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A Free-Boundary Problem for a Degenerate Parabolic System

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1. INTRODUCTION

We shall be concerned with the first initial-boundary-value problem for non-negative solutions of a system of nonlinear partial differential equations of the form

$$\begin{aligned} &\frac{\partial}{\partial t}\,\alpha(\theta(x,t)) + h(\theta(x,t) - \varphi(x,t)) \ni f_1(x,t), \\ &\frac{\partial}{\partial t}\,\beta(\varphi(x,t)) - \Delta\varphi(x,t) + h(\varphi(x,t) - \theta(x,t)) \ni f_2(x,t) \end{aligned} \qquad (1.1)$$

and a related free-boundary problem of Stefan type. Here Δ denotes the Laplacean in the spatial variable $x \in \mathbb{R}^m$, $h > 0$, and the pair α, β are maximal monotone graphs in $\mathbb{R} \times \mathbb{R}$. If the first equation were to contain the term "$-k\Delta\theta(x,t)$" with $k > 0$, then the system (1.1) would be parabolic. The situation we consider here with $k = 0$ is accordingly a degenerate parabolic system.

Although the system (1.1) with the (possibly multi-valued) nonlinear monotone graphs α, β is of mathematical interest in its own right, we present in Section 2 an extensive discussion of how such a system arises as a model of heat conduction in a composite material consisting of two components in which a change of phase occurs in the second component. This model is

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described by (1.1) with β obtained in the special form $\beta(x) = bx + LH(x)$, where $b > 0$, L is the latent heat of fusion and $H(\cdot)$ is the multi-valued Heaviside step function. Certain models of diffusion through fractured porous media lead to the same system.

Our results on (1.1) are organized as follows. In Section 3 we prove that the first initial-boundary-value problem for (1.1) is well-posed when the data satisfy certain integrability conditions and β is defined everywhere on $\mathbb{R}$. If the data are non-negative then the solution is likewise non-negative; this property is essential for the model problem discussed in Section 2. We make extensive use of the theory of maximal monotone operators in Hilbert space to which we refer to [2, 3].

Certain properties of the solution are obtained when we restrict attention to the case where α has a lower linear bound and $\beta = bI + LH$ as in the model problem. In Section 4 we show that if the data are essentially bounded then the solution of (1.1) is essentially bounded. Additional conditions on the data are shown to imply that θ and φ are continuous. In order to obtain these regularity results we found it very useful to treat the problem as an equation of evolution rather than to have formulated it as a variational inequality.

In Section 5 we exhibit an explicit lower bound on the first component of the solution of (1.1). This implies that the set of points where this function is positive (the positivity set) is non-decreasing with time. Finally, we show that the positivity set of the first component contains that of the second component, and an example is given to show that this containment may be proper.

2. Diffusion in Heterogeneous Media

We begin with a mathematical description of diffusion processes within a medium consisting of two components. A fundamental assumption is that the first component occurs in small isolated parts that are suspended in the second component. This situation arises in thermal conduction through rocky soil, since the rocks are isolated within the soil. It also occurs in the diffusion of liquid or gas through a porous media that has been fractured, since the blocks of the medium are isolated from one another by the system of fissures. Next we shall formulate a free-boundary problem of Stefan type that results from a change of phase in the second component of the medium. This arises in the model of heat conduction through the moisture in rocky soil since the soil moisture may freeze or thaw with a corresponding release of latent heat; there is no moisture in the rocks. If we consider diffusion in a fractured medium in which the system of fissures is only partially saturated then we can think of the fissures as containing holes in which a certain amount (per volume) of the liquid or gas is trapped and can no longer take part in the diffusion. Such a diffusion process is formally equivalent to the preceding heat conduction problem. Finally, we give a weak formulation of this one-phase Stefan problem for a two-component medium.

Consider the conduction of heat through a heterogeneous medium $G \subset \mathbb{R}^m$ consisting of two components. As our first model for such a process we take the system of equations in the region $\Omega \equiv G \times (0, \infty)$,

$$
\begin{aligned}
a\frac{\partial\theta}{\partial t} - k\Delta\theta + h(\theta - \varphi) &= f_1, \\
b\frac{\partial\varphi}{\partial t} - \Delta\varphi + h(\varphi - \theta) &= f_2,
\end{aligned}
\tag{2.1}
$$

where θ and φ are the temperatures in the first and second components, respectively. Each is a function of position $x \in G$ and time $t > 0$ and is obtained at a point x by averaging the temperature of the corresponding component in a neighborhood which contains a sufficiently large number of pieces of both components. The constants a, b are specific heats of the respective components, k is the conductivity of the first, the conductivity of the second component is normalized to unity, and the positive number h is related to the surface area common to the two components. Thus h is a measure of the homogeneity of the material. The system (2.1) is just a pair of classical heat conduction equations together with a linear coupling to model the simplest exchange between components. Our basic assumption that the first component occurs in small parts isolated by the second component implies that $k = 0$ in (2.1). That is, the particles of the first component may store heat ($a > 0$) or may exchange with the surrounding second component ($h > 0$), but they cannot pass heat directly to other first-component particles ($k = 0$). This is the sense in which the system (2.1) is *degenerate parabolic*.

