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THE SUPER-STEFAN PROBLEM

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Abstract—The classical Stefan free-boundary problem describes the conduction of heat through a medium in which a phase change occurs at a prescribed temperature. Here is considered the more general case in which the freezing temperature lies strictly below the melting temperature, so the entropy is given by a hysteresis functional which permits super-cooling or super-heating of the medium. The model is developed and formulated as an evolution system in L^1, and well-posedness results are described. This model and its connection to the evolution equation are made possible by our representation of the hysteresis functional as an ordinary differential equation subject to a constraint.

1. INTRODUCTION

The objective here is to develop a model of a free-boundary problem of Stefan type in which the freezing and thawing occur at different temperatures. This comprises the Super-Stefan problem which permits the super-heating or super-cooling of a material. We shall also sketch the proof of well-posedness of the initial-boundary-value problem with resulting hysteresis memory effects distributed over the spatial region. That is, the previous history of the system contributes to the present value of the functional, and hence to the current evolution rate. Hysteresis effects also arise in the magnetization of a ferromagnetic material. Additional examples can be found in biology, chemistry, and economics.

The method developed here entails coupling a semilinear parabolic equation with an ordinary differential equation that generates the hysteresis term in a rather clever way so as to preserve the monotone structure of the whole system and thereby permit the direct application of standard methods of convex analysis and monotone operator theory. Previous treatments of hysteresis nonlinearities have been achieved with formal operators which are inserted in problems for partial differential equations in an ad hoc manner. The well-posedness results are obtained for the stationary system which represents the resolvent of the evolution operator by combining methods of convex analysis with L^1 estimates for semilinear partial differential equations. The integral solution is then recovered by an application of the Crandall–Liggett theorem.

2. THE SUPER-STEFAN MODEL

We begin with a mathematical description of heat conduction through a medium in which a change of phase occurs in the vicinity of a given temperature. We develop the example of the melting and refreezing of water/ice in a porous medium, and the hysteresis effects result from the assumption that the melting and the freezing temperatures are *different*: the ice melts at a slightly positive temperature and freezing occurs after a slightly negative temperature is reached. Since this example is so important and has a substantial history (see Section 4), we shall develop it in some detail in the style of [1].

In order to specify our model of phase change with super-heating or super-cooling, we begin with the description of the relation between phase, energy, and temperature. Let a and b be given numbers with $a < 0 < b$. Begin with a unit volume of ice at temperature $u < a$ and apply a uniform heat source of intensity F. The temperature increases according to the relation

$e = c(u)$, where $e \equiv Ft$ is the accumulated *internal energy*, until it reaches the temperature $u = b > 0$. Then the temperature remains at $u = b$ until L units of additional heat have been added; $L > 0$ is the *latent heat.* During this period there is a fraction w of water coexisting with the ice, and w increases at the constant rate F/L. The *water fraction* w, $0 \le w \le 1$, is the phase variable. When all the ice has melted, $w = 1$ and the temperature u begins to rise again according to $e = c(u) + L$. If the process is reversed by drawing heat out of the unit volume at a constant rate, F, the temperature falls according to $e = c(u) + L$ until it reaches $u = a < 0$, then w decreases at the rate F/L until it reaches $w = 0$, and thereafter the temperature u falls with $e = c(u)$. Note that the freezing took place at $u = a$ and the melting at $u = b$. If $a = b$ this is just the traditional Stefan problem, but above we permit superheated ice and supercooled water. It is here that hysteresis occurs. Denote by $H(\cdot)$ the *Heaviside graph*: $H(u) = \{0\}$ if $u < 0$, $H(0) = [0, 1]$, and $H(u) = \{1\}$ if $u > 0$. The relation between energy and temperature is given by $e \in c(u) + LH(u - b)$ when u is increasing and by $e \in c(u) + LH(u - a)$ when it is decreasing. The difference $e - c(u)$ is just L times the *simple relay*: $w \in R(u)$ means $w = H(u - b)$ if u is increasing from below a, $w \in H(u - a)$ if u is decreasing from above b; also w remains constant for $a < u < b$, since there is no phase change until the threshold values are reached.

