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Degenerate Parabolic Initial-Boundary Value Problems*

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1. INTRODUCTION

We consider a class of implicit linear evolution equations of the form

$$\frac{d}{dt}\mathscr{M}u(t) + \mathscr{L}u(t) = f(t), \qquad t > 0, \tag{1.1}$$

in Hilbert space and their realizations in function spaces as initial-boundary value problems for partial differential equations which may contain degenerate or singular coefficients. The Cauchy problem consists of solving (1.1) subject to the initial condition $\mathscr{M}u(0) = h$. We are concerned with the case where the solution is given by an *analytic* semigroup; it is this sense in which the Cauchy problem is *parabolic*. Sufficient conditions for this to be the case are given in Theorem 1; this is a refinement of previously known results [15] to the linear problem and it extends the related work [13] to the (possibly) degenerate situation under consideration. Specifically, we do not assume $\mathscr{M}$ is invertible, but only that it is symmetric and non-negative.

Our primary motivation for considering the Cauchy problem for (1.1) is to show that certain classes of mixed initial-boundary value problems for partial differential equations are well-posed. Theorem 2 shows that if the operators $\mathscr{M}$ and $\mathscr{L}$ have additional structure which is typical of those operators arising from (possibly degenerate) parabolic problems then the evolution equation (1.1) is equivalent to a partial differential equation

$$\frac{d}{dt}(Mu(t)) + Lu(t) = F(t) \tag{1.2}$$

(obtained by restricting (1.1) to test functions) and a complementary boundary condition

$$\frac{d}{dt}(\partial_m u(t)) + \partial_l u(t) = g(t) \tag{1.3}$$

in an appropriate space of boundary values. These boundary conditions have

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a precise meaning even though no regularity results are claimed for the corresponding (stationary) elliptic problem.

The second objective here is to permit (possibly) *both* of the operators $\mathscr{M}$ and $\mathscr{L}$ to be degenerate, i.e., to correspond to partial differential operators whose coefficients are only assumed non-negative. For the classical diffusion equation

$$\frac{\partial}{\partial t}(c(x)\, u(x, t)) - \frac{\partial}{\partial x}\left(k(x)\, \frac{\partial u(x, t)}{\partial x}\right) = F(x, t) \tag{1.4}$$

our requirements on $\mathscr{M}$ are met if $c(x) \geqslant 0$ and if $k(x) > 0$ at each point in the region with (possibly) $k(x) \to 0$ at specified rates as x approaches the boundary.

The inclusion of such problems with degenerate elliptic parts is made possible by use of appropriate weighted Sobolev spaces [4,9]. In Section 3 we give sufficient conditions for such spaces to satisfy the structural requirements introduced in Section 2. See [1, 10, 11, 12] for related results with applications to degenerate elliptic problems.

Examples of initial-boundary value problems to which the abstract results apply are given in Section 4. Specifically, we discuss the classical problems for the equation (1.4) in higher dimensions as well as for a third order *pseudoparabolic* equation [3] which may "degenerate" to (1.4). The last example given is a problem for the diffusion equation which contains a (degenerate) elliptic-parabolic equation on a lower dimensional submanifold in the region. Such problems can arise in diffusion problems with singularities [2]; our results show how the appropriate "strong formulation" of such a problem depends on the degeneracy of the coefficients. Additional results for degenerate parabolic equations are given in [6, 8, 14].

The following standard notation will be used. For an interval I of real numbers, Banach space $\mathscr{X}$ and integer $m \geqslant 0$, by $C^m(I, \mathscr{X})$ we denote the space of m times continuously differentiable functions from I to $\mathscr{X}$. Such functions whose mth order derivative is Hölder continuous with exponent δ, $0 < \delta \leqslant 1$, are denoted by $C^{m+\delta}(I, \mathscr{X})$.

For a complex-valued function f on the open set G in Euclidean space $\mathbb{R}^n$, we denote by $\int_G f = \int_G f(x)\, dx$ the Lebesgue integral; dx is Lebesgue measure on G. Similarly ds is surface measure on the boundary, ∂G (of dimension $n - 1$) and $d\xi$ denotes measure on the $(n - 2)$-dimensional boundary of ∂G. $L^p(\Omega)$ is the usual Lebesgue space over any measurable set Ω. Partial derivative in the x_j-direction is given by $\partial_j = \partial/\partial x_j$. The space of m-times continuously differentiable complex-valued functions on G is $C^m(G)$; such functions with compact support in G are denoted by $C_0^{\,m}(G)$. $H^1(G)$ is the Hilbert space of Sobolev consisting of those functions in $L^2(G)$ all of whose first order derivatives belong to $L^2(G)$. For information on these and related spaces we refer to [7].

2. Degenerate Parabolic Cauchy Problem

We begin with an abstract evolution equation for which the Cauchy problem is resolved by an analytic semigroup.

Theorem 1. *Let W be a (complex) seminormed space whose seminorm is obtained from the non-negative symmetric sesquilinear form $x, y \mapsto \mathscr{M}x(y)$ associated with the given linear map $\mathscr{M}$ of W into the dual W' of conjugate-linear continuous functionals on W. Let V be a Hilbert space dense and continuously embedded in W and let $\mathscr{L}$ be continuous and linear from V into V'. Assume that for some real number λ, $\lambda\mathscr{M} + \mathscr{L}$ is V-elliptic: there is a $c > 0$ such that*

$$\operatorname{Re}(\lambda\mathscr{M} + \mathscr{L})\, x(x) \geqslant c \, \| x \|_V^2, \qquad x \in V. \tag{2.1}$$

Then for each $h \in W'$ and each Hölder continuous $f \in C^\delta([0, \infty), W')$, $0 < \delta \leqslant 1$, there is a unique $u \in C^0((0, \infty), V)$ such that $\mathscr{M}u \in C^0([0, \infty), W') \cap C^1((0, \infty), W')$, $\mathscr{M}u(0) = h$, and

$$\frac{d}{dt}\mathscr{M}u(t) + \mathscr{L}u(t) = f(t), \qquad t > 0. \tag{2.2}$$

Proof. First note that u is a solution of (2.2) if and only if the function defined by $v(t) \equiv e^{-\lambda t}u(t)$ is a solution of the corresponding problem with $\mathscr{L}$ replaced by $\lambda\mathscr{M} + \mathscr{L}$. Thus we may take $\lambda = 0$ in (2.1); the Lax-Milgram-Lions theorem then asserts that $\mathscr{L}$ is a bijection of V onto V'.