Suppose there is a solid–liquid phase change in the second component at the temperature $\varphi = 0$. We consider here the (one-phase) situation wherein $\varphi \geqslant 0$ everywhere. The region Ω is separated into a conducting region Ω_+ where $\varphi > 0$ and a non-conducting region Ω_0 where $\varphi = 0$; these correspond to completely melted and partially frozen parts, respectively. We need not assume that Ω_0 consists exclusively of ice but only that it is a mixture of ice and water in thermal equilibrium at the melting temperature. At each point (x, t) of Ω we introduce the fraction of water, $\xi(x, t)$; note that $\xi \in H(\varphi)$ in Ω, where $H(\cdot)$ is the maximal monotone Heaviside graph given by $H(s) = 1$ for $s > 0$, $H(0) = [0, 1]$, and $H(s) = 0$ for $s < 0$. The two regions are separated at time t by an interface $S(t)$. If we let $\mathbf{n}$ be the unit normal on

$S(t)$ directed towards Ω_0 and V be the speed of $S(t)$ along $\mathbf{n}$, then we obtain the condition

$$\frac{\partial \varphi}{\partial n} = -LV(1-\xi) \qquad \text{on} \quad S(t), \tag{2.2}$$

where $\partial\varphi/\partial n = \nabla_x \cdot \mathbf{n}$ is the heat flux across $S(t)$ and $(1-\xi)$ is the fraction of ice. Moreover, if $\mathbf{N} = (N_1, N_2, ..., N_m, N_t)$ denotes the unit normal on the interface $S = \bigcup\{(S(t), t)\}$, we find that (2.2) is equivalent to

$$\nabla_x \varphi \cdot (N_1, ..., N_m) = LN_t(1-\xi). \tag{2.3}$$

Each of (2.2) and (2.3) is called the *interface* or *free-boundary condition.*

It is worthwhile to recall the simple experiment in which one applies a uniform heat source of intensity F to a unit volume of ice at temperature $\varphi = 0$. The temperature remains at zero until L units of heat have been added. During this period there is a fraction ξ of water coexisting with the ice and ξ increases at a constant rate F/L. When all the ice has melted, $\xi = 1$ and the temperature φ begins to rise at the rate F/b. The constants L and b are the latent heat and specific heat, respectively. We can summarize the above by stating that the rate of increase of the internal energy or *enthalpy* $v = b\varphi + L\xi$ is given by F. Later we shall see that not only is enthalpy the natural variable to determine the state of the process but that it is mathematically the proper variable by which to describe the evolution of the process.

We can now formulate our problem. With the notation above we seek a triple of non-negative real-valued functions θ, φ, ξ on Ω which satisfy the following:

$$a\frac{\partial \theta}{\partial t} + h(\theta - \varphi) = f_1 \tag{2.4}$$

and

$$\xi \in H(\varphi), \qquad \text{in} \quad \Omega, \tag{2.5}$$

$$b\frac{\partial \varphi}{\partial t} - \Delta\varphi + h(\varphi - \theta) = f_2 \qquad \text{in} \quad \Omega_+, \tag{2.6}$$

$$L\frac{\partial \xi}{\partial t} = f_2 + h\theta \qquad \text{in} \quad \Omega_0, \tag{2.7}$$

$$\frac{\partial \varphi}{\partial n} + LV(1-\xi) = 0 \qquad \text{on} \quad S, \tag{2.8}$$

$$\varphi = 0 \qquad \text{on} \quad \partial G \times (0, \infty), \tag{2.9}$$

$$\theta(x, 0) = \theta_0(x), \qquad \varphi(x, 0) = \varphi_0(x),$$
$$\xi(x, 0) = \xi_0(x) \qquad \text{on} \quad G. \tag{2.10}$$

The data consist of the strictly positive numbers a, b, h, L and the non-negative functions f_1, f_2 on Ω and θ_0, φ_0, ξ_0 on G for which we assume $\xi_0(x) \in H(\varphi_0(x))$ for all $x \in G$. As before, we have set $\Omega_+ = \{(x, t) \in \Omega : \varphi(x, t) > 0\}$ and $\Omega_0 = \{(x, t) \in \Omega : \varphi(x, t) = 0\}$. The unknown interface S between Ω_+ and Ω_0 is the primary difficulty in the problem.

It is approrpiate to obtain a weak formulation of the problem (2.4)–(2.10). This is necessary even with smooth data because the free boundary S may vary in a discontinuous manner and it is also convenient because it casts the problem into the form of an evolution equation in Hilbert space. Thus we first compute $\partial v/\partial t - \Delta\varphi$ in the sense of distributions on Ω. For each test function $\psi \in C_0^\infty(\Omega)$ we obtain

$$\left\langle \frac{\partial v}{\partial t} - \Delta\varphi, \psi \right\rangle = -\int_\Omega \left(v \frac{\partial \psi}{\partial t} + \varphi(\Delta\psi) \right) =$$

$$\int_{\Omega_+} \left(\frac{\partial}{\partial t} (b\varphi + L) - \Delta\varphi \right) \psi$$

$$+ \int_{\partial\Omega_+} \{(\psi\, \nabla\varphi - \varphi\, \nabla\psi) \cdot (N_1, \dots, N_m) - (b\varphi + L)\, N_t \psi\}$$

$$+ \int_{\Omega_0} L \frac{\partial \xi}{\partial t} \psi - \int_{\partial\Omega_0} L\xi N_t \psi$$

$$= \int_{\Omega_+} \left(b \frac{\partial \varphi}{\partial t} - \Delta\varphi \right) \psi + \int_{\Omega_0} \left(L \frac{\partial \xi}{\partial t} \right) \psi$$

$$+ \int_S (\nabla\varphi \cdot (N_1, \dots, N_m) + L(\xi - 1)\, N_t)\, \psi.$$

We have assumed that the interface S and the restrictions of φ and ξ to Ω_+ and to Ω_0 are sufficiently smooth to apply Gauss' theorem. This calculation shows that

$$\frac{\partial v}{\partial t} - \Delta\varphi + h(\varphi - \theta) = f_2$$

in $\mathscr{D}'(\Omega)$ if and only if (2.6), (2.7) and (2.8) hold. From these remarks we obtain the following weak or generalized formulation of the two-component Stefan problem: given $T > 0$ and the non-negative functions f_1, f_2 on Ω and

θ_0, φ_0, ξ_0 on G with $\xi_0 \in H(\varphi_0)$, find a non-negative triple of functions which satisfy

$$\theta \varepsilon H^1(0, T; L^2(G)), \quad \varphi \in L^2(0, T; H_0^1(G)), \quad v \in H^1(0, T; H^{-1}(G)), \tag{2.11}$$

$$a \frac{d\theta}{dt} + h(\theta - \varphi) = f_1 \qquad \text{in} \quad L^2(0, T; L^2(G)), \tag{2.12}$$

$$\frac{dv}{dt} - \Delta\varphi + h(\varphi - \theta) = f_2 \qquad \text{in} \quad L^2(0, T; H^{-1}(G)), \tag{2.13}$$

$$\varphi = (bI + LH)^{-1}(v) \qquad \text{in} \quad L^2(0, T; H_0^1(G)), \tag{2.14}$$

$$\theta(0) = \theta_0 \quad \text{and} \quad v(0) = b\varphi_0 + L\xi_0 \qquad \text{in} \quad L^2(G). \tag{2.15}$$

Certainly a smooth solution of (2.11)–(2.15) for which the level set S is a smooth manifold necessarily satisfies (2.4)–(2.10).