We shall formulate a free-boundary problem which describes heat conduction through a domain G in Euclidean space R^m subject to the constitutive assumptions above on the hysteresis relation between energy, phase, and temperature. This will be called the *Super-Stefan problem.* Denote the boundary of G by ∂G and set $\Omega = G \times (0, \infty)$. The temperature at the point $x \in G$ and the time $t > 0$ is $u(x, t)$ and the smooth monotone functions $c(u)$, $k(u)$ are given with $c(0) = k(0) = 0$; their derivatives $c'(u)$, $k'(u)$ denote the *specific heat* and *conductivity*, respectively, of ice–water at temperature u. The phase change from water to ice occurs at $u = a < 0$ and from ice to water at $u = b > 0$. The space–time region Ω is then separated into an always-ice region Ω_- where $u < a$, an always-water region Ω_+ where $u > b$, and a region Ω_0 where $a \le u \le b$ and in which the phase depends on it's preceding history. In the presence of a distributed source of intensity $F(x, t)$, there may also arise regions Ω_a where $u = a$ and Ω_b where $u = b$ consisting of a mixture of ice and water. Such *mushy regions* may then persist into Ω_0. However, we shall postpone mention of these additional history-dependent regions until after we have described the simpler case with no mush.

Let $w(x, t)$ be the fraction of water at $(x, t) \in \Omega$, and note that according to our constitutive assumptions above we have $w \in R(u)$. The energy is given by $e = c(u) + Lw$. Let S_- be the boundary of Ω_- in Ω and S_+ the boundary of Ω_+ in Ω. The *unit normal* $N = (N_1, \ldots, N_m, N_t)$ on $S_- \cup S_+$ is oriented out of Ω_- and Ω_+, and hence into Ω_0. We shall denote by $[g]$ the *saltus* or jump in values of the function g across the boundaries, S_- and S_+, in the direction of N: for $(x, t) \in S_- \cup S_+$

$$[g(x, t)] = \lim_{h \to 0^+} \{g((x, t) + hN) - g((x, t) - hN)\}.$$

The strong form of the *Super-Stefan problem* is to find a pair of functions u and w on Ω for which

$$\frac{\partial}{\partial t} c(u) - \Delta k(u) = 0 \quad \text{in} \quad \Omega_- \cup \Omega_0 \cup \Omega_+, \tag{1a}$$

$$w \in R(u) \quad \text{in } \Omega, \tag{1b}$$

$$[\nabla k(u)] \cdot (N_1, \ldots, N_m) = LN_t[w] \quad \text{on} \quad S_- \cup S_+, \tag{1c}$$

$$u(x, 0) = u_0, \qquad x \in G, \tag{2a}$$

$$w(x, 0) = w_0(x) \in [0, 1], \quad \text{where} \quad a \le u_0(x) \le b, \tag{2b}$$

$$u(s, t) = 0, \qquad s \in \partial G, \qquad t > 0. \tag{3}$$

For the moment one should assume $w_0(x)$ is either identically zero or identically one; this is only to avoid introducing a mushy region. The classical (possibly nonlinear) heat equation (1a) determines the temperature where $u \neq a$ and $u \neq b$. The water fraction is given by the hysteresis functional (1b) which was described above, so we have $w = 0$ in Ω_-, $w = 1$ in Ω_+, and w is either 0 or 1 in Ω_0 according to whether the temperature was last below a or above b, respectively. Let n be the unit vector in the direction $(N_1, \ldots, N_m)$, and let V be the *velocity* of S_- or S_+ at time t in the direction of n. By dividing (1c) by $(N_1^2 + \cdots + N_m^2)^{1/2}$, we obtain

$$\left[\frac{\partial}{\partial n} k(u)\right] + LV[w] = 0 \quad \text{on} \quad S_- \cup S_+$$

which is equivalent to (1c). This means the difference in heat flux across the *free boundary* S_+ determines the velocity V of that boundary by melting the fraction of ice $1 - w = -[w]$ with latent heat L, and similarly the velocity of S_- is determined by the freezing of the fraction of water $w = [w]$. The Dirichlet boundary condition (3) is used here for simplicity, but any of the usual types can just as easily be attained.