Let K be the kernel of $\mathscr{M}$, let W/K be the corresponding quotient space, and denote by H the completion of W/K. Regard the quotient map q: $W \to W/K$ as a norm-preserving injection of W into H and denote the corresponding dual map by q^*: $H' \to W'$. Note that q^* is an isomorphism. If $\mathscr{M}_0$: $H \to H'$ is the Riesz map associated with the scalar-product on H inherited from W, then we easily check that $\mathscr{M}$ factors according to

$$\mathscr{M}x = q^*\mathscr{M}_0 q(x), \qquad x \in W. \tag{2.3}$$

In order to simultaneously factor $\mathscr{L}$ we consider the subspace $D \equiv \{x \in V: \mathscr{L}x \in W'\}$ where we identify $W' \subset V'$. Then for each $x \in D$ we have

$$| \mathscr{L}x(y)| \leqslant \text{const.}(\mathscr{M}y(y))^{1/2}, \qquad y \in V.$$

If $x \in K \cap D$, then setting $y = x$ above an using (2.1) we obtain $x = 0$. Thus $K \cap D = \{0\}$ and there is a unique linear map $\mathscr{L}_0$ of $D_0 \equiv q[D]$ onto H' for which

$$\mathscr{L}x = q^*\mathscr{L}_0 q(x), \qquad x \in D. \tag{2.4}$$

Finally, we define the linear map $A_0 \equiv \mathscr{M}_0^{-1}\mathscr{L}_0$ from D_0 onto H.

We shall verify that $-A_0$ is the generator of an analytic semigroup on H. Since $\mathscr{M}_0$ is the Riesz map for H we obtain

$$(A_0x, y)_H = \mathscr{M}_0A_0x(y) = \mathscr{L}_0x(y), \qquad x \in D_0, \quad y \in H.$$

Setting $x = y = q(z)$ in the above and using (2.3) and (2.4), we obtain

$$(A_0x, x)_H = \mathscr{L}z(z), \qquad z \in D, \quad x = q(z).$$

Since $\mathscr{L}$ is V-elliptic, this shows that A_0 is *sectorial* [5, p. 280]. From the identity

$$q^*\mathscr{M}_0(I + A_0)\, q(z) = (\mathscr{M} + \mathscr{L})(z), \qquad z \in D$$

it follows that $I + A_0$ maps D_0 onto H, so A is *m-sectorial* and, hence, $-A_0$ generates an analytic semigroup on H [5, pp. 490–493]. This implies that the Cauchy problem

$$v'(t) + A_0v(t) = (q^*\mathscr{M}_0)^{-1} f(t), \qquad t > 0, \tag{2.5}$$

has a unique solution $v \in C^0([0, \infty), H) \cap C^1((0, \infty), H)$. For each $t > 0$ we have $v(t) \in D_0$, the domain of the generator, $-A_0$, so there is a unique $u(t) \in D$ for which $q(u(t)) = v(t)$ and $\mathscr{L}u(t) = q^*\mathscr{L}_0v(t)$. The function u so obtained is the desired solution of the Cauchy problem for (2.2).

To verify uniqueness, note that a solution u of (2.2) with $f \equiv 0$ and $\mathscr{M}u(0) = 0$ satisfies

$$\frac{d}{dt}\mathscr{M}u(t)(u(t)) = -2 \operatorname{Re} \mathscr{L}u(t)(u(t)) \leqslant 0,$$

so $\mathscr{M}u(t)(u(t)) \equiv 0$. Thus $\mathscr{L}u(t) = \mathscr{M}u(t) = 0$ for $t \geqslant 0$ and (2.1) gives $u(t) \equiv 0$. ∎

Remarks. (1) If W is a Hilbert space the Theorem 1 coincides with a result in [13].

(2) We were able to factor $\mathscr{L}$ in the form (2.4) and so obtain a *function* $\mathscr{L}_0$. The preceding technique extends to nonlinear situations and others where $\mathscr{L}_0$ and A_0 may be multi-valued [3, 15].

Our next objective is to describe sufficient additional structure on the spaces and operators in Theorem 1 to permit us to characterize the solution of (2.2) by means of an abstract partial differential equation (1.2) and an abstract boundary condition (1.3).

The Spaces. Let W be a seminormed space and V_1 be a Hilbert space with V_1 continuously embedded in W; V and $V_0 \subset V$ are (closed) subspaces of V_1 with V dense in W. Thus we identify $W' \subset V'$ by restriction. Denote by W_0 the closure of V_0 in W; then we can similarly identify $W_0' \subset V_0'$. The dual space W' is the direct sum of W_0' and the *annihilator* of W_0, $W_0^\perp \equiv \{h \in W' : h(w) = 0$ for all $w \in W_0\}$. This is denoted by $W' = W_0' \oplus W_0^\perp$ and identifies W_0' as a subspace of W'.

The Trace. The *trace* operator γ is a continuous linear surjection of V onto the Hilbert space B of *boundary values.* Assume V_0 is the kernel of γ; then the corresponding induced map $\hat{\gamma}$: $V/V_0 \to B$ is an isomorphism by the open-mapping theorem. Since $(V/V_0)'$ is (isometrically) isomorphic to $V_0^\perp$, the dual of $\hat{\gamma}$ gives an isomorphism γ^* of B' onto $V_0^\perp$ defined by $\gamma^*(g) = g \circ \gamma$ for all $g \in B'$.

