau-(\varepsilon/2)(1-y^2)\,u_{\xi\xi\tau}+\varepsilon u u_\xi = \mathcal{O}(\varepsilon^2). \tag{9}$$

From (7) we obtain

$$v_{\tau\xi\xi}+u_{\xi\xi\xi} = \mathcal{O}(\varepsilon), \tag{10}$$

so (7) is equivalent to

$$v_\tau-(\varepsilon/2)(y^2-(1/3))\,v_{\tau\xi\xi}+u_\xi+\varepsilon(vu)_\xi = \mathcal{O}(\varepsilon^2). \tag{11}$$

Equations (9) and (11) give us our basic model correct to second order in ε for long waves of small amplitude

$$\left\{\begin{aligned} & u_\tau-(\varepsilon/2)(1-y^2)\,u_{\xi\xi\tau}+v_\xi+\varepsilon u u_\xi = 0, \\ & v_\tau-(\varepsilon/2)(y^2-1/3)\,v_{\xi\xi\tau}+u_\xi+\varepsilon(vu)_\xi = 0. \end{aligned}\right. \tag{I}$$

The system (I) is a variation on that given by Boussinesq [2] and reduces to it when the horizontal velocity u is averaged over $0 < y < 1$. [From (8) it follows that the average horizontal velocity is attained where $y = 1/\sqrt{3}$ and then the second term in the second equation of (I) is eliminated.] We shall show in Section 6 that various initial-boundary value problems for (I) are well-posed locally in time. *See* [7] for a similar hyperbolic model correct to second order.

3. Unidirectional Waves

We shall simplify the system (I) by considering only solutions whose main part consists of waves travelling to the right. In order to motivate the appropriate assumption, note that to lowest order in ε (I) becomes

$$u_\tau + v_\xi = 0, \qquad v_\tau + u_\xi = 0$$

and the solution is given by

(12)
$$u = f(\xi - \tau) + g(\xi + \tau), \qquad v = f(\xi - \tau) - g(\xi + \tau).$$

The quantity $u - v = 2g(x+t)$ is a measure of that component of the solution (12) which consists of the leftward travelling wave. For a rightward moving solution of (I) we may not expect $u - v \equiv 0$ but will assume $u - v = \mathcal{O}(\varepsilon)$. Then we have

$$\varepsilon(vu)_\xi = \varepsilon(v^2)_\xi + \mathcal{O}(\varepsilon^2),$$

and when the above is substituted in (11), the pair (9) and (11) give us our second model correct to second order

(II)
$$\begin{cases} u_\tau - (\varepsilon/2)(1-y^2)\,u_{\xi\xi\tau} + v_\xi + \varepsilon\, u u_\xi = 0, \\ v_\tau - (\varepsilon/2)(y^2 - 1/3)\,v_{\xi\xi\tau} + u_\xi + 2\varepsilon\, v v_\xi = 0. \end{cases}$$

The system (II) is a modification of (I) in which the equations are coupled only through linear terms. It results from the assumption $u - v = \mathcal{O}(\varepsilon)$ which we associate with essentially rightward moving waves. We shall show in Section 7 that certain initial-boundary value problems for (II) are well-posed globally in time. Leftward travelling waves are similarly modelled by a system like (II) but with a change of sign in the last term.

Finally we note that if v is eliminated from (II) and terms of order $\mathcal{O}(\varepsilon^2)$ or smaller are neglected, then we obtain a form of Boussinesq's equation ([2], [3]):

(13)
$$u_{\tau\tau} - (\varepsilon/3)\,u_{\xi\xi\tau\tau} + (\varepsilon/2)(u^2)_{\xi\tau} - (u + \varepsilon u^2)_{\xi\xi} = 0.$$

Note further that (13) is independent of y. By substituting the lowest order approximation of (13), $u_{\tau\tau} = u_{\xi\xi} + \mathcal{O}(\varepsilon)$, we obtain another variant of (13) in the form

(14)
$$u_{\tau\tau} - (\varepsilon/3)\,u_{\xi\xi\xi\xi} + (\varepsilon/2)(u^2)_{\xi\tau} - (u + \varepsilon u^2)_{\xi\xi} = 0.$$

See [11] (Sect. 13.11) for a discussion of (13) and (14).

4. Nearly-Stationary Waves

We shall find conditions under which the systems (I) and (II) can be reduced to a single equation. As before, we consider only a rightward moving wave, and so we assume $u - v = \mathcal{O}(\varepsilon)$ throughout this Section. Our objective is to show that the indicated reduction is possible if and only if a second condition is satisfied, namely,

(15)
$$(D_\tau + D_\xi)(u - v) = \mathcal{O}(\varepsilon^2).$$

The assumption (15) is not always clearly mentioned in the reduction of (I) or (II) to a single equation and we briefly explore its consequences below.