Remarks. The condition $b > 0$ arises later in the discussion of properties of solutions so we briefly indicate the significance of this assumption. The constant b is a measure of the storage capacity of the second component and it depends on the type of material and also the percentage present in the second component. Similarly, the constant L is determined by the type and percentage of this material in the second component of the medium. The essential interest here is in the change of phase phenomenon so we are concerned with the case of a sufficient percentage of the second component material being present to permit $L > 0$. The corresponding physically significant case is that of $b > 0$; otherwise we would be considering the unlikely case of a material with positive latent heat of fusion but with null heat capacity. Neventheless, most of our results to follow are obtained from the weaker assumptions that $L \geqslant 0$ and $b \geqslant 0$.

The type of the problem we have called degenerate parabolic. In the system of partial differential equations (2.1) with $k = 0$ it is of interest to consider the case of $b = 0$ [1, 7]; one can then reduce it to the single partial differential equation

$$\frac{\partial}{\partial t}(a\varphi - (a/h)\Delta\varphi) - \Delta\varphi = f_1 + f_2 + (a/h)\frac{\partial}{\partial t} f_2,$$

which is of *pseudo-parabolic* type [6, 22]. This is distinctly not the case for the free-boundary problem considered here. An elimination of θ from (2.12), (2.13) leads to the evolution equation

$$(a/h)\frac{d^2 v}{dt^2} + \frac{d}{dt}(v + a\varphi - (a/h)\Delta\varphi) - \Delta\varphi = f_1 + f_2 + (a/h)\frac{df_2}{dt}. \tag{2.16}$$

The pairs of equations (2.14), (2.16) gives an equation for φ which is of second order in time-derivatives, definitely not pseudo-parabolic unless both $b=0$ and $L=0$. Thus, even in the case of $b=0$ where the local description of the problem contains a pseudo-parabolic equation (cf. (2.4) and (2.6) in Ω_+), the free-boundary problem with $L>0$ is not of this type. Problems where the phase is determined by the first component can be pseudoparabolic; see [10, 18].

3. Existence and Uniqueness of the Weak Solution

We shall prove that the weak formulation of the Stefan problem (2.11)–(2.15) is well-posed. This will be achieved by showing that the problem corresponds to an evolution equation whose solutions are determined by a nonlinear semigroup of contractions and that the generator of this semigroup is a subgradient operator.

The existence and uniqueness of a generalized solution of the Stefan problem is contained in the following.

Theorem 1. *Let α and β be maximal monotone graphs on $\mathbb{R}\times\mathbb{R}$ and let j and k be proper convex lower-semi-continuous functions whose subgradients are given by $\partial j=\alpha^{-1}$ and $\partial k=\beta^{-1}$. Assume*

$$u_0\in L^2(G),\quad j(u_0)\in L^1(G),\quad v_0\in L^1(G)\cap H^{-1}(G),\quad k(v_0)\in L^1(G),$$
$$f_1\in L^2(0,T;L^2(G)),\qquad f_2\in L^2(0,T;H^{-1}(G)),$$

and that the domain of β is equal to $\mathbb{R}$. Then there exists a unique quadruple of functions which satisfy

$$\begin{aligned} &u\in H^1(0,T;L^2(G)), && v\in H^1(0,T;H^{-1}(G)),\\ &\theta\in L^2(0,T;L^2(G)), && \varphi\in L^2(0,T;H_0^1(G)), \end{aligned}\tag{3.1}$$

$$\frac{du}{dt}+h(\theta-\varphi)=f_1\qquad \text{in}\quad L^2(0,T;L^2(G)),\tag{3.2}$$

$$\frac{dv}{dt}-\Delta\varphi+h(\varphi-\theta)=f_2\qquad \textit{in}\quad L^2(0,T;H^{-1}(G)),\tag{3.3}$$

$$u\in\alpha(\theta),\qquad v\in\beta(\varphi)\qquad \textit{a.e. in } \Omega,\tag{3.4}$$

$$u(0)=u_0,\qquad v(0)=v_0\qquad \textit{a.e. in } G.\tag{3.5}$$

(a) *If in addition there is a pair $\theta_0\in L^2(G)$, $\varphi_0\in H_0^1(G)$ for which*

$u_0 \in \alpha(\theta_0)$ and $v_0 \in \beta(\varphi_0)$ a.e. in G, and if $f_1 \in H^1(0, T; L^2(G))$, $f_2 \in H^1(0, T; H^{-1}(G))$, then

$$\frac{du}{dt} \in L^\infty(0, T; L^2(G)), \qquad \frac{dv}{dt} \in L^\infty(0, T; H^{-1}(G)),$$

$$\theta \in L^\infty(0, T; L^2(G)), \qquad \varphi \in L^\infty(0, T; H_0^1(G)).$$

(b) *If in addition $\alpha(0) \ni 0$, $\beta(0) \ni 0$ and each of the functions f_1, f_2, u_0 and v_0 is non-negative, then each of u, v, θ and φ is non-negative.*

Proof. Let V be the product space $L^2(G) \times H_0^1(G)$ which has the dual $V^* = L^2(G) \times H^{-1}(G)$. Define $B \in \mathscr{L}(V, V^*)$ by

$$Bu(v) = \int_G \{h(u_1 - u_2)(v_1 - v_2) + \nabla u_2 \cdot \nabla v_2\},$$

$$u = [u_1, u_2], \qquad v = [v_1, v_2] \in V.$$

Renorm V with the equivalent norm $(Bu(u))^{1/2}$ so that $B : V \to V^*$ is the corresponding Riesz isomorphism of the Hilbert space V onto its dual. Note that B is given in $\mathscr{D}'(G)$ in the form

$$B([u_1, u_2]) = [h(u_1 - u_2), h(u_2 - u_1) - \Delta u_2], \qquad [u_1, u_2] \in V.$$