The Operators. Let $\mathscr{L}$: $V_1 \to V'$ be given and define the corresponding *formal operator* L: $V_1 \to V_0'$ by setting Lu equal to the restriction to V_0 of $\mathscr{L}u$ for each $u \in V_1$. We can define $D_0 \equiv \{u \in V_1 : Lu \in W_0'\}$ since $W_0' \subset V_0'$. Then for each $u \in D_0$ we have $Lu \in W_0' \subset W' \subset V'$, so $\mathscr{L}u - Lu \in V_0^\perp$; this gives $\mathscr{L}u - Lu = \gamma^*(\partial_l u)$ for some $\partial_l u \in B'$. That is, there is a unique ∂_l: $D_0 \to B'$ such that

$$\mathscr{L}u(v) - Lu(v) = \partial_l u(\gamma v), \qquad u \in D_0, \quad v \in V. \tag{2.6}$$

Let $\mathscr{M}$: $W \to W'$ be given and define the formal operator M: $W' \to W_0'$ by setting Mw equal to the restriction of $\mathscr{M}w$ to W_0 for each $w \in W$. The restriction of $\mathscr{M}w - Mw$ to V then belongs to $V_0^\perp$, hence, equals $\gamma^*(\partial_m w)$ for some $\partial_m w \in B'$. Thus, there is a unique ∂_m: $W \to B'$ such that

$$\mathscr{M}w(v) - Mw(v) = \partial_m w(\gamma v), \qquad w \in W, \quad v \in V. \tag{2.7}$$

The identities (2.6) and (2.7) are *abstract Green's formulas.*

Before proceeding to our characterization of the solution of the Cauchy problem for (2.2), we consider the characterization of the solution of the corresponding stationary or *elliptic* problem. Thus, assume we are given the spaces, trace and operators as above, and assume that $\lambda\mathscr{M} + \mathscr{L}$ is V-coercive for some real number λ; cf. (2.1). Then the Lax-Milgram-Lions theorem shows that $\lambda\mathscr{M} + \mathscr{L}$ is an isomorphism of V onto V'. Let $d \in V_1$, $F \in W_0'$ and $g \in B'$ be given and set $f \equiv (\mathscr{L} + \lambda\mathscr{M})d + F + \gamma^*(g) \in V'$. There is a unique $\tilde{u} \in V$ such that $(\lambda\mathscr{M} + \mathscr{L})\tilde{u} = f$; set $u = \tilde{u} + d$. Then u is the unique solution of

$$u \in V_1, \qquad u - d \in V, \tag{2.8}$$

$$(\lambda\mathscr{M} + \mathscr{L})u = F + \gamma^*(g) \qquad \text{in} \quad V'. \tag{2.9}$$

By applying the equation (2.9) to points in V_0 we obtain

$$\lambda Mu + Lu = F \qquad \text{in} \quad W_0', \tag{2.10}$$

hence, $u \in D_0$. From (2.6)–(2.10) we obtain

$$\lambda\partial_m u + \partial_l u = g \qquad \text{in} \quad B'. \tag{2.11}$$

These computations show that the problem (2.8), (2.9) is equivalent to (2.8), (2.10), (2.11). In the applications below, (2.8) is a stable boundary condition, (2.10) is a partial differential equation in a space of distributions, and (2.11) is a complementary boundary condition.

Our results on the well-posedness of the degenerate parabolic Cauchy problem are given in the following.

THEOREM 2. *Assume we are given the spaces, the trace, and the operators as above. Assume the seminorm on W is obtained from the symmetric and non-negative operator $\mathscr{M}$ and that, for some real number λ, $\lambda\mathscr{M} + \mathscr{L}$ is V-elliptic (cf. (2.1)). Let $d \in C^{1+\delta}([0, \infty), V_1)$, $g \in C^{1+\delta}([0, \infty), B')$, $F \in C^{\delta}([0, \infty), W_0')$ and $h \in W'$. Then there exists exactly one $u \in C^0((0, \infty), V_1)$ with $\mathscr{M}u \in C^0([0, \infty), W') \cap C^1((0, \infty), W')$ such that*

$$\mathscr{M}u(0) = h \quad \text{in} \quad W', \tag{2.12}$$

and

$$\frac{d}{dt} Mu(t) + Lu(t) = F(t) \quad \text{in} \quad W_0', \tag{2.13}$$

$$u(t) - d(t) \in V, \tag{2.14}$$

$$\frac{d}{dt} \partial_m u(t) + \partial_l u(t) = g(t) \quad \text{in} \quad B' \tag{2.15}$$

for each $t > 0$.

Proof. Our plan is to apply Theorem 1 to obtain the solution of a problem similar to (2.2) and then to show this solution is characterized by (2.12)–(2.15). We may assume $\lambda = 0$ just as in the proof of Theorem 1; thus, $\mathscr{L}$ is an isomorphism of V onto V'.

Since $d(t) \in V_1$ for each $t \geqslant 0$, there exists a $\tilde{u}(t) \in V$ such that $\mathscr{L}\tilde{u}(t) = -\mathscr{L}d(t)$; the continuity of $\mathscr{L}$: $V_1 \to V'$ and $\mathscr{L}^{-1}$: $V' \to V$ shows that $\tilde{u} \in C^{1+\delta}([0, \infty), V)$. Setting $u_1(t) \equiv \tilde{u}(t) + d(t)$ for $t \geqslant 0$ gives us $u_1 \in C^{1+\delta}([0, \infty), V_1)$ such that $u_1(t) - d(t) \in V$, $\mathscr{L}u_1(t)(v) = 0$ for $v \in V$, and

$$\frac{d}{dt} (\mathscr{M}u_1(t)) + \mathscr{L}u_1(t) = \mathscr{M}u_1'(t), \qquad t \geqslant 0. \tag{2.16}$$

Similarly, for $t \geqslant 0$ we have $g(t) \circ \gamma \in V'$ and we can define $u_2(t) \equiv \mathscr{L}^{-1}(g(t) \circ \gamma)$. Then $u_2 \in C^{1+\delta}([0, \infty), V)$ and it satisfies

$$\frac{d}{dt} (\mathscr{M}u_2(t)) + \mathscr{L}u_2(t) = \mathscr{M}u_2'(t) + g(t) \circ \gamma, \qquad t \geqslant 0. \tag{2.17}$$

The right sides of (2.16) and (2.17) are in $C^{\delta}([0, \infty), W')$. Thus the function defined by $f(t) \equiv F(t) - \mathscr{M}u_1'(t) - \mathscr{M}u_2'(t)$, $t \geqslant 0$, belongs to $C^{\delta}([0, \infty), W')$

so we may appeal to Theorem 1 for a function $u_3 \in C^0((0, \infty), V)$ such that $\mathscr{M}u_3 \in C^0([0, \infty), W') \cap C^1((0, \infty), W')$, $\mathscr{M}u_3(0) = h - \mathscr{M}u_1(0) - \mathscr{M}u_2(0)$, and

$$\frac{d}{dt}(\mathscr{M}u_3(t)) + \mathscr{L}u_3(t) = f(t), \qquad t > 0. \tag{2.18}$$

Define the function u by $u(t) = u_1(t) + u_2(t) + u_3(t)$ for $t \geqslant 0$. It then follows from the above that $u \in C^0((0, \infty), V_1)$ with $\mathscr{M}u \in C_0([0, \infty), W') \cap C^1((0, \infty), W')$ statisfying (2.12) and for each $t > 0$, (2.14) and

$$\frac{d}{dt}(\mathscr{M}u(t)) + \mathscr{L}u(t) = F(t) + g(t) \circ \gamma \tag{2.19}$$

in V'.