To first see how the condition (15) arises, we substitute $v = u+\mathcal{O}(\varepsilon)$ in the first order terms of (9) and (11). The resulting equations are added to yield

$$(D_\tau+D_\xi)(u+v)-(\varepsilon/3)u_{\xi\xi\tau}+3\varepsilon uu_\xi = \mathcal{O}(\varepsilon^2).$$

From this it follows that the equation

$$u_\tau+u_\xi-(\varepsilon/6)u_{\xi\xi\tau}+(3/2)\varepsilon uu_\xi = 0 \tag{16}$$

is correct to second order in ε if and only if (15) holds. In view of the first order approximation

$$u_\tau+v_\xi = \mathcal{O}(\varepsilon),$$

the equation

$$u_\tau+u_\xi+(\varepsilon/6)u_{\xi\xi\xi}+(3/2)\varepsilon uu_\xi = 0 \tag{17}$$

is similarly correct to second order in ε if and only if (15) holds. *See* [4] for a review of (17), ([8], [10]) for results on (16), and ([1], [11]) for a comparison of the two.

The condition expressed by (15) is that the quantity $u-v$ has slow growth... of order ε^2... in the direction of the wave. That is, if we view the wave while moving to the right with unit velocity, then $u-v$ will have an apparent wavelength of order $1/\varepsilon$. Thus, essential change in the shape of the wave will occur in time intervals of order $1/\varepsilon$. Furthermore, it is time intervals of this order that the terms of order ε^2 which we dropped above may contribute to the change in waveform. These remarks suggest that in order to study the long time effect of neglected terms on the shape of the wave in the vicinity of the rightward progressing wave, we should rescale in the coordinates

$$s = \xi-\tau, \qquad T = \varepsilon\tau$$

in which we have

$$D_\tau+D_\xi = \varepsilon D_T$$

and the new variables defined by

$$u(\xi, \tau) \equiv U(s, T),$$

$$v(\xi, \tau) \equiv U(s, T)+\varepsilon V(s, T).$$

Then we have the two conditions

$$u-v = \mathcal{O}(\varepsilon), \qquad D_\tau+D_\xi = \mathcal{O}(\varepsilon) \tag{18}$$

which imply (15). According to (18), we are looking at a rightward moving wave which has nearly-constant shape, i. e., a nearly-stationary wave.

The preceeding change-of-variable in (9) and (11) gives

$$\varepsilon U_T+(\varepsilon/2)(1-y^2)U_{sss}+\varepsilon V_s+\varepsilon UU_s = \mathcal{O}(\varepsilon^2),$$

$$\varepsilon U_T+(\varepsilon/2)(y^2-1/3)U_{sss}-\varepsilon V_s+2\varepsilon UU_s = \mathcal{O}(\varepsilon^2),$$

and when these are added and divided by 2ε, we obtain

(19) $$U_T+(1/6)U_{sss}+(3/2)UU_s=0$$

correct to first order in ε. It is in the variables

$$s=\varepsilon^{1/2}(x-t), \qquad T=\varepsilon^{3/2}t$$

that the nonlinearity is balanced with the dispersion in the model. This is reflected by the lack of dependence of (19) on ε and can be motivated directly by the dispersion relation [4]. In particular, (19) is the natural model in which to study asymptotic behavior $t \gg 1$ of nearly-stationary waves progressing to the right. For the computation of profiles of such waves over shorter time intervals... of length $\mathcal{O}(1/\varepsilon^{1/2})$... either of (16) or (17) seems equally appropriate. *See* [1] or [11] (p. 463) for further discussion.

INITIAL-BOUNDARY VALUE PROBLEMS FOR THE APPROXIMATE EQUATIONS. — We first consider the Cauchy problem for the abstract evolution equation

(20) $$\mathcal{M}w'(\tau)+F(w(\tau))=0$$

in Hilbert space. For our applications to the approximate equations for water waves we shall determine $\mathcal{M}$ so that

(21) $$\mathcal{M}w=[u-(\varepsilon/2)(1-y^2)u_{\xi\xi}, v-(\varepsilon/2)(y^2-1/3)v_{\xi\xi}], \qquad w=[u, v]$$

in the sense of distributions. Thus, if F is given by

(22) $$F(w)=[v_\xi+\varepsilon uu_\xi, u_\xi+\varepsilon(vu)_\xi], \qquad w=[u, v],$$

then the system I is a realization of (20) in an appropriate space of pairs of functions. Similarly, if F is given by

(23) $$F(w)=[v_\xi+\varepsilon uu_\xi, u_\xi+2\varepsilon vv_\xi], \qquad w=[u, v],$$

then the system II is realized by (20) in an appropriate function space. Thus, existence-uniqueness-stability results for the abstract Cauchy problem will yield corresponding results for generalized solutions of initial-boundary value problems for the system I and II. A similar program for (16) was presented in [1] et [10].