Next we consider the function $J : V^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\begin{aligned} J(u) &= \int_G (j(u_1) + k(u_2)) && \text{if} \quad u_1 \in L^2,\ j(u_1) \in L^1,\ u_2 \in L^1 \cap H^{-1}, \\ & && \qquad k(u_2) \in L^1, \\ &= +\infty && \text{otherwise},\ u = [u_1, u_2] \in V^*. \end{aligned}$$

From [3, pp. 115, 123] we find that J is a proper, convex and lower-semi-continuous function on V^*. Furthermore, the subgradient of J is determined as follows:

$$g \in \partial J(u) \qquad \text{with} \quad g = [g_1, g_2] \qquad \text{and} \qquad u = [u_1, u_2] \qquad \text{in} \quad V^*$$

if and only if for some $v = [v_1, v_2] \in V$ we have

$$g = B(v) \qquad \text{and} \qquad v_1 \in \partial j(u_1), \qquad v_2 \in \partial k(u_2) \quad \text{a.e. in } G.$$

These computations are immediate from the corresponding results of [3] on the components of V^*.

It is useful to characterize ∂J explicitly as a composition of operators in $\mathscr{D}'(G)$. Thus we define $A : V \to V^*$ by $A = [\alpha, \beta]$; that is,

$$u \in A(v) \qquad \text{with} \quad u = [u_1, u_2] \in V^* \qquad \text{and} \qquad v = [v_1, v_2] \in V$$

if and only if $u_1 \in \alpha(v_1)$ and $u_2 \in \beta(v_2)$ a.e. in G. (Note that A^{-1} is the subgradient of J computed from the Banach space V^* to its dual $V^{**} = V$ and A is the corresponding subgradient of the conjugate of J [11].) From the computations above we have the representation $\partial J = B \circ A^{-1}$ as desired.

Given the subgradient operator ∂J on the Hilbert space V^*, it is well known [2, 3] that the initial-value problem

$$\frac{dw(t)}{dt} + \partial J(w(t)) = f(t), \qquad \text{a.e.} \quad t \in [0, T],$$

$$w(0) = w_0 \tag{3.6}$$

has a unique solution $w \in H^1(0, T; V^*)$ whenever $w_0 \in \operatorname{dom}(J)$ and $f \in L^2(0, T; V^*)$ are given. Furthermore, if $w_0 \in \operatorname{dom}(\partial J)$ and $f \in H^1(0, T; V^*)$ then this solution satisfies $dw/dt \in L^\infty(0, T; V^*)$. These remarks, with the identifications $w(t) = [u(t), v(t)] \in V^*$, $w_0 = [u_0, v_0]$, $f(t) = [f_1(t), f_2(t)]$ and

$$[\theta(t), \varphi(t)] = B^{-1}(f(t) - w'(t)) \in A^{-1}(w(t)),$$

show that (3.6) is equivalent to (3.1)–(3.5) and thereby establish all but (b) of Theorem 1. For the proof of (b) we first change the data as follows: (i) Set $j(s) = j(0)$ for $s < 0$ and leave the values as originally given for $s \geqslant 0$; thus $\operatorname{dom}(\alpha) \subset [0, +\infty)$. (ii) Add to $\beta(s)$ the quantity s for those $s < 0$ and leave the values as originally given for $s \geqslant 0$; thus β is strictly monotone on $(-\infty, 0]$. Since u_0 is non-negative the hypotheses of Theorem 1 still hold so there is exactly one solution of (3.1)–(3.5) with the modified data; we denote it by u, v, θ, φ as before. Since the domain of α contains only non-negative numbers, it follows that $\theta \geqslant 0$. Our plan is to show that the remaining three functions are non-negative.

Next we consider Eq. (3.3) written with right side $h\theta + f_2$ and initial condition v_0 being non-negative. This equation is of independent interest.

LEMMA 1. *Let $\Lambda \equiv h - \Delta$ be the indicted Riesz map of the Hilbert space $H_0^1(G)$ nto its dual, $H^{-1}(G)$, and let $H^{-1}(G)$ have the scalar-product corresponding to Λ. Let γ be a maximal monotone graph on $\mathbb{R} \times \mathbb{R}$ which contains the origin and whose range is all of $\mathbb{R}$.*

(a) *The operator $\Lambda \circ \gamma$ is maximal monotone on $H^{-1}(G)$ with the* *domain* $\{v \in H^{-1} \cap L^1$: *there is a* $\varphi \in H_0^1(G)$ *with* $\varphi(x) \in \gamma(v(x))$ *a.e.* $x \in G\}$.

(b) *If* $C \equiv \{f \in H^{-1}(x) : f \geqslant 0\}$, *then* $[I + \lambda A \circ \gamma]^{-1}(C) \subset C$ *for every* $\lambda > 0$.

Proof. Part (a) follows from Theorem 17 of [3], where it is shown that $A \circ \gamma$ is a subgradient on $H^{-1}(G)$. To verify (b), let $(I + \lambda A \circ \gamma)(v) = f$ in $H^{-1}(G)$ with $f \geqslant 0$. By truncation and regularization we obtain a sequence $f_n \in L^1(G) \cap C^\infty(G) \cap C$ with $f_n \to f$ in $H^{-1}(G)$. Since $A \circ \gamma$ is maximal monotone, the corresponding sequence $v_n \equiv [I + \lambda A \circ \gamma]^{-1} f_n$ converges to v in $H^{-1}(G)$. From Proposition 5 of [5] it follows that each $v_n \in C$, so we have $v \in C$.