To establish the existence of a solution, it suffices to verify that (2.19) implies (2.13) and (2.15). First apply (2.19) to points in V_0; this implies (2.13) and, hence, that $u(t)$ belongs to the domain D_0 of the abstract boundary operator ∂_l. Thus we may subtract (2.13) from (2.19) and use (2.6) and (2.7) to obtain (2.15). These computations can be reversed to show (2.13) and (2.15) are equivalent to (2.19). If each of the functions d, g and F is zero, then any solution u of (2.13)–(2.15) is also a solution of (2.2) with $f \equiv 0$; if $h = 0$ then the uniqueness result from Theorem 1 shows $u \equiv 0$. These remarks prove the uniqueness for the linear problem (2.13)–(2.15). ∎

3. Weighted Sobolev Spaces

We wish to apply Theorem 2 in situations where the ellipticity of the operator $\mathscr{L}$ is permitted to go to zero on the boundary of the domain. Thus, it is necessary to consider function spaces of Sobolev type where the norm is weighted in a corresponding manner. We shall show that these spaces and their corresponding trace maps onto boundary values satisfy the assumptions of Section 2 when the degeneration of the ellipticity near each boundary point is of the order of some power of the distance to the boundary. This power is between zero and one and may depend on the boundary point.

Let G be an open bounded and connected subset of Euclidean space $\mathbb{R}^n$ and assume it lies locally on one side of its boundary, ∂G. Suppose ∂G is a C^1-manifold. That is, each point $x \in \partial G$ has an $\mathbb{R}^n$-neighborhood N_x and a C^1 bijection φ_x of N_x onto the cube $Q^n \equiv \{x \in \mathbb{R}^n : |x_j| \leqslant 1\}$ for which $\varphi[N_x \cap G] = Q_+{}^n \equiv \{x \in Q^n : x_n > 0\}$ and $\varphi[N_x \cap \partial G] = Q_0{}^n \equiv \{x \in Q^n : x_n = 0\}$. Let $\rho(x)$ be the distance from $x \in \bar{G}$ to ∂G and $0 \leqslant \alpha < 1$. We first consider the space $W(\alpha)$ obtained by completing $C^1(\bar{G})$ with the norm

$$\| u \|_{W(\alpha)} \equiv \left\{ \int_G (| u(x)|^2 + \rho^\alpha(x) \, | \nabla u(x)|^2) \, dx \right\}^{1/2}.$$

Here ∇u denotes the *gradient* of u on G. This generalized Sobolev space is described in [4, 9]; there it is shown that the embedding $W(\alpha) \hookrightarrow L^2(G)$ is compact and the trace operator γ: $W(\alpha) \to L^2(\partial G)$ is continuous.

Assume we are given a pair of functions

$$\begin{gathered} c(\cdot),\, k(\cdot) \in L^1(G), \qquad c(x) \geqslant 0 \quad \text{and} \quad k(x) \geqslant \epsilon\rho^\alpha(x) \\ \text{a.e. on } G, \quad \text{and} \quad c(\cdot) \text{ is non-zero in } L^1(G) \end{gathered} \tag{3.1}$$

for some $\epsilon > 0$. We define V to be the completion of $C^1(\bar{G})$ with the norm

$$\| u \|_V = \left\{ \int_G (c(x) \mid u(x)|^2 + k(x) \mid \nabla u(x)|^2)\, dx \right\}^{1/2}. \tag{3.2}$$

LEMMA 1. *$V \subset W(\alpha)$ and the embedding is continuous.*

Proof. The continuity of the embedding is not lost by letting $c(\cdot)$ and $k(\cdot)$ be larger, so it suffices to prove the result for the case of $c(\cdot) \in L^\infty(G)$ and $k(x) = \epsilon\rho^\alpha(x)$. With these assumptions, $\| \cdot \|_V$ is a continuous norm on $W(\alpha)$ and satisfies

$$\| u \|_V^2 \geqslant \epsilon \int_G \rho^\alpha(x) \mid \nabla u(x)|^2\, dx, \qquad u \in W(\alpha).$$

Suppose $V \hookrightarrow W(\alpha)$ is not continuous. Then there is a sequence $\{v_n\}$ in $W(\alpha)$ such that $\| v_n \|_V \to 0$ and $\| v_n \|_{W(\alpha)} = 1$ for $n \geqslant 1$. $W(\alpha)$ is weakly compact and the embedding $W(\alpha) \hookrightarrow L^2(G)$ is compact, so by passing to a subsequence (again denoted by $\{v_n\}$) we have weak-lim $v_n = v$ in $W(\alpha)$, hence, in V, and strong-lim $v_n = v$ in $L^2(G)$. Since $\| \cdot \|_V$ is weakly lower semicontinuous we have $\| v \|_V \leqslant \liminf \| v_n \|_V = 0$. This shows $v = 0$ so $v_n \to 0$ in $L^2(G)$. The above implies that $v_n \to 0$ in $W(\alpha)$, a contradiction. ∎

Consider hereafter the restriction of the trace operator from $W(\alpha)$ to V; Lemma 1 shows that γ: $V \to L^2(\partial G)$ is continuous. Define V_0 to be the closure in V of the subspace $C_0^\infty(G)$ of test functions on G. We clearly have $V_0 \subset \ker(\gamma)$, the kernel of γ, but we need the equality $V_0 = \ker(\gamma)$ to apply Theorem 2. Thus we shall seek conditions on V which imply $V_0 \supset \ker(\gamma)$.