In order to obtain a weak formulation of the Super-Stefan problem, we consider a solution u, w of (1) for which $u \in H^1(\Omega)$, u is smooth in each of Ω_-, Ω_0, Ω_+, and discontinuities in ∇u drive the surfaces S_- and S_+ according to (1c). We begin by computing $\frac{\partial e}{\partial t} - \Delta k(u)$ in the sense of distributions on Ω. Thus, for $\varphi \in C_0^\infty(\Omega)$ we have

$$\left\langle \frac{\partial e}{\partial t} - \Delta k(u), \varphi \right\rangle \equiv \int_\Omega \{-(c(u) + Lw)\varphi_t - k(u)\,\Delta\varphi\}.$$

Since $k(u)$ belongs to $H_0^1(\Omega)$ we have

$$\begin{aligned}\int_\Omega \{-(c(u) + Lw)\varphi_t + \nabla k(u) \cdot \nabla\varphi\} = &\int_{\Omega_-} \{-c(u)\varphi_t + \nabla k(u) \cdot \nabla\varphi\} \\ &+ \int_{\Omega_0} \{-(c(u) + Lw)\varphi_t + \nabla k(u) \cdot \nabla\varphi\} \\ &+ \int_{\Omega_+} \{-(c(u) + L)\varphi_t + \nabla k(u) \cdot \nabla\varphi\}.\end{aligned}$$

From Gauss' theorem we can write these three successive integrals in the respective forms

$$\int_{\Omega_-} (c(u)_t - \Delta k(u))\varphi + \int_{S_-} \{\nabla k(u) \cdot (N_1, \ldots, N_m) - c(a^-)N_t\}\varphi,$$

$$\begin{aligned}\int_{\Omega_0} ((c(u) + Lw)_t - \Delta k(u))\varphi - &\int_{S_-} \{\nabla k(u) \cdot (N_1, \ldots N_m) - (c(a^+) + Lw)N_t\}\varphi \\ &- \int_{S_+} \{\nabla k(u) \cdot (N_1, \ldots, N_m) - (c(b^-) + Lw)N_t\}\varphi,\end{aligned}$$

and

$$\int_{\Omega_+} (c(u)_t - \Delta k(u))\varphi + \int_{S_+} \{\nabla k(u) \cdot (N_1, \ldots, N_m) - (c(b^+) + L)N_t\}\varphi.$$

By adding these and using the observations that $w_t = 0$ in Ω_0 and $c(\cdot)$ is continuous we obtain

$$\left\langle \frac{\partial e}{\partial t} - \Delta k(u), \varphi \right\rangle = \int_{\Omega_- \cup \Omega_0 \cup \Omega_+} (c(u)_t - \Delta k(u))\varphi + \int_{S_- \cup S_+} (-[\nabla k(u)] \cdot (N_1, \ldots, N_m) + L[w]N_t)\varphi.$$

Hence, it follows that (1a) and (1c) are equivalent to

$$\frac{\partial}{\partial t}(c(u) + Lw) - \Delta k(u) = 0,$$

where (1) holds and w, u are appropriately smooth.

Conversely, let a distributed source $F \in L^1(\Omega)$ be given in addition to $c(\cdot)$, $k(\cdot)$, L, u_0, w_0, and the hystersis functional R as above. The *generalized Super-Stefan problem* is to find a pair $u \in L^1(\Omega)$, $w \in L^\infty(\Omega)$ for which

$$\frac{\partial}{\partial t}(c(u) + Lw) - \Delta k(u) = F \quad \text{in } \mathscr{D}'(\Omega), \tag{4a}$$