Let F be a maximal monotone operator on a Hilbert space H; let $v_0 \in \overline{\mathrm{dom}(F)}$ and $f \in L^1(0, T; H)$. Then there exists a unique *weak solution* of the initial-value problem [2, p. 64]

$$\frac{dv}{dt} + F(v) \ni f \qquad \text{on} \quad [0, T], \qquad v(0) = v_0. \tag{3.7}$$

By a weak solution we mean a uniform limit of strong solutions v_n corresponding to data v_0^n and f_n with $v_0^n \to v_0$ and $f_n \to f$ in H and $L^1(0, T; H)$, respectively. This existence result is proved by choosing the sequences above with each $v_0^n \in \mathrm{dom}(F)$ and each f_n a step-function with values from the range of f [2, p. 65].

LEMMA 2. *Let C be a closed cone in H. If $v_0 \in C$, $f(t) \in C$ for all $t \in [0, T]$, and if $[I + \lambda F]^{-1}(C) \subset C$ for all $\lambda > 0$, then the weak solution v of* (3.7) *satisfies $v(t) \in C$ for all $t \in [0, T]$.*

Proof. By the preceding remarks it suffices to consider the case of $v_0 \in \mathrm{dom}(F)$ and a step-function f given on a partition $0 = a_1 < \cdots < a_n = T$ by $f = y_i \in C$ on $[a_{i-1}, a_i)$. The solution is given inductively by $v(0) = v_0$ and $v(t) = S_i(t - a_i)\, v(a_i)$ on $[a_{i-1}, a_i]$, where S_i is the semigroup generated by $-(F - y_i)$. By [2, Proposition 4.5] it suffices to show $[I + \lambda(F - y_i)]^{-1}(C) \subset C$, for then we have $v(t) \in C$ for all $t \in [0, T]$. Thus, let $x = [I + \lambda(F - y_i)]^{-1} y$ with $y \in C$ and $\lambda > 0$. It follows directly that $x = (I + \lambda F)^{-1}(\lambda y_i + y)$. Since $\lambda y_i + y \in C$ we have $x \in C$ and we are done.

To obtain $v \geqslant 0$ in (3.3) we apply Lemma 1 with $\gamma = \beta^{-1}$ and then apply Lemma 2 with $F = A \circ \gamma$. Since β is strictly monotone on $(-\infty, 0]$ it follows from (3.4) that $\varphi \geqslant 0$. Finally, writing u as the sum of its positive and negative parts, $u = u^+ - u^-$, we obtain, from (3.2),

$$(1/2)\frac{d}{dt}\int_G (u^-)^2 = -\left(u^-, \frac{du}{dt}\right)_{L^2(G)} = h(u^-, \theta)_{L^2(G)} - (u^-, h\varphi + f_1)_{L^2(G)}.$$

Since u and θ have the same sign and $h\varphi + f_1$ is non-negative, the right side is non-positive so $u^- = 0$. Thus all four of u, v, θ, φ are non-negative. It follows that this quadruple is a solution of the original problem without the modified data. By uniqueness this is the solution of the original problem and (b) is established.

Remarks. The essential point in the first part of the proof of Theorem 1 is to reduce the problem to the evolution equation (3.6) whose solution is the pair $[u(t), v(t)]$ of "enthalpy" functions associated with the weak solution. It is this sense in which enthalpy is the natural variable for the problem.

For the special case of $f_2 \in L^2(0, T; L^2(G))$ we can give an alternate proof of part (b) of Theorem 1 as follows. Approximate β by a smooth β_n for which the corresponding solutions $[u_n, v_n]$ can be shown to be non-negative by direct L^2-estimates on (3.2) and (3.3). Then using methods of [9] we can let $n \to \infty$ to obtain the non-negativity of $[u, v]$. However, the proof given above permits the more general data of the existence result, and we also obtain the corresponding the corresponding well-known non-negativity result for the abstract porous media equation

$$\frac{dv}{dt} + (h - \Delta)\,\gamma(v) \ni f(t)$$

in $H^{-1}(G)$, where $h \geqslant 0$ and γ is maximal monotone. We could not find this result in the literature.

The evolution equation (3.6) is of the form

$$\frac{d}{dt} B^{-1} w(t) + A^{-1}(w(t)) \ni \tilde{f}(t),$$

where B^{-1} is positive self-adjoint and A^{-1} is maximal monotone from a Hilbert space to its dual. Various generalizations and related equations have been discussed in [4, 6, 9, 10, 15, 16, 20, 21].

4. Boundedness and Continuity of the Weak Solution

We shall prove that the "temperatures" θ and φ in the weak solution are bounded when the data in the problem are bounded. We also give sufficient conditions for θ and φ to be continuous. These results are obtained in the following special case of Theorem 1 which contains the weak formulation of the one-phase two-component Stefan problem.

THEOREM 2. *In addition to the conditions of Theorem* 1(b) *we assume the following:*

(i) *There is a number $a > 0$ such that $r \geqslant as$ for all $r \in \alpha(s)$;*

(ii) *the maximal monotone β is of the form $\beta = bI + LH$, where I is the identity, H is the Heaviside graph and both b and L are non-negative;*

(iii) *the initial data and forcing terms are essentially bounded: u_0, $v_0 \in L^\infty(G)$ and $f_1, f_2 \in L^\infty(\Omega)$. Then the functions u, v, θ, φ are bounded on Ω.*

(a) *If in addition we have $|r_1 - r_2| \geqslant a\,|s_1 - s_2|$ for all $r_1 \in \alpha(s_1)$ and $r_2 \in \alpha(s_2)$, and if the functions u_0 and $\int_0^T f_1(\cdot, t)\,dt$ are uniformly (Hölder) continuous on G, then θ is uniformly (respectively, Hölder) continuous on Ω.*

(b) *If $b > 0$ and φ_0 is uniformly continuous on G, then φ is uniformly continuous on Ω.*