5. A Nonlinear Evolution Equation

Let V be a real Hilbert space. Denote its continuous dual by V′, the value of $f \in V'$ at $x \in V$ by $\langle f, x\rangle$, the norm on V by $\|\cdot\|$, and its innerproduct by $(.,.)_V$. Thus, the norm on V′ is given by $\|f\|_{V'} \equiv \sup\{|\langle f, x\rangle| : x \in V, \|x\| \leqq 1\}$. Assume we are given a continuous linear $\mathcal{M}: V \to V'$ which is coercive. That is, there is a $k>0$ such that

$$\langle \mathcal{M}x, x\rangle \geqq k\|x\|^2, \qquad x \in V.$$

It follows that $\mathcal{M}$ is a bijection of V onto V′ with a continuous inverse. Finally, suppose we have a (nonlinear) function $F: V \to V'$ which is uniformly Lipschitz on bounded sets. That is, for each bounded set B in V there is a $Q > 0$ such that

(24) $$\|F(x)-F(y)\|_{V'} \leqq Q\|x-y\|, \qquad x, y \in B.$$

Since $\mathcal{M}^{-1} : V' \to V$ is bounded, it follows that the composite function $\mathcal{M}^{-1} F : V \to V$ is similarly uniformly Lipschitz on bounded sets in V.

For each integer $k \geqq 0$, interval $[a, b]$ and Hilbert space $\mathcal{X}$, we denote by $C^k([a, b], \mathcal{X})$ the space of $\mathcal{X}$-valued functions on $[a, b]$ which are (strongly) continuous together with their (strong) derivatives through order k. Then a *weak solution* of (20) on $[a, b]$ is a $w \in C^1([a, b], V)$ which satisfies (20) at every $\tau \in [a, b]$. The following result on local solutions of (20) follows from the classical successive-approximations proof.

THEOREM 1. — *Let* V *be a Hilbert space,* $\mathcal{M} : V \to V'$ *be continuous, linear and coercive, and let* $F : V \to V'$ *be uniformly Lipschitz on bounded sets. Then for each* $w_0 \in V$ *there is a unique weak solution of* (20) *on some interval* $[-c, c]$ *which satisfies* $w(0) = w_0$. *It is sufficient to choose* $c = kb/Q(1+b)$, *where* Q *and* b *are chosen so that* (24) *holds for* $B = \{x \in V : \|x-w_0\| \leqq b\}$ *and that* $\|F(w_0)\|_{V'} \leqq Q$.

The preceding result gives a solution local in time whereas we shall also want to establish existence of a solution over long intervals. Our intended applications do not permit the imposition of linear growth rates on F. Note that (22) and (23) contain quadratic terms and such growth rates can lead to blow-up in finite time. However, the following situation will be shown to be appropriate for the system II.

THEOREM 2. — *In addition to the hypotheses of Theorem* 1, *assume* $\mathcal{M}$ *is symmetric,* i. e.,

$$\langle \mathcal{M}x, y\rangle = \langle \mathcal{M}y, x\rangle, \qquad x, y \in V,$$

and for some constants $K, L \geqq 0$ *we have*

(25) $$\langle F(x), x\rangle \geqq -K\|x\|^2 - L\|x\|, \qquad x \in V.$$

Then for each $w_0 \in V$ *there exists a* $c > 0$ *and a unique weak solution* $w(.)$ *of* (20) *on* $[-c, +\infty)$ *with* $w(0) = w_0$. *This solution satisfies the estimate*

(26) $$\|w(t)\| \leqq (e^{tK/k}/k^{1/2})(\langle \mathcal{M}w_0, w_0\rangle + (L/k^{1/2})t)^{1/2}, \qquad t \geqq 0.$$

Proof. — First we establish the *a priori* estimate (26) on any weak solution of (20) on $[0, \infty)$. Since $\mathcal{M}$ is symmetric and (25) holds, we find that the function $\sigma(\tau) \equiv \langle \mathcal{M}w(\tau), w(\tau)\rangle$ is continuously differentiable and satisfies

$$\sigma'(\tau) \leqq (2K/k)\sigma(\tau) + (2L/k^{1/2})(\sigma(\tau))^{1/2}.$$

That is, the function

$$\psi(\tau) \equiv \exp(-2K\tau/k).\sigma(\tau)$$

satisfies the differential inequality

$$\psi'(\tau) \leqq \exp(-K\tau/k).(2L/k^{1/2}).\psi(\tau)^{1/2},$$

and this is integrated to obtain the estimate

$$\psi(\tau) \leqq \psi(0) + (L/k^{1/2})\tau, \qquad \tau \geqq 0.$$

This implies the desired result.