$$w \in R(u) \quad \text{in } \Omega, \tag{4b}$$

(2) holds, and $k(u) \in L^2(0, T; H_0^1(G))$. This last condition implies (3) if $k'(0) > 0$. Assume we have a solution for which the corresponding sets Ω_-, Ω_0, Ω_+, Ω_a, and Ω_b are smoothly bounded, i.e. to which Gauss' theorem applies. Let S_- be the boundary between Ω_- and Ω_a, and S_+ the boundary between Ω_+ and Ω_b. Denote the boundary between Ω_a and Ω_0 by S_a and the boundary between Ω_b and Ω_0 by S_b. The normal N is directed out of Ω_- and Ω_+ and directed into Ω_0; this is consistent with the preceding discussion when Ω_a and Ω_b are empty, and in that case $S_- = S_a$, $S_+ = S_b$. Assume u is smooth in each of Ω_-, Ω_0, Ω_+. Then from the calculations leading to (4) above, we obtain the following:

$$\frac{\partial}{\partial t}c(u) - \Delta k(u) = F \quad \text{in} \quad \Omega_- \cup \Omega_0 \cup \Omega_+, \tag{5a}$$

$$Lw_t = F \quad \text{in} \quad \Omega_a \cup \Omega_b, \tag{5b}$$

$$w \in R(u) \quad \text{in } \Omega, \tag{5c}$$

$$[\nabla k(u)] \cdot (N_1, \ldots, N_m) = LN_t[w] \quad \text{on} \quad S_- \cup S_a \cup S_b \cup S_+. \tag{5d}$$

Of course we have $u = a$ in Ω_a and $u = b$ in Ω_b by definition, and these complement (5a). Likewise from (5c) we have $w = 0$ in Ω_-, $w = 1$ in Ω_+, and $w_t = 0$ in Ω_0, and these complement (5b). According to (5d), the frozen region Ω_- will not increase unless $F > 0$ in Ω_a, and a dual statement holds for the melted region Ω_+. Also $\left[\frac{\partial u}{\partial n}\right] > 0$ on $S_a \cup S_b$, so the velocity direction there is determined by the difference in water fractions $[w]$ across these boundaries.

We close with some remarks on some modifications of the enthalpy functional (1b). The simple relay $R(u)$ described above is an idealization. Specifically, during the phase change the temperature u does not likely remain exactly constant but increases at a very small rate. Thus it is reasonable to replace the Heaviside relation by a (single-valued) monotone function which closely approximates it. Conversely, if one could manage to force temperatures past either of the phase-change temperatures, the phase w, would not be expected to respond instantly, but only at a very high rate. We shall permit both of these modifications in our problem below, and it happens that we must require at least one of them in order to get a good theory by the simplest method. Another construction is required to recover the ideal situation that was originally described.

3. THE PARABOLIC HYSTERESIS PROBLEM

Our objective here is to reduce the Super-Stefan problem to an evolution equation in Banach space for which there is a theory of well-posedness for the initial value problem and thereby to show that this problem is well-posed. For the sake of completeness we first give a brief description of this theory; the estimates necessary to apply this theory will also show that standard numerical schemes are applicable. This reduction will be achieved by showing that the simple relay hysteresis functional (4b) is duplicated by an ordinary differential equation with convex constraint. Additional remarks on alternative models and related results are given in the final section.

First we briefly review the theory of evolution equations in a Banach space X as it applies to our system below. A (possibly multi-valued) operator or relation $\mathbb{C}$ in X is a collection of related pairs $[x, y] \in X \times X$ denoted by $y \in \mathbb{C}(x)$; the domain $D(\mathbb{C})$ is the set of all such x and the range $Rg(\mathbb{C})$ consists of all such y. For such an operator $\mathbb{C}$ we shall consider the *evolution equation*

$$x'(t) + \mathbb{C}(x(t)) \ni F(t) \quad \text{in } X.$$

The *Cauchy problem* is to find a solution $x(t)$ on the interval $(0, T)$ for which $x(0) = x_0$, where $x_0 \in \overline{D(\mathbb{C})}$ and $F:(0, T) \to x$ are given.