Proof. Let u, v, θ, φ be the solution of (3.1)–(3.5); by assumption (ii) we may write $v = b\varphi + L\xi$, $\xi \in H(\varphi)$, in Ω. For each $\varepsilon > 0$ we consider the Steklov averages

$$\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} \varphi(s)\,ds, \qquad \theta_\varepsilon(t) = \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} \theta(s)\,ds,$$

where φ and θ are extended as $\varphi(0)$, $\theta(0)$, respectively, on $(-\varepsilon, 0)$. It is known that $\lim_{\varepsilon\to 0} \varphi_\varepsilon = \varphi$ (etc.) in $L^2(0, T; H_0^1(G))$ [14, p. 85]. By integrating (3.3) over $[t-\varepsilon, t]$ we obtain

$$\begin{aligned} &b\varphi_\varepsilon'(t) + (L/\varepsilon)(\xi(t) - \xi(t-\varepsilon)) + h\varphi_\varepsilon(t) - \Delta\varphi_\varepsilon(t) \\ &\quad = h\theta_\varepsilon(t) + (1/\varepsilon)\int_{t-\varepsilon}^{t} f_2(s)\,ds. \end{aligned}$$

We shall apply this to $(\varphi(t) - k)^+$, integrate over $(0, t) \times G$ where the superscript plus denotes the positive part of the indicated function in $H_0^1(G)$ and the number k is chosen by

$$k \equiv \max\left\{\|\varphi_0\|_{L^\infty(G)};\ \frac{2}{a}\|u_0\|_{L^\infty(G)} + \frac{1}{h}\left(\|f_1\|_{L^\infty(\Omega)} + 2\,\|f_2\|_{L^\infty(\Omega)}\right)\right\},$$

and take the limit as $\varepsilon \downarrow 0$. To this end we obtain

$$\lim_{\varepsilon\to 0}\int_0^t\int_G \varphi_\varepsilon'(t)(\varphi(t) - k)^+ = \tfrac{1}{2}\,\|(\varphi(t) - k)^+\|_{L^2(G)}^2 - \tfrac{1}{2}\,\|(\varphi_0 - k)^+\|_{L^2(G)}^2$$

and the last term vanishes since $k \geqslant \|\varphi_0\|_{L^2(G)}$:

$$(\xi(t) - \xi(t-\varepsilon))(\varphi(t) - k)^+ \geqslant (\xi(t) - 1)(\varphi(t) - k)^+ = 0$$

since $k \geqslant 0$ and $\xi(t) \in H(\varphi(t))$; and

$$\lim_{\varepsilon \to 0} \int_0^t \int_G (h\varphi_\varepsilon - \Delta\varphi_\varepsilon)(\varphi - k)^+$$

$$= \int_0^t \int_G (h\,|(\varphi - k)^+|^2 + hk(\varphi - k)^+ + |\nabla(\varphi - k)^+|^2).$$

Thus we obtain

$$\frac{b}{2}\,\|(\varphi(t) - k)^+\|^2_{L^2(G)} + hk \int_0^t \int_G (\varphi - k)^+$$

$$+ h \int_0^t \int_G |(\varphi - k)^+|^2 + \int_0^t \int_G |\nabla(\varphi - k)^+|^2$$

$$\leqslant h \int_0^t \int_G \theta(\varphi - k)^+ + \int_0^t \int_G \left(\frac{1}{\varepsilon} \int_{\tau - \varepsilon}^{\tau} f_2\right) (\varphi(\tau) - k)^+ \, d\tau\, dx.$$

This leads immediately to the estimate

$$k \int_0^t \int_G (\varphi - k)^+ + \int_0^t \int_G |(\varphi - k)^+|^2$$

$$\leqslant \int_0^t \int_G \theta(\varphi - k)^+ + \frac{1}{h} \|f_2\|_{L^2(\Omega)} \int_0^t \int_G (\varphi - k)^+. \tag{4.1}$$

Next we estimate the first term on the right side of (4.1). Integrate (3.2) over $(0, t)$ and use (i) and $\theta \geqslant 0$ to obtain

$$a\theta(t) \leqslant u(t) + h \int_0^t \theta = h \int_0^t \varphi + \int_0^t f_1 + u_0. \tag{4.2}$$

From (4.2) it easily follows that

$$a\theta(t) \leqslant h \int_0^t (\varphi - k)^+ + t(hk + \|f_1\|_{L^\infty(\Omega)}) + \|u_0\|_{L^\infty(G)}.$$

Now apply this to $(\varphi - k)^+$ and integrate to obtain

$$a \int_0^t \int_G \theta(\varphi - k)^+ \leqslant h \int_0^t \int_G \left\{(\varphi(\tau) - k)^+ \int_0^\tau (\varphi - k)^+ \right\} dx\, d\tau$$

$$+ [t(hk + \|f_1\|_{L^\infty(\Omega)}) + \|u_0\|_{L^\infty(G)}] \int_0^t \int_G (\varphi - k)^+.$$

Note that

$$\int_0^t \int_G \left\{(\varphi(\tau)-k)^+ \int_0^\tau (\varphi-k)^+\right\} dx\, d\tau$$
$$\leqslant \frac{1}{2}\left(\int_0^t \|(\varphi-k)^+\|_{L^2(G)}\right)^2 \leqslant \frac{t}{2}\int_0^t \|(\varphi-k)^+\|^2_{L^2(G)}$$

so we have from above

$$\int_0^t \int_G \theta(\varphi-k)^+ \leqslant \frac{ht}{2a}\int_0^t \|(\varphi-k(^+\|^2_{L^2(G)}$$
$$+\left[\frac{t}{a}(hk+\|f_1\|_{L^\infty(\Omega)})+\frac{1}{a}\|u_0\|_{L^\infty(G)}\right]\int_0^t\int_G(\varphi-k)^+. \tag{4.3}$$

If we use (4.3) in (4.1) we obtain

$$\left(1-\frac{ht}{2a}\right)\int_0^t \|(\varphi-k)^+\|^2_{L^2(G)}$$
$$\leqslant \left[\left(\frac{th}{a}-1\right)k+\frac{t}{a}\|f_1\|_{L^\infty}+\frac{1}{a}\|u_0\|_{L^\infty}+\frac{1}{h}\|f_2\|_{L^\infty}\right]\int_0^t\int_G(\varphi-k)^+.$$

Thus, if $0 \leqslant t \leqslant a/2h$, then by our choice of k the right side is non-positive and the left side is necessarily zero.