Next we construct a weak solution on an interval $[-c, T]$, where $T > 0$ is arbitrary. For any such $T > 0$, it follows from (26) that there is a $\beta > 0$ such that a weak solution of (20) on an interval $[0, a]$ with $0 < a \leqq T$ and $w(0) = w_0$ satisfies $\| w(\tau) \| \leqq \beta$. Let $B = \{ x \in V : \| x \| \leqq 3\beta \}$ and choose Q so large that (24) holds and $Q \geqq \sup \{ \| F(x) \|_{V'} : \| x \| \leqq \beta \}$. Finally, choose $c_\beta \equiv k\, 2\beta / Q(1+2\beta)$. It follows from Theorem 1 that for every s, $0 \leqq s \leqq T$, and for every $w_s \in V$ with $\| w_s \| \leqq \beta$ there is an unique weak solution of (20) on the interval $[s - c_\beta, s + c_\beta]$ satisfying $w(s) = w_s$. Thus, the solution of (24) with $w(0) = w_0$ is obtained by a finite number of extensions as indicated on an interval $[-c_\beta, T + c_\beta]$. Since $T > 0$ was arbitrary, the desired result follows.

COROLLARY. — *In addition to the hypotheses of Theorem* 1, *assume* $\mathcal{M}$ *is symmetric and*

(27)
$$\langle F(x), x \rangle = 0, \qquad x \in V.$$

Then for each $w_0 \in V$ *there exists a unique weak solution* $w(.)$ *of* (20) *on* $(-\infty, \infty)$ *with* $w(0) = w_0$. *This solution satisfies*

(28)
$$\langle \mathcal{M} w(t), w(t) \rangle = \langle \mathcal{M} w_0, w_0 \rangle, \qquad -\infty \leqq t < \infty.$$

A stronger notion of a solution of (20) also will occur in our applications. To describe it, we let H be a Hilbert space with inner product $(.,.)_H$ and assume V is a dense subset of H for which the identity $V \hookrightarrow H$ is continuous. We identify H with its dual H' by the Riesz map and thereby obtain a continuous identification of H in V' by duality. Set $D(M) \equiv \{ x \in V : \mathcal{M} x \in H \}$ and denote by M the restriction of $\mathcal{M}$ to $D(M)$. Note that $D(M)$ is a Hilbert space with the inner product

$$(x, y)_{D(M)} \equiv (Mx, My)_H, \qquad x, y \in D(M).$$

The identity $D(M) \hookrightarrow V$ is continuous and this implies that

$$C^k([a, b], D(M)) \subset C^k([a, b], V)$$

for any interval $[a, b]$ and integer $k \geqq 0$. We define a *strong solution* of (20) on $[a, b]$ to be a $w \in C^1([a, b], D(M))$ which satisfies (20) at each $\tau \in [a, b]$. Thus, under the hypotheses of Theorem 1, weak and strong solutions of (20) on $[a, b]$ are distinguished by each term in (20) belonging to $C^0([a, b], V')$ and $C^0([a, b], H)$, respectively.

We give a sufficient condition for a weak solution to be a strong solution. *See* [9] for more general results.

THEOREM 3. — *In addition to the hypotheses of Theorem* 1, *assum*[2] $F : V \to H$ *is continuous. Let* w *be a weak solution of* (20) *on* $[a, b]$. *Then* w *is a strong solution of* (20) *on* $[a, b]$ *if and only if* $w(s) \in D(M)$ *for some* $s \in [a, b]$.

Proof. – The function $F(w(.))$ belongs to $C^0([a, b], H)$, so (20) implies $w' \in C^0([a, b], D(M))$. From the continuity of the embedding $D(M) \hookrightarrow V$ and the fundamental theorem applied to $w' \in C^1([a, b], V)$, it follows that

$$\int_s^\tau w' = w(\tau) - w(s), \qquad \tau, s \in [a, b],$$

where the above left side is (e. g.) the Riemann integral in D (M). The desired result follows immediately.