We first consider the special case of the half-space, $G = Q_+^n$; let $u \in \ker(\gamma)$ with the support of u contained inside Q; our objective is to prove $u \in V_0$. Each $x \in Q_+^n$ is denoted by $x = (y, x_n)$ with $y \in Q^{n-1}$ and $0 < x_n < 1$; set $k(x) = k(y, x_n)$. For integer $j \geqslant 1$, choose the function $\theta_j \in C^1(\mathbb{R})$ to satisfy $\theta_j(s) = 0$, $s \leqslant 1/j$, $\theta_j(s) = 1$, $s \geqslant 2/j$, and $0 \leqslant \theta_j'(s) \leqslant 2j$. Since the product $\theta_j(x_n)\, u(y, x_n)$ has support in Q_+^n it follows by a standard mollifier approximation that $\theta_j u \in V_0$. Thus it suffices to show

$$\lim_{j\to\infty}(\theta_j u) = u \qquad \text{in} \quad V.$$

Since $c^{1/2}u \in L^2(Q_+{}^n)$ we obtain

$$\theta_j(c^{1/2}u) = c^{1/2}(\theta_j u) \to c^{1/2}u \qquad \text{in} \quad L^2(Q_+{}^n)$$

by dominated convergence. Similarly, for $1 \leqslant i \leqslant n-1$,

$$k^{1/2} \frac{\partial}{\partial x_i}(\theta_j u) = \theta_j k^{1/2} \frac{\partial u}{\partial x_i} \to k^{1/2} \frac{\partial u}{\partial x_i} \qquad \text{in} \quad L^2(Q_+{}^n).$$

Also we have

$$k^{1/2} \frac{\partial}{\partial x_n}(\theta_j u) = k^{1/2}\theta_j' u + k^{1/2}\theta_j \frac{\partial u}{\partial x_n},$$

and

$$k^{1/2}\theta_j \frac{\partial u}{\partial x_n} \to k^{1/2} \frac{\partial u}{\partial x_n} \qquad \text{in} \quad L^2(Q_+{}^n),$$

as before, so it suffices to show

$$k^{1/2}\theta_j' u \to 0 \qquad \text{in} \quad L^2(Q_+{}^n). \tag{3.3}$$

Since $\gamma(0) = 0$ we obtain for $(y, x_n) \in Q_+{}^n$

$$u(y, x_n) = \int_0^{x_n} \frac{\partial u(y, t)}{\partial x_n}\, dt = \int_0^{x_n} \frac{1}{k^{1/2}(y, t)}\, k^{1/2}(y, t) \frac{\partial u(y, t)}{\partial x_n}\, dt,$$

and the Cauchy-Schwartz inequality gives

$$| u(y, x_n)|^2 \leqslant \int_0^{x_n} \frac{dt}{k(y, t)} \int_0^{x_n} k(y, t) \mid \partial_n u(y, t)|^2\, dt. \tag{3.4}$$

Setting $\psi(y, x_n) \equiv (\theta_j'(x_n))^2\, k(y, x_n) \int_0^{x_n} dt/k(y, t)$, we multiply (3.4) by $(\theta_j')^2 k$ and integrate to obtain

$$\int_0^{2/j} (\theta_j'(x_n))^2\, k(y, x_n) \mid u(y, x_n)|^2\, dx_n$$

$$\leqslant \int_0^{2/j} \psi(y, x_n) \int_0^{x_n} k(y, t) \mid \partial_n u(y, t)|^2\, dt\, dx_n\,.$$

Interchanging the order of integration shows this last term equals

$$\int_0^{2/j} \int_t^{2/j} \psi(y, x_n)\, dx_n (k(y, t) \mid \partial_n u(y, t)|^2)\, dt$$

and thus we have

$$\int_0^1 (\theta_j'(x_n))^2\, k(y, x_n) \mid u(y, x_n)|^2\, dx_n$$
$$\leqslant \sup_{\xi\in Q^{n-1}} \int_0^{2/j} \psi(\xi, x_n)\, dx_n \cdot \int_0^{2/j} k(y, t) \mid \partial_n u(y, t)|^2\, dt.$$

Integrating this inequality over Q^{n-1} gives the estimate

$$\int_{Q_+^n} (\theta_j')^2\, k(x) \mid u(x)|^2\, dx$$
$$\leqslant \sup_{\xi\in Q^{n-1}} \int_0^{2/j} \psi(\xi, x_n)\, dx_n \cdot \int_{Q^{n-1}+[0,2/j]} k(x) \mid \partial_n u(x)|^2\, dx.$$

Thus, for (3.3) to hold it is sufficient to have

$$\sup_{\substack{y\in Q^{n-1}\\ j\geqslant 0}} \int_0^{2/j} (\theta_j'(x_n))^2\, k(y, x_n) \int_0^{x_n} \frac{dt}{k(y, t)}\, dx_n < \infty.$$

But we note that $\theta_j' \leqslant 2j$ so we need only to show that for some constant K

$$\int_0^{2/j} k(y, x_n) \int_0^{x_n} \frac{dt}{k(y, t)}\, dx_n \leqslant K/j^2, \qquad j \geqslant 1, \quad y \in Q^{n-1}. \tag{3.5}$$

A sufficient condition for (3.5) can be described as follows. Suppose there are a pair of positive constants c_0, c_1 and a function $\alpha(\cdot)$: $Q^{n-1} \to \mathbb{R}$ with $0 \leqslant \alpha(y) < 1$ for each $y \in Q^{n-1}$ such that

$$c_0 t^{\alpha(y)} \leqslant k(y, t) \leqslant c_1 t^{\alpha(y)}, \qquad 0 \leqslant t \leqslant 1.$$

Then the left side of (3.5) is bounded by $2c_1/c_0(1 - \alpha(y))j^2$, so for (3.5) to hold it is sufficient to have

$$0 \leqslant \alpha(y) \leqslant \alpha < 1$$

for all $y \in Q^{n-1}$. The preceding proves $V_0 = \ker \gamma$ in this essential special case. The general situation will now be described.

THEOREM 3. *Let the bounded domain G be given as above and let $0 \leqslant \alpha < 1$. Suppose there is a function $\alpha(\cdot)$ on ∂G for which $0 \leqslant \alpha(s) \leqslant \alpha$ for each $s \in \partial G$. Assume the functions $c(\cdot)$ and $k(\cdot)$ are given and satisfy* (3.1). *Furthermore, suppose there is at each point of ∂G a neighborhood N in $\mathbb{R}^n$ and constants $0 < c(N) < C(N)$ such that*

(i) *for each $x \in N \cap G$ there is a unique $x_0 \in \partial G$ such that $\| x_0 - x \|_{\mathbb{R}^n} = \rho(x)$,* *and*

(ii) *for each* $x \in N \cap G$,

$$c(N) \leqslant k(x)/(\rho(x))^{\alpha(x_0)} \leqslant C(N). \tag{3.6}$$

Define $V_{c,k}$ *to be the Hilbert space obtained by completing* $C^1(\bar{G})$ *in the norm* (3.2). *Then there is a continuous trace map* γ *of* $V_{c,k}$ *into* $L^2(\partial G)$, *determined by* $\gamma(u) = u\,|_G$ *for* $u \in C^1(\bar{G})$; *the kernel of* γ *equals* V_0, *the closure in* V *of* $C_0^\infty(G)$; *and the range of* γ *is dense in* $L^2(\partial G)$.