In summary, we have shown that with k as given above we have

$$\|\varphi(t)\|_{L^\infty(G)} \leqslant k,$$
$$\|u(t)\|_{L^\infty(G)} \leqslant \frac{a}{2h}(hk+\|f_1\|_{L^\infty(\Omega)})+\|u_0\|_{L^\infty(G)} \equiv k_1, \tag{4.4}$$
$$\|\theta(t)\|_{L^\infty(G)} \leqslant (k_1/a)\, e^{1/2},$$

for a.e. $t \in [0, a/2h|$. The first is immediate from our preceding calculations, the second follows from (4.2), and the third is obtained from (4.2) and Gronwall's inequality. From the dependence of k on the data it is clear that the estimates (4.4) on $G \times (0, a/2h)$ can be extended to give a bound on u, θ and φ on all of Ω in a finite number of steps.

In order to prove (a) we first consider the functions

$$\Phi(x,t)=\int_0^t \varphi(x,s)\,ds, \qquad \Theta(x,t)=\int_0^t \theta(x,s)\,ds.$$

Integrate (3.3) to see that Φ is a weak solution of

$$b\frac{\partial\Phi}{\partial t}-\Delta\Phi+h\Phi=h\Theta+\int_0^t f_2+L\xi_0-L\xi,\qquad b\Phi(x,0)=0,$$

$$\Phi(\cdot,t)\in H_0^1(G),\qquad 0\leqslant t\leqslant T.$$

If $b>0$ then from [14, Theorem 1.1, p. 419] we conclude that Φ is uniformly Hölder continuous on Ω. If $b=0$ then from [13, Theorem 14.1, p. 201] we conclude that Φ is uniformly Hölder continuous on G, uniformly in $t\in[0,T]$. Since φ is bounded, Φ is trivially Lipschitz in t. Thus it follows that Φ is uniformly Hölder continuous on Ω.

Next we integrate (3.2) to get

$$u(x,t)+\int_0^t h(\theta(x,s)-\varphi(x,s))\,ds=\int_0^t f_1(x,s)\,ds+u_0(x).$$

By taking the difference of this identity at $x=x_1,\ x_2\in G$ we obtain

$$\begin{aligned}|u(x_1,t)-u(x_2,t)|&\\ \leqslant h\int_0^t|\theta(x_1,s)-\theta(x_2,s)|\,ds+\Phi(x_1,t)+\int_0^t f_1(x_1,s)\,ds+u_0(x_1)&\\ -\Phi(x_2,t-\int_0^t f_1(x_2,s)\,ds-u_0(x_2).&\end{aligned}\tag{4.5}$$

By our assumptions in (a) the left side of (4.5) bounds the quantity $a\,|\theta(x_1,t)-\theta(x_2,t)|$ and the function

$$x\mapsto\Phi(x,t)+\int_0^t f_1(x,s)\,ds+u_0(x)$$

has a modulus of continuity $\sigma(\cdot)$ which is independent of $t\in[0,T]$, so we have

$$a\,|\theta(x_1,t)-\theta(x_2,t)|\leqslant h\int_0^t|\theta(x_1,s)-\theta(x_2,s)|\,ds+\sigma(x_1-x_2).$$

By Gronwall's inequality it follows that θ has the same modulus of continuity, σ, in x. From (3.2) follows the uniform Lipschitz continuity in t of u and then the assumption in (a) shows that θ is uniformly Lipschitz in t. This finishes the proof of (a).

The proof of (b) is an immediate corollary of [8, Theore 5.3, p. 69]. The point is that φ is an essentially bounded weak solution of

$$b\frac{\partial\varphi}{\partial t}+L\frac{\partial\xi}{\partial t}-\Delta\varphi=h(\theta-\varphi)+f_2,\qquad \xi\in H(\varphi),$$

with $\partial\varphi/\partial t\in L^2(\Omega)$. This last inclusion follows from $b>0$; see [9] or [14, p. 501].

Remarks. The boundary ∂G of the region G is assumed to satisfy "Condition A" of [13] in both (a) and (b) of Theorem 2. That is, there is a pair of positive numbers a_0 and θ_0 such that for any sphere B_r with center on ∂G of radius $r\leqslant a_0$ and for any component $\hat{G}_r$ of the intersection $G_r=B_r\cap G$ it follows that $\operatorname{mes}(\hat{G}_r)\leqslant(1-\theta_0)\operatorname{mes}(B_r)$. Without such a restriction on the smoothness of the boundary of G we obtain local or interior continuity results as above.

It is not known whether φ is continuous in Ω in the case $b=0$.

5. Additional Properties of the Weak Solution

Under rather general conditions on the data in the weak formulation of our problem it follows that the positivity set of the enthalpy u is increasing with time. This is equal to the positivity set of the temperature φ. An example shows this containment may be proper.

The preceding properties of the weak solution will be obtained in part from the following comparison result.

Lemma 3. *Suppose $t_0<T$ and for each $t\in[t_0,T]$ we are given a pair $\gamma_1(t,\cdot)$, $\gamma_2(t,\cdot)$ of graphs on $\mathbb{R}\times\mathbb{R}$ such that $\gamma_2(t,\cdot)$ is monotone and*

for each $s_1\in\gamma_1(t,r)$ there exists an $s_2\in\gamma_2(t,r)$ for which $s_2\leqslant s_1$. (5.1)

Let the pair of absolutely continuous functions u_1, $u_2:[t_0,T]\to\mathbb{R}$ satisfy $u_1(t_0)\leqslant u_2(t_0)$ and

$$u_1'(t)+\gamma_1(t,u_1(t))\ni 0,\qquad u_2'(t)+\gamma_2(t,u_2(t))\ni 0,$$

for a.e. $t\in[t_0,T]$. Then $u_1(t)\leqslant u_2(t)$ for $t_0\leqslant t\leqslant T$.