COROLLARY. – *A weak solution* w *of* (20) *on* $[a, b]$ *satisfies* $w(\tau) - w(s) \in D(M)$ *for all* $\tau, s \in [a, b]$.

6. Local Solvability of I

We shall present initial-boundary value problems for the system (I) and show they are well-posed over a sufficiently short time interval. Let $J \equiv (a, b)$ be an interval with $-\infty \leqq a < b \leqq +\infty$, and denote by $L^2(J)$ the Hilbert space of (equivalence classes of) real-valued square-summable functions on J. For each integer $k \geqq 0$, let $H^k(J)$ be the Hilbert space of Sobolev consisting of those $\varphi \in L^2(J)$ for which all the derivatives $D^j \varphi$, $0 \leqq j \leqq k$, are also in $L^2(J)$. (Derivatives are taken in the sense of distributions. *See* [6].) Each $\varphi \in H^1(J)$ is (represented by) an absolutely continuous function and we have

$$\| \varphi \|_{L^\infty(J)} \leqq \| \varphi \|_{H^1(J)}, \qquad \varphi \in H^1(J), \tag{29}$$

where $\| . \|_{L^\infty(J)}$ denotes the essential supremum on J and the $H^1(J)$-norm is given by

$$\| \varphi \|_{H^1(J)} = (\| \varphi \|_{L^2(J)}^2 + \| D\varphi \|_{L^2(J)}^2)^{1/2}.$$

Let V be a closed subspace of the product $H^1(J) \times H^1(J) \equiv \{ [\varphi_1, \varphi_2] : \varphi_1, \varphi_2 \in H^1(J) \}$ with the inherited inner product and norm,

$$\| \varphi \|_V \equiv \| \varphi \|_{H^1(J) \times H^1(J)} = (\| \varphi_1 \|_{H^1(J)}^2 + \| \varphi_2 \|_{H^1(J)}^2)^{1/2}, \qquad \varphi = [\varphi_1, \varphi_2] \in V.$$

We also will assume $C_0^\infty(J) \times C_0^\infty(J) \subset V$. Since $C_0^\infty(J)$ is dense in $L^2(J)$, it follows that we may identify $L^2(J) \times L^2(J) \subset V'$ and simultaneously $L^2(J) \times L^2(J) \subset \mathscr{D}'(J) \times \mathscr{D}'(J)$ [Here $\mathscr{D}'(J)$ denotes the space of distributions on J.] Finally, we set $H = L^2(J) \times L^2(J)$. Note that we have already identified $H = H'$ in the above.

Let y be a real number and define a continuous and symmetric $\mathscr{M} : V \to V'$ by

$$\begin{aligned} \langle \mathscr{M} \varphi, \psi \rangle \equiv{}& (\varphi_1, \psi_1)_{L^2(J)} + (\varepsilon/2)(1-y^2)(D\varphi_1, D\psi_1)_{L^2(J)} \\ &+ (\varphi_2, \psi_2)_{L^2(J)} + (\varepsilon/2)(y^2 - 1/3)(D\varphi_2, D\psi_2)_{L^2(J)}, \\ &\varphi = [\varphi_1, \varphi_2], \quad \psi = [\psi_1, \psi_2] \in V. \end{aligned}$$

We shall always assume $1/3 < y^2 < 1$; then $\mathscr{M}$ is coercive, hence, satisfies the hypothesis in Theorems 1, 2, and 3. For each $\varphi \in V$, let $\mathscr{M}_0 \varphi$ denote the restriction to $C_0^\infty(J) \times C_0^\infty(J)$ of $\mathscr{M} \varphi \in V'$. Thus $\mathscr{M}_0 \varphi \in \mathscr{D}'(J) \times \mathscr{D}'(J)$ is given by [*cf.*, (21)]:

$$\mathscr{M}_0 \varphi = [\varphi_1 - (\varepsilon/2)(1-y^2) D^2 \varphi_1, \varphi_2 - (\varepsilon/2)(y^2 - 1/3) D^2 \varphi_2].$$

Whenever $\varphi \in V$ and $\mathcal{M}_0 \varphi \in H \equiv L^2(J) \times L^2(J)$, then $\varphi \in H^2(J) \times H^2(J)$ and we obtain *Green's formula*

$$
(30) \qquad \begin{cases} \langle \mathcal{M}\varphi, \psi \rangle - (\mathcal{M}_0 \varphi, \psi)_H \\ \quad = (\varepsilon/2)\{(1-y^2)D\varphi_1(\xi)\psi_1(\xi)+(y^2-1/3)D\varphi_2(\xi)\psi_2(\xi)\}\Big|_{\xi=a}^{\xi=b}, \\ \qquad\qquad \psi \in V. \end{cases}
$$