Proof (continued). By a partition-of-unity and corresponding coordinate transformations the general situation is reduced to the special case above. See [9, pp. 749-750] for the relevant details. Thus we have that γ is defined and continuous and that its kernel is as claimed. In order to prove the claim about the range $Rg(\gamma)$, it suffices to show for the special case $G = Q_+^n$ that $Rg(\gamma) \supset C_0^\infty(Q^{n-1})$. But if $\varphi \in C_0^\infty(Q^{n-1})$ and $\psi \in C_0^\infty(-1, 1)$ satisfies $\psi(0) = 1$, then $\varphi(y)\,\psi(x_n) \equiv u(x)$ $(x = (y, x_n))$ belongs to V and $\gamma(u) = \varphi$. ∎

Remarks. (3) Since ∂G is a C^1-manifold, the condition (i) in Theorem 3 is already true for neighborhoods chosen sufficiently small.

(4) The dual space V_0' is a space of distributions on G; we can identify $V_0 \subset \mathscr{D}'(G)$.

(5) We shall define B to be the range of γ. It suffices for our purposes to note that $B \subset L^2(\partial G) \subset B'$; a more precise description of B can be given e.g., when $\alpha(s) = \alpha$ for all $s \in \partial G$ [4].

4. Examples

We present some applications of the preceding results to a variety of initial-boundary value problems for partial differential equations. The objective is to illustrate various types of problems which can be included so we do not attempt the most general results in any sense. The examples include the elliptic-parabolic equation (1.4) subject to boundary conditions of first, second or third type, a parabolic-pseudoparabolic equation, and a problem with elliptic-parabolic constraints on a submanifold. In the following, the domain G in $\mathbb{R}^n$ and the functions $c(\cdot)$ and $k(\cdot)$ are as given in Theorem 3. The unit outward normal vector on ∂G is denoted by ν.

(a) *Elliptic-parabolic equation.* Let Γ_0 and Γ_1 be disjoint measurable subsets of ∂G whose union equals ∂G. Let $V_1 = V_{c,k}$, V_0 be the closure of $C_0^\infty(G)$ in $V_{c,k}$, and V be the subspace of those $v \in V_1$ (with trace) satisfying $v = 0$ a.e. on Γ_0. Since the trace operator is "local" we can identify $B \subset L^2(\Gamma_1) \subset B'$

where B is just the range of the trace on V. Let $\sigma \in L^\infty(\Gamma_1)$ with $\sigma(s) \geqslant 0$ a.e on Γ_1 and define

$$\mathscr{L}u(v) \equiv \int_G k(x)\, \nabla u(x) \cdot \nabla \overline{v(x)}\, dx + \int_{\Gamma_1} \sigma(s)\, u(s)\, \overline{v(s)}\, ds, \qquad u, v \in V_1 .$$

The corresponding formal operator is given by

$$Lu = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(k(x) \frac{\partial u}{\partial x_j} \right) \in V_0' \subset \mathscr{D}'(G)$$

and the complementary boundary operator in (2.6) is given by

$$\partial_l u = \left(k \frac{\partial u}{\partial \nu} + \sigma u \right) \Big|_{\Gamma_1}$$

for those u sufficiently smooth. Here $\partial u / \partial \nu$ is the directional derivative along the outward normal ν. Let W be the seminorm space consisting of $V_{c,k}$ with the seminorm induced by

$$\mathscr{M}u(v) \equiv \int_G c(x)\, u(x)\, \overline{v(x)}\, dx, \qquad u, v \in W \equiv V_{c,k} .$$

Then V_0 is dense in W so $W_0 = W$, $\partial_m = 0$, and the formal operator is

$$Mu = c(\cdot)\, u(\cdot).$$

Note that $W' = W_0' = \{c^{1/2} v : v \in L^2(G)\}$.

Assume the following data is given:

$$H \in L^2(G), \qquad f \in C^\delta([0, \infty), L^2(G))$$
$$d \in C^{1+\delta}([0, \infty), V_{c,k}),\ g \in C^{1+\delta}([0, \infty), L^2(\Gamma_1)), \qquad 0 < \delta \leqslant 1.$$

Then set $h(x) = c^{1/2}(x)\, H(x) \in W'$ and $F(t) = c^{1/2}(\cdot) f(t)$ for $t \geqslant 0$. From Theorem 2 we obtain existence and uniqueness of a solution to

$$\lim_{t \to 0} \{c^{1/2}(\cdot)\, u(\cdot, t)\} = H(\cdot) \qquad \text{in} \quad L^2(G), \tag{4.1}$$

$$\frac{\partial}{\partial t} (c(x)\, u(x, t)) - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(k(x) \frac{\partial u(x, t)}{\partial x_j} \right) = c^{1/2}(x) f(x, t) \qquad \text{in} \quad \mathscr{D}'(G) \tag{4.2}$$

$$u(\cdot, t) = d(\cdot, t) \qquad \text{in} \quad L^2(\Gamma_0), \tag{4.3}$$

and

$$k(\cdot) \frac{\partial u(\cdot, t)}{\partial \nu} + \sigma(\cdot)\, u(\cdot, t) = g(\cdot, t) \qquad \text{in} \quad L^2(\Gamma_1) \tag{4.4}$$

Note that (4.3) is the non-homogeneous boundary condition of first type and (4.4) is of second type ($\sigma(s) = 0$) or third type ($\sigma(s) > 0$). See [6, 8, 14] for related results.