The operator $\mathbb{C}$ is called *accretive* if for all $y_1 \in \mathbb{C}(x_1)$, $y_2 \in \mathbb{C}(x_2)$ and $\varepsilon > 0$

$$\|x_1 - x_2\| \le \|x_1 - x_2 + \varepsilon(y_1 - y_2)\|.$$

This is equivalent to requiring that $(I + \varepsilon\mathbb{C})^{-1}$ be a non-expansive function on $Rg(I + \varepsilon\mathbb{C})$ for every $\varepsilon > 0$. If, in addition, $Rg(I + \varepsilon\mathbb{C}) = X$ for some (equivalently, for all)$\varepsilon > 0$, then $\mathbb{C}$ is called m-accretive. For such an operator, one can approximate the derivative in the evolution equation by a backward-difference quotient of step size $h > 0$ and the function $F(t)$ by the step function $F^h(t)$ $(= F_k^h$ for $kh \le t < (k + 1)h)$ and get a unique solution $\{x_k^h : 1 \le k\}$ of

$$\frac{x_k^h - x_{k-1}^h}{h} + \mathbb{C}(x_k^h) \ni F_k^h, \qquad k = 1, 2, \ldots,$$

with $x_0^h = x_0$. Since $\mathbb{C}$ is m-accretive, this scheme is uniquely solved recursively to obtain x_k^h and, hence, the piecewise-constant approximate solution $x^h(t)$ $(= x_x^h$ for $kh \le t < (k + 1)h)$ of the Cauchy problem. The fundamental result is the following.

C–L THEOREM. *Assume* $\mathbb{C}$ *is m-accretive,* $x_0 \in \overline{D(\mathbb{C})}$, $F \in L^1([0, T], X)$ *and that* $F^h \to F$ *in* $L^1([0, T], X)$. *Then* $x^h \to x(\cdot)$ *uniformly as* $h \to 0$ *and* $x(\cdot) \in C([0, T], X)$.

Thus $x(\cdot)$ is an obvious candidate for a solution of the Cauchy problem. It can be uniquely characterized as an *integral solution.* This rather technical characterization does not require any differentiability of the solution. However, if F is Lipschitz continuous and $x_0 \in D(\mathbb{C})$, it is known that x is also Lipschitz continuous. For an introduction to the abstract Cauchy problem in Banach space and its applications to initial-boundary-value problems for partial differential equations, see [2]. For further details, refinements and perspective, see [3, 4].

Next we describe our representation of the hysteresis functional (4b) by means of an ordinary differential equation subject to a constraint. This will permit us to write the system (5) as an evolution equation in L^1. To do so we use the *maximal monotone graph,* sgn, defined by $\mathrm{sgn}(y) = \{-1\}$ if $y < 0$, $\mathrm{sgn}(0) = [-1, 1]$, and $\mathrm{sgn}(y) = \{1\}$ if $y > 0$. To be precise, we shall use the *inverse graph* sgn^{-1} obtained by reflection of the coordinates about the diagonal. Thus, the graph $y \in \mathrm{sgn}^{-1}(x)$ is equivalent to

$$-1 \le x \le 1, \qquad (1 - x)(1 + x)y = 0, \quad \text{and} \quad xy \ge 0.$$

That is, we have exactly one of

$$\begin{gathered} -1 < x < 1 \quad \text{and} \quad y = 0, \\ x = -1 \quad \text{and} \quad y \le 0, \quad \text{or} \\ x = 1 \quad \text{and} \quad y \ge 0, \end{gathered}$$

and this shows why such graphs arise naturally in problems containing *variational inequalities.*

We construct a hysteresis model from an ordinary differential equation as follows. Let a maximal monotone graph $b(\cdot)$ be given; our hysteresis model will be of the type *generalized* *play* described by horizontal translates of $w \in b(u)$. (The simple relay from Section 2 is obtained as a special case by choosing $b = H$.) Thus, we introduce a new variable, v, in order to represent the phase constraints:

$$w \in b(v), \qquad u - 1 \leq v \leq u + 1.$$

Finally, we use the sgn^{-1} graph to realize these constraints. Let $u(t)$ be a time-dependent input to this generalized play model, and let $w(t)$ be the corresponding output or response. There is at each time a corresponding phase variable $v(t)$ which is related to $w(t)$ and $u(t)$ as above, and so it is required that $w(t)$ be non-decreasing when $v(t) = u(t) - 1$, non-increasing when $v(t) = u(t) + 1$, and stationary ($w'(t) = 0$) in the interior region, $u - 1 < v < u + 1$. This is equivalent to requiring that $w(t)$, $v(t)$ satisfy