If φ is such that the right side of (30) vanishes for all $\psi \in V$, then $\mathcal{M}\varphi = \mathcal{M}_0\varphi \in H$. Conversely, if $\mathcal{M}\varphi \in H$, then $\mathcal{M}_0\varphi = \mathcal{M}\varphi \in H$ and the right side of (30) vanishes for all $\psi \in V$. These remarks characterize D (M) in V in terms of regularity and boundary conditions of its elements. Also, we have shown M is the restriction of $\mathcal{M}_0$ to those φ which satisfy the "variational" boundary conditions obtained from (30).

We define the nonlinear function $F : V \to H$ as suggested by (22):

$$
F(\varphi) \equiv [D\varphi_2 + \varepsilon\varphi_1 D\varphi_1,\ D\varphi_1 + \varepsilon D(\varphi_1\varphi_2)], \qquad \varphi = [\varphi_1, \varphi_2] \in V.
$$

The estimates

$$
\begin{aligned}
\|F(\varphi)-F(\psi)\|_H &\leq \|D(\varphi_2-\psi_2)+\varepsilon(\varphi_1 D\varphi_1-\psi_1 D\psi_1)\|_{L^2(J)} \\
&\quad + \|D(\varphi_1-\psi_1)+\varepsilon D(\varphi_1\varphi_2-\psi_1\psi_2)\|_{L^2(J)} \\
&\leq \|D(\varphi_2-\psi_2)\|_{L^2(J)}+\varepsilon\|\varphi_1 D(\varphi_1-\psi_1)+(\varphi_1-\psi_1)D\psi_1\|_{L^2(J)} \\
&\quad + \|D(\varphi_1-\psi_1)\|_{L^2(J)}+\varepsilon\|(\varphi_1-\psi_1)D\varphi_2+\psi_1 D(\varphi_2-\psi_2)\|_{L^2(J)} \\
&\quad + \varepsilon\|(\varphi_2-\psi_2)D\varphi_1+\psi_2 D(\varphi_1-\psi_1)\|_{L^2(J)} \\
&\leq (1+\varepsilon\|\psi_1\|_{L^\infty(J)}+\varepsilon\|D\varphi_1\|_{L^2(J)})\|D(\varphi_2-\psi_2)\|_{L^2(J)} \\
&\quad + (1+\varepsilon\|\varphi_1\|_{L^\infty(J)}+\varepsilon\|D\psi_1\|_{L^2(J)} \\
&\qquad + \varepsilon\|D\varphi_2\|_{L^2(J)}+\varepsilon\|\psi_2\|_{L^\infty(J)}).\|D(\varphi_1-\psi_1)\|_{L^2(J)}
\end{aligned}
$$

and (29) show that $F : V \to H$ is uniformly Lipschitz on bounded sets. Thus, F satisfies the hypotheses of Theorems 1 and 3.

Example 1: *Bounded Channel.* – Suppose $J = (a, b)$ is a bounded interval and choose $V = \{\varphi = [\varphi_1, \varphi_2] : \varphi_1, \varphi_2 \in H^1(J),\ \varphi_1(a) = \varphi_1(b) = 0\}$. Then the remarks following (30) show $\varphi \in D(M)$ if and only if $\varphi \in V$, $\varphi \in H^2(J) \times H^2(J)$, and

$$
D\varphi_2(a) = D\varphi_2(b) = 0.
$$

Theorem 1 implies that for each pair $[u_0, v_0] = w_0 \in V$ there exists a unique weak solution $w(t) = [u(t), v(t)]$ of (20) on some interval $-c < t < c$ with $w(0) = w_0$. This pair of functions on $J \times (-c, c)$ is a generalized solution of the system (I) with the initial conditions

$$
u(\xi, 0) = u_0(\xi), \quad v(\xi, 0) = v_0(\xi), \qquad \xi \in J
$$

and the boundary conditions

$$
\begin{gathered}
u(a, \tau) = u(b, \tau) = 0, \\
D_\xi D_\tau v(a, \tau) = D_\xi D_\tau v(b, \tau) = 0, \qquad -c < \tau < c.
\end{gathered}
$$

These boundary conditions follow, respectively, from the inclusions $w(\tau) \in \mathrm{V}$ and $w'(\tau) \in \mathrm{D}(\mathrm{M})$. If (and only if) $w_0 \in \mathrm{D}(\mathrm{M})$, then w is a strong solution for which we obtain the more restrictive boundary conditions

$$u(a, \tau) = u(b, \tau) = 0,$$
$$\mathrm{D}_\xi v(a, \tau) = \mathrm{D}_\xi v(b, \tau) = 0, \qquad -c < \tau < c.$$

Such boundary conditions occur in the description of gravity waves in a channel which is bounded between two walls. At each wall, the (horizontal) velocity u is zero and the slope of the disturbance v is zero. (We have not been able to establish global existence results in this situation. This may be related to the interaction of waves reflecting from the boundaries.)