Proof. Suppose there is a $t_2\in(t_0,T]$ such that $u_1(t_2)>u_2(t_2)$. Define $t_1\equiv\operatorname{lub}\{t\in[t_0,T]:u_1(t)\leqslant u_2(t)\}$ and note that $u_1(t_1)=u_2(t_1)$ and $u_1(t)\geqslant u_2(t)$ for all $t\in[t_1,t_2]$. For each $t\in[t_1,t_2]$ for which $-u_u'(t)\in\gamma_1(t,u_1(t))$

there is by (5.1) an $s(t) \in \gamma_2(t, u_1(t))$ with $s(t) \leqslant -u_1'(t)$. For such a t we obtain

$$(u_1'(t) - u_2'(t))(u_1(t) - u_2(t)) \leqslant (-s(t) - u_2'(t))(u_1(t) - u_2(t))$$

and this last quantity is non-positive because $\gamma_2(t, \cdot)$ is monotone. Thus the function $(u_1(t) - u_2(t))^2$ has a non-positive derivative on $[t_1, t_2]$, it therefore vanishes on $[t_1, t_2]$ and this contradicts the choice of t_2.

Suppose we are in the situation of Theorem 1(b). For a.e. $x \in G$ the function $u_2(t) \equiv u(x, t)$ is an absolutely continuous solution of (3.2). In order to apply Lemma 3 to (3.2), (3.4), define

$$\gamma_1(t, u) = (h/a)\, u^+ - f_1(x, t), \qquad \gamma_2(t, u) = h\alpha^{-1}(u) - h\varphi(x, t) - f_1(x, t).$$

We will assume $a > 0$ and that

$$\text{for each } r \geqslant 0 \text{ there exists an } s \in \alpha^{-1}(r) \text{ such that } as \leqslant r. \tag{5.2}$$

This implies that (5.1) holds for our choice of γ_1, γ_2. Let u_1 be the solution of

$$u_1'(t) + (h/a)\, u_1(t) = f_1(x, t), \qquad 0 \leqslant t \leqslant T,$$

with $u_1(t_0) = u(x, t_0)$. From Lemma 3 we obtain the first part of the following.

THEOREM 3. *In addition to the conditions of Theorem* 1(b) *we assume there is an* $a > 0$ *for which* (5.2) *holds. Then the first component of the solution of* (3.1)–(3.5) *satisfies*

$$u(x, t) \geqslant e^{-(h/a)(t - t_0)} \cdot u(x, t_0) + \int_{t_0}^{t} e^{-(h/a)(t-s)} f_1(x, s)\, ds,$$
$$0 \leqslant t_0 \leqslant t \leqslant T, \tag{5.3}$$

for almost every $x \in G$. *Thus the set* $S_t^+(u) \equiv \{x \in G : u(x, t) > 0\}$ *is increasing with* t. *Furthermore, the set* $S^+(u) \equiv \{(x, t) \in \Omega : u(x, t) > 0\}$ *contains the interior of* $S^+(f_1)$ *and*

$$\bigcup \{S_t^+(\varphi) : 0 < t < t_1\} \subset S_{t_1}^+(u), \qquad 0 < t_1 \leqslant T. \tag{5.4}$$

Proof. The inequality (5.3) follows from the preceding remarks and it immediately implies the monotonicity of $S_t^+(u)$ and the inclusion of the interior of $S^+(f_1)$ in $S^+(u)$. We verify (5.4). Let $x_1 \notin S_{t_1}^+(u)$, that is, $u(x_1, t_1) = 0$, so by (5.3) we have $u(x_1, t) = 0$ for all $0 \leqslant t \leqslant t_1$. Thus $\partial u(x_1, t)/\partial t = 0$ for $0 < t < t_1$. From (5.2) we obtain (see below) $\theta(x_1, t) = 0$

for $0 \leqslant t \leqslant t_1$ so (3.2) implies $\varphi(x_1, t) = 0$ for $0 \leqslant t < t_1$. That is, $x_1 \notin S_t^+(\varphi)$ for all $0 < t < t_1$.

COROLLARY. *In the situation of Theorem* 2(b) *we have* $S^+(\varphi) \subset S^+(u)$.

Proof. Since φ is continuous this follows from (5.4).

Remarks. The condition (5.2) is actually equivalent to the assumption (i) in Theorem 2: $r \geqslant as$ for all $(s, r) \in \alpha$. To see this, note that if $(s_0, r_0) \in \alpha$ with $r_0 < as_0$, then we can choose $r_1 \equiv (1/2)(r_0 + as_0) \geqslant 0$ and from (5.2) a s_1 with $(s_1, r_1) \in \alpha$ and $as_1 \leqslant r_1$. But then $s_1 \leqslant (1/2)(r_0/a + s_0) < s_0$ and $r_1 > r_0$, contradicting the monotonicity of α. Thus (5.2) implies that all of α lies above the graph of $r = as$.

As a consequence of the above remark it follows from (3.4) that $u(x, t) \geqslant a\theta(x, t)$, hence $S^+(\theta) \subset S^+(u)$. If in addition we have $\alpha(0) = \{0\}$, then $S^+(\theta) = S^+(u)$.

In the case of our original problem, (2.11)–(2.15), we have $S^+(\varphi) \subset S^+(\theta)$: Thus $\theta > 0$ in the region Ω_+ where the water is completely melted. The following example shows that we do not necessarily have $S^+(\varphi) = S^+(\theta)$.

EXAMPLE. Define $\theta(t) = e^{-t}$, $\varphi = 0$ and $v(t) = 1 - e^{-t}$ for $t \geqslant 0$. This triple of non-negative functions is the solution of (2.11) – (2.15) with $L = a = h = 1$ and arbitrary $b \geqslant 0, f_1 = f_2 = 0$, and $\theta_0 = 1$, $\varphi_0 = \xi_0 = 0$. This solution is independent of $x \in G$. In the thermal conduction model of Section 2, this example corresponds to the situation wherein a small amount of heat uniformly distributed in the first component is all absorbed as latent heat to convert the second component from solid ice to water at temperature equal to zero.

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