$$w(t) \in b(v(t)), \qquad w'(t) + \mathrm{sgn}^{-1}(v(t) - u(t)) \ni 0.$$

Thus we are led to ordinary differential equations of the form

$$w(t) \in b(v(t)), \qquad w'(t) + c(v(t) - u(t)) \ni 0$$

with maximal monotone graphs $b(\cdot)$ and $c(\cdot)$ as models of hysteresis in which the output is the solution $w(t)$ with input $u(t)$. We explore below conditions on the pair of graphs for which an appropriate Cauchy problem is well-posed.

In view of our development above of an ordinary differential equation as a hysteresis model, we shall consider the Cauchy problem for the system

$$\frac{\partial}{\partial t} a(u) - \Delta u - c(v - u) \ni f \tag{6a}$$

$$\frac{\partial}{\partial t} b(v) + c(v - u) \ni g \tag{6b}$$

with (2) and (3). Such a system results from (5) after a substitution for the strictly monotone $k(u)$; we need only to take $b = H$, $c = \mathrm{sgn}^{-1}$ and set $g \equiv 0$. Under appropriate assumptions to be given below we can show that the dynamics of the system (6) is determined by a nonlinear semigroup of contractions on the Banach space $L^1(G) \times L^1(G)$. This semigroup is constructed by the C–L Theorem above from an operator $\mathbb{C}$ for which the *resolvent equation*, $(I + \varepsilon\mathbb{C})([a, b]) \ni [f, g]$ with $\epsilon > 0$, takes the form

$$a(u) - \varepsilon\, \Delta u - \varepsilon c(v - u) \ni f \tag{7a}$$

$$b(v) + \varepsilon c(v - u) \ni g \tag{7b}$$

in the state space $L^1(G) \times L^1(G)$ with $u \in H_0^1(G)$, $v \in L^2(G)$ when f and g are in $L^2(G)$. In order to motivate the necessary estimates we consider the special case of functions $a(\cdot)$, $b(\cdot)$, $c(\cdot)$. To get the variational form of this problem, multiply the equations by appropriate test functions φ and ψ on G and integrate to obtain

$$\int_G (a(u)\varphi + \varepsilon\, \nabla u \cdot \nabla\varphi + b(v)\psi + \varepsilon c(v - u)(\psi - \varphi))\, \mathrm{d}x = \int_G (f\varphi + g\psi)\, \mathrm{d}x.$$

By choosing $\varphi = \mathrm{sgn}(u)$, $\psi = \mathrm{sgn}(v)$ we *formally* obtain the stability estimate

$$\|a(u)\|_{L^1(G)} + \|b(v)\|_{L^1(G)} \leq \|f\|_{L^1(G)} + \|g\|_{L^1(G)}.$$

By employing the corresponding argument to differences of solutions, we find that the map $[f, g] \to [a(u), b(u)]$ given by $(I + \varepsilon\mathbb{C})^{-1}$ is a contradiction on $L^1(G) \times L^1(G)$. The proof of the following result is contained in [6].

L–S Theorem. *Assume $a(\cdot)$ and $b(\cdot)$ are continuous monotone functions, $a(0) = b(0) = 0$,*

$$\int_0^s a(t)\,\mathrm{d}t \geq k_1 s^2 - k_2, \qquad \int_0^s b(t)\,\mathrm{d}t \leq K_1 s^2 + K_2, \qquad s \in \mathbb{R}$$

for some $k_1 > 0$. Assume $c(\cdot)$ is a maximal monotone graph whose antiderivative is quadratically lower-bounded. Then the dynamics of the system (6) is given by a nonlinear semigroup of contractions in $L^1(G) \times L^1(G)$. That is, if f and g belong to $L^1([0, T], L^1(G))$, and if $a_0, b_0 \in L^2(G)$ with $a_0(x) \in a(u_0(x))$, $b_0(x) \in b(v(x))$ at a.e. $x \in G$ for some pair $u_0 \in H_0^1(G) \cap H^2(G)$, $v_0 \in L^2(G)$ for which $c_0(x) \in c(v_0(x) - u_0(x))$ at a.e. $x \in G$ for some $c_0 \in L^2(G)$, then there is a unique generalized solution (= integral solution) of the system (6) with $\lim_{t\to 0} a(u(t)) = a_0$ and $\lim_{t\to 0} b(v(t)) = b_0$ in $L^1(G)$.