7. Global Solvability of II

Two examples of initial-boundary value problems for the system (II) will be presented. Each is shown to possess the existence-uniqueness-stability results characteristic of problems which are well-posed over long time intervals.

Define $\mathscr{M}$ as in Section 6 and assume as before that $1/3 < y^2 < 1$. We define the non-linear function $\mathrm{F} : \mathrm{V} \to \mathrm{H}$ as suggested by (23):

$$\mathrm{F}(\varphi) \equiv [\mathrm{D}\varphi_2 + \varepsilon\varphi_1 \mathrm{D}\varphi_1, \mathrm{D}\varphi_1 + 2\varepsilon\varphi_2 \mathrm{D}\varphi_2], \qquad \varphi = [\varphi_1, \varphi_2] \in \mathrm{V}.$$

Estimates similar to those of Section 6 show that F is uniformly Lipschitz on each bounded set in V. Thus, the hypotheses of Theorems 1 and 3 are satisfied. To check the estimate (25), we have the computation

$$\begin{aligned}\langle \mathrm{F}(\varphi), \varphi \rangle &= (\mathrm{D}\varphi_2 + \varepsilon\varphi_1 \mathrm{D}\varphi_1, \varphi_1)_{\mathrm{L}^2(\mathrm{J})} + (\mathrm{D}\varphi_1 + 2\varepsilon\varphi_2 \mathrm{D}\varphi_2, \varphi_2)_{\mathrm{L}^2(\mathrm{J})} \\ &= \int_a^b \mathrm{D}\{\varphi_1\varphi_2 + (\varepsilon/3)(\varphi_1)^3 + (2\varepsilon/3)(\varphi_2)^3\}.\end{aligned}$$

From this it follows that (27) is satisfied if we have

$$\varphi(a) = \varphi(b), \qquad \varphi \in \mathrm{V}. \tag{31}$$

The condition (31) holds in the two important cases of periodic and Dirichlet boundary conditions. The two examples to follow illustrate these respective cases.

Example 2: Periodic Waves. — Let the interval $\mathrm{J} = (a, b)$ be bounded and choose $\mathrm{V} = \{\varphi = [\varphi_1, \varphi_2] : \varphi_j \in \mathrm{H}^1(\mathrm{J}), \varphi_j(a) = \varphi_j(b), j = 1, 2\}$. From Green's formula we find that $\mathrm{D}(\mathrm{M}) = \{\varphi \in \mathrm{V} : \varphi_j \in \mathrm{H}^2(\mathrm{J}), \mathrm{D}\varphi_j(a) = \mathrm{D}\varphi_j(b), j = 1, 2\}$. Since (27) is true, the Corollary to Theorem 2 implies that for each $w_0 = [u_0, v_0] \in \mathrm{V}$, there exists a unique weak solution of (20) on $(-\infty, \infty)$ for which $w(0) = w_0$ and (28) holds. The pair of functions u, v on $\mathrm{J} \times (-\infty, \infty)$ given by $w(\tau) = [u(., \tau), v(., \tau)]$ is the unique generalized solution of the system (II) with the initial conditions

$$u(\xi, 0) = u_0(\xi), \quad v(\xi, 0) = v_0(\xi), \qquad a \leqq \xi \leqq b,$$

and the boundary conditions

$$(32)\quad \begin{cases} u(a,\tau)=u(b,\tau), \qquad v(a,\tau)=v(b,\tau), \\ D_\xi D_\tau(u(a,\tau)-u(b,\tau)) = D_\xi D_\tau(v(a,\tau)-v(b,\tau))=0, \qquad -\infty<\tau<\infty. \end{cases}$$

Furthermore, Theorem 3 shows that if $w_0 \in D(M)$ then the (strong) solution satisfied the stronger set of boundary conditions

$$(33)\quad \begin{cases} u(a,\tau)=u(b,\tau), \qquad v(a,\tau)=v(b,\tau), \\ D_\xi u(a,\tau)=D_\xi u(b,\tau), \quad D_\xi v(a,\tau)=D_\xi v(b,\tau), \qquad -\infty<\tau<\infty. \end{cases}$$