(b) *Parabolic-pseudoparabolic equation.* Let the space V_1, V, V_0, B and the operator $\mathscr{L}$ be given as in (a). Let $m \in L^1(G)$ satisfy

$$0 \leqslant m(x) \leqslant K_1 k(x), \quad \text{a.e. } x \in G; \tag{4.5}$$

set $W = V_{c,k}$ and define

$$\mathscr{M}u(v) \equiv \int_G (c(x)\, u(x)\, \overline{v(x)} + m(x)\, \nabla u(x) \cdot \overline{\nabla v(x)})\, dx, \qquad u, v \in W.$$

The formal operator M: $W \to W_0'$ is

$$Mu = cu - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(m(x)\frac{\partial u}{\partial x_j}\right) \in \mathscr{D}'(G)$$

and the complementary boundary operator is given by

$$\partial_m u = m \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_1}$$

on smooth functions. Assume we are given the following data:

$$H \in L^2(G),\ d \in C^{1+\delta}([0, \infty), V_{c,k}),\ g \in C^{1+\delta}([0, \infty), L^2(\Gamma_1)),$$

and

$$f_j \in C^{\delta}([0, \infty), L^2(G)), \qquad 0 \leqslant j \leqslant n, \quad 0 < \delta \leqslant 1.$$

Set $h(\cdot) = c^{1/2}(\cdot)H \in W'$ and

$$F(t)(v) \equiv \int_G \Big(c^{1/2}(x)\, f(x, t)\, \overline{v(x)}$$

$$+ \sum_{j=1}^{n} m^{1/2}(x)\, f_j(x, t)\, \frac{\partial \overline{v(x)}}{\partial x_j}\, dx, \qquad t \geqslant 0, \quad v \in W.$$

Then Theorem 2 applies to the problem consisting of the initial conditions (4.1) and $\partial_m u(\cdot, 0) = 0$ in $W_0{}^{\perp}$, the equation

$$\frac{\partial}{\partial t}\left(c(x)\, u(x, t) - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(m(x)\frac{\partial u}{\partial x_j}\right)\right) - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(k(x)\frac{\partial u}{\partial x_j}\right)$$

$$= c^{1/2}(x)\, f(x, t) - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(m^{1/2}(x)\, f_j(x, t)) \qquad \text{in } \mathscr{D}'(G), \tag{4.6}$$

and the boundary conditions (4.3) and

$$\frac{\partial}{\partial t}\left(m(\cdot)\,\frac{\partial u(\cdot,t)}{\partial \nu}\right)+k(\cdot)\,\frac{\partial u(\cdot,t)}{\partial \nu}+\sigma(\cdot)\,u(\cdot,t)=g(\cdot,t) \qquad \text{in} \quad B'. \tag{4.7}$$

Note that the initial condition (4.1) is attained in a stronger seminorm in (b) than was so in (a). Also, the boundary condition (4.4) contains only terms belonging to $L^2(\Gamma_1)$ whereas the terms in (4.7) belong to the larger space B'. See [3, 13] for related results and applications.

Remarks. (6) By use of the change of variable used in Theorem 1 it follows we may weaken the assumptions on $k(\cdot)$ and $m(\cdot)$ in this example. Specifically we can drop (4.5) and replace (3.6) by a similar estimate with $m+k$ substituted for k. If $m(x)\geqslant 0$, this is a weaker hypothesis.

(7) If we have the estimate

$$0\leqslant m(x)\leqslant K_2\rho(x)^{\alpha_1}, \qquad x\in G$$

for some $\alpha_1\geqslant 1$, then $W_0=W$ (see [4]) and $\partial_m=0$. Then (4.4) and (4.7) are equivalent.

(c) *Singular Surface.* Suppose the domain G and the partition Γ_0, Γ_1 of ∂G are as above; let $\Gamma\subset\Gamma_1$ be an $n-1$ dimensional C^1-manifold which for simplicity is flat. That is, $\Gamma\subset\mathbb{R}^{n-1}$; assume $\partial\Gamma$ is a C^1 manifold of dimension $n-2$ and Γ lies locally on one side of $\partial\Gamma$. Denote by ν_Γ the unit outward normal to Γ along $\partial\Gamma$. Suppose we are given a pair of non-negative functions $c\in L^\infty(\Gamma)$, $k\in L^1(\Gamma)$ and k satisfies estimates on Γ analogous to (3.6). Let $V_k(\Gamma)$ be the Hilbert space obtained by completing $C^1(\bar{\Gamma})$ in the norm

$$\|w\|_\Gamma=\int_\Gamma(|w(t)|^2+k(t)\,|\nabla_0 w(t)|^2)\,dt^{1/2}$$

where ∇_0 is the gradient in the $n-1$ variables on Γ. Thus, Theorem 3 describes the trace of $V_k(\Gamma)$ into $L^2(\partial\Gamma)$.

For our Hilbert spaces we take $V_1\equiv\{v\in H^1(G): v\,|_\Gamma\in V_k(\Gamma)\}$ with the norm $(\|v\|^2_{H^1(G)}+\|v\|_\Gamma{}^2)^{1/2}$ and $V\equiv\{v\in V: v\,|_{\Gamma_0}=0\}$. The closure in V of $C_0^\infty(G)$ is the usual Sobolev space $H_0^1(G)$, and the range of the trace operator on V is given by $B\equiv\{w\in H^{1/2}(\Gamma_1): w\,|_\Gamma\in V_k(\Gamma)\}$.

Let W be the space V_1 with the scalar-product

$$\mathscr{M}u(v)\equiv\int_G u(x)\,\overline{v(x)}\,dx+\int_\Gamma c(s)\,u(s)\,\overline{v(s)}\,ds, \qquad u,v\in W,$$

and corresponding operator $\mathscr{M}$: $W \to W'$. Then we have $W_0' = L^2(G)$, $Mu = u$ on G and $\partial_m u = cu$ on Γ. Finally, we define

$$\mathscr{L}u(v) \equiv \int_G \nabla u \cdot \nabla \bar{v}\, dx + \int_\Gamma k(s)\, \nabla_0 u \cdot \nabla_0 \bar{v}\, ds, \qquad u, v \in V_1 .$$

The formal operator is given by the Laplace operator, $Lu = -\Delta u \in H^{-1}(G)$, and the complementary boundary operator by

$$\partial_l u(w) = \int_{\Gamma_1} \left(\frac{\partial u}{\partial \nu}\, \bar{w} + k(s)\, \nabla_0 u \cdot \nabla_0 \bar{w}\right) ds, \qquad u \in D_0 , \qquad w \in B$$

where $\partial u/\partial \nu \in H^{-1/2}(\Gamma_1) \equiv H^{1/2}(\Gamma_1)'$ is a distribution on Γ_1. (Recall that $C_0^\infty(\Gamma_1)$ is dense in $H^{1/2}(\Gamma_1)$; see [7, p. 60].) The function k is extended as zero from Γ to Γ_1.