Remark. Since $b(\cdot)$ is required to be a function, this Theorem does not apply to the original Super-Stefan problem with the Heaviside functional. However it does work for the more physically likely case where $b(\cdot)$ is a monotone function approximating the Heaviside graph. Also, we obtain essentially the same results as in the Theorem but with (multi-valued) maximal monotone graphs $a(\cdot)$ and $b(\cdot)$ if we require that $c(\cdot)$ be a function. If we choose $c(\cdot)$ to be a monotone function approximating the sgn^{-1} graph, for example, the Yosida approximation $(\mathrm{sgn} + \varepsilon I)^{-1}$ with $\varepsilon > 0$, then we obtain a *dynamic hysteresis model* which is a rate-dependent approximation to the hysteresis functional (1b). (See [5].) Each of these two modifications should be regarded as a *regularization* of the Super-Stefan problem. By an averaging construction due to Preisach, one can obtain the original form of the problem as well as a very general class of hysteresis models. See [6].

4. REFERENCES AND COMMENTS

The Super-Stefan problem provides an example of a simple but basic form of hysteresis in which the output (phase) depends not only on the current value of the input (temperature) but on the history of the input. For an excellent well-motivated introduction to this topic, see the monograph [5]. Hysteresis is a very common and general concept, and one should consult the recent survey [7] for a concise description of recent results on the development and application of mathematical models of hysteresis. Due to the complex description of the operators traditionally used to represent hysteresis [8], their addition to systems of differential equations has led to substantial technical problems for the development of a good theory. We have shown above that it is possible to model some forms of hysteresis in a fashion that is strikingly compatible with standard methods for differential equations.

The model problem of a parabolic diffusion equation coupled to hysteresis, which we introduce above as the system (6), has been studied extensively by A. Visintin. In the case of a linear equation he has proved existence by backward-difference discretization and compactness methods applied to the monotone steps that describe the discretization. In [9] is covered the (regularized) case without jumps in the defining functions of the hysteresis model, and this is extended in [10] to the case with jumps. The (nondegenerate) semilinear case with a general Preisach hysteresis is covered in [11]. The uniqueness of a solution is established in [12].

The device of representing hysteresis by (systems of) ordinary differential equations with constraints is certainly not new, but the special form appearing in (6) was developed in [13] based on the idea in the earlier work [14]. The extension to (possibly degenerate) semilinear equations with rather general Preisach hysteresis models is given in [6] as an application of evolution equations in general Banach space (as in Section 3). A related construction is recently described in [15] with corresponding results announced to appear; the distinction there is that the coupling function there is specified as a function of two variables instead of a function of the difference of the variables.

Systems of the general form of (6) appear in many other contexts in which (6b) is frequently a local storage or capacity in immobile (nondiffusive) sites. A similar quasilinear system in which all three of the monotone functions $a(\ )$, $b(\ )$, and $c(\ )$ have power growth rates was developed in [16]. The device used there to prove regularity of solutions, i.e. to show the difference scheme also converges in a dual Sobolev space, could be used to get properties of solutions here. Also the technique of approximating the generalized solutions by smooth solutions of a corresponding problem with regularized functions is applicable here. See [17] and [18] for additional related systems. For control of Stefan problems by hysteresis functionals, see [19] and [20].

Finally we would like to mention that the term 'super-cooled' is used in other contexts, especially in variations on the classical Stefan problem, where it has a different meaning from what is given here and is not directly related to hysteresis. For example, in the classical strong formulation of the Stefan problem in which