We could call (32) and (33) *weak periodicity* and *strong periodicity*, respectively, of the functions u, v. Note that each of these two conditions is equivalent to requiring that the (unique) $b-a$ periodic extensions of u and v belong (locally) to $H^1(R)$ and $H^2(R)$, respectively. The weakly (strongly) periodic boundary conditions apply to gravity waves in an infinite channel which are observed to be unidirectional and weakly (respectively, strongly) periodic at some reference initial time $\tau=0$. Note that (28) implies a conservation of "energy":

$$\|u(\tau)\|^2_{L^2(J)}+\|v(\tau)\|^2_{L^2(J)} +(\varepsilon/2)(1-y^2)\|Du(\tau)\|^2_{L^2(J)}+(\varepsilon/2)(y^2-1/3)\|Dv(\tau)\|^2_{L^2(J)}=\text{Const.},$$
$$-\infty<\tau<\infty.$$

The initial data must of course satisfy the condition $u_0-v_0=\mathcal{O}(\varepsilon)$ which is implied by the assumption which we associated with essentially rightward moving waves.

Example 3: *An Undular Bore.* — We shall use the system (II) to describe the development of a long wave of small elevation which forms a gentle transition between a uniform flow and still water. This occurs, for example, when a stream of water is sent into a long channel of still water in such a way that the transition between still water and the resulting deeper water has a gentle slope.

Let $J=(-\infty,\infty)$ and $V=H^1(J)\times H^1(J)$. Recall that each $\varphi\in H^1(J)$ is asymptotically null : $\lim_{x\to-\infty}\varphi(x)=\lim_{x\to\infty}\varphi(x)=0$ [6]. Let $h:J\to[0,1]$ be absolutely continuous and monotone decreasing with $Dh\in L^2(J)$, $\lim_{x\to-\infty}h(x)=1$, and $\lim_{x\to\infty}h(x)=0$. We shall consider a generalized solution u, v of (II) for which

$$(34)\quad \begin{cases} \lim_{\xi\to-\infty} u(\xi,\tau)=\lim_{\xi\to-\infty} v(\xi,\tau)=1, \\ \lim_{\xi\to\infty} u(\xi,\tau)=\lim_{\xi\to\infty} v(\xi,\tau)=0, \qquad 0\leqq\tau<\infty \end{cases}$$

and both $u(\xi,0)$ and $v(\xi,0)$ are given close to $h(\xi)$ for $\xi\in J$. Since $h(.)$ is not in $H^1(J)$, we consider instead the pair of functions

$$U(\xi,\tau)=u(\xi,\tau)-h(\xi), \qquad V(\xi,\tau)=v(\xi,\tau)-h(\xi).$$

Then it follows that the pair u, v is a solution of (II) if and only if the pair U, V is a solution of the system resulting from (20) when $\mathscr{M}$ is chosen as in Section 6 and F is defined by

$$F(\varphi) = [D(\varphi_2+h)+\varepsilon(\varphi_1+h)D(\varphi_1+h),\ D(\varphi_1+h)+2\varepsilon(\varphi_2+h)D(\varphi_2+h)],$$
$$\varphi = [\varphi_1, \varphi_2] \in V.$$

It follows from a computation similar to that above that the estimate (25) holds with constants K and L depending on $h(.)$. Thus, for each pair $U_0, V_0 \in H^1(J)$ it follows from Theorem 2 that there is a unique weak solution U, V of (20) on $[0, \infty)$ with $U(\xi, 0) = U_0(\xi)$, $V(\xi, 0) = V_0(\xi)$. The pair of functions $u \equiv U+h$, $v \equiv V+h$ is then the unique generalized solution on $J \times [0, \infty)$ of the system (II) with the initial conditions

$$u(\xi, 0) = U_0(\xi)+h(\xi), \quad v(\xi, 0) = V_0(\xi)+h(\xi), \qquad \xi \in J$$

and the boundary conditions (34). These boundary conditions follow since for each τ, $0 \leqq \tau < \infty$, both of $U(., \tau)$ and $V(., \tau)$ are asymptotically null. Finally from Theorem 3 and the observation that $D(M) = H^2(J)$ we see $u_\xi(., \tau)$ and $v_\xi(., \tau)$ are asymptotically null at every $\tau \geqq 0$ if $U_0(.)$, $V_0(.)$ and $h(.)$ all belong to $H^2(J)$.

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