An essential point of this example is the characterization of the solution of the equation $\partial_l u = g$ in B' (cf. (2.15)) so we do the computation in the simpler (stationary) case. Thus let $g_1 \in L^2(\Gamma_1)$ and $g_0 \in L^2(\partial\Gamma)$ be given and define $g \in B'$ by

$$g(w) \equiv \int_{\Gamma_1} g_1(s)\, \overline{w(s)}\, ds + \int_{\partial\Gamma} g_0(\xi)\, \overline{w(\xi)}\, d\xi, \qquad w \in B. \tag{4.8}$$

(We denote by "$w(\xi)$" the trace on $\partial\Gamma$ from $V_k(\Gamma)$.) Consider a solution u of

$$\partial_l u(w) = g(w), \qquad w \in B. \tag{4.9}$$

Since (4.9) holds for all $w \in C_0^\infty(\Gamma_1)$ we obtain

$$\frac{\partial u}{\partial \nu} - \nabla_0 \cdot k\, \nabla_0 u = g_1 \qquad \text{in} \quad H^{-1/2}(\Gamma_1), \tag{4.10}$$

where ∇_0 is the divergence on $\Gamma \subset \mathbb{R}^{n-1}$. When this is substituted in (4.9) there follows

$$\int_{\partial\Gamma} k(\xi)\, \frac{\partial u(\xi)}{\partial \nu_\Gamma}\, \overline{w(\xi)}\, d\xi = \int_{\partial\Gamma} g_0(\xi)\, \overline{w(\xi)}\, d\xi, \qquad w \in B.$$

Thus we can show that (4.9) is equivalent to (4.10) and

$$k\, \frac{\partial u}{\partial \nu_\Gamma} = g_0 \qquad \text{in} \quad L^2(\partial\Gamma) \tag{4.11}$$

in the same sense that (2.9) is equivalent to (2.10) and (2.11). The operator on the left side of (4.11) is the complementary boundary operator constructed from the operator ∂_l and the trace of $V_k(\Gamma)$ into $L^2(\partial\Gamma)$.

Assume the following data is given:

$$H_1 \in L^2(G), \qquad H_2 \in L^2(\Gamma),$$
$$F \in C^\delta([0, \infty), L^2(G)), \qquad 0 < \delta \leqslant 1,$$
$$d \in C^{1+\delta}([0, \infty), V_1),$$
$$g_1 \in C^{1+\delta}([0, \infty), L^2(\Gamma_1)), \; g_0 \in C^{1+\delta}([0, \infty), L^2(\partial\Gamma)).$$

Then define $h \in W'$ and $g \in C^{1+\delta}([0, \infty), B')$ by

$$h(w) \equiv \int_G H_1(x)\, \overline{w(x)}\, dx + \int_\Gamma c^{1/2}(s)\, H_2(s)\, \overline{w(s)}\, ds, \qquad w \in V_1,$$

$$g(t)(w) \equiv \int_{\Gamma_1} g_1(s, t)\, \overline{w(s)}\, ds + \int_{\partial\Gamma} g_0(\xi, t)\, \overline{w(\xi)}\, d\xi, \qquad w \in B, \quad t \geqslant 0.$$

Recall that the embedding $L^2(\Gamma_1) \hookrightarrow B$ and the trace map $B \to L^2(\partial\Gamma)$ are continuous. Theorem 2 shows there exists a unique solution to the problem (2.12)–(2.15) with the data given above. Thus, we have shown that the following problem is well-posed:

$$\lim_{t\to 0} u(x, t) = H_1(x) \quad \text{in } L^2(G), \qquad \lim_{t\to 0} c^{1/2}(s)\, u(s, t) = H_2(s) \quad \text{in } L^2(\Gamma), \tag{4.12}$$

$$\frac{\partial}{\partial t} u(x, t) - \Delta u(x, t) = F(x, t) \qquad \text{in} \quad L^2(G), \qquad t > 0, \tag{4.13}$$

$$u(s, t) = d(s, t) \qquad \text{in} \quad L^2(\Gamma_0), \qquad t > 0, \tag{4.14}$$

$$\frac{\partial}{\partial t}(c(s)\, u(s, t)) + \frac{\partial u(s, t)}{\partial \nu} - \nabla_0 \cdot (k(s)\, \nabla_0 u(s, t))$$
$$= g_1(s, t) \qquad \text{in} \quad H^{-1/2}(\Gamma_1), \quad t > 0, \tag{4.15a}$$

$$k(\xi)\, \frac{\partial u(\xi, t)}{\partial \nu_\Gamma} = g_0(\xi) \qquad \text{in} \quad L^2(\partial\Gamma), \qquad t > 0. \tag{4.15.b}$$

In the same manner one can handle similar problems where the submanifold Γ may extend into the interior of the region G. (See [14] for the details.) Such problems arise from models of diffusion in a region G in which the submanifold Γ approximates a narrow fracture of width $w(s)$ at each $s \in \Gamma$ [2]. Then the coefficients $c(s)$ and $k(s)$ both contain a factor of $w(s)$ and therefore must be allowed to vanish as $s \to \partial\Gamma$. Thus the degeneracy arises from the geometry of the problem as well as (possibly) the properties of the material.

Remark. (8) When $c(s) = cw(s)$ and $k(s) = kw(s)$ for c, $k > 0$ as above, we have shown the characterization of the solution includes the boundary condition (4.15.b) so long as the width satisfies (3.6) on Γ, i.e., does not vanish too quickly along $\partial\Gamma$. Conversely, if w satisfies the estimate

$$0 \leqslant w(s) \leqslant K_2 \operatorname{dist}(s, \partial\Gamma)^{\alpha_1}, \qquad s \in \Gamma$$

for some $\alpha_1 \geqslant 1$, then $C_0^\infty(\Gamma_1)$ is dense in B [4] and (4.9) is equivalent to (4.10). (Cf. Remark 7.) Then (4.15.b) is deleted from the statement of the problem. This important distinction of the two cases can be expressed locally on $\partial\Gamma$ and seems to have not been observed before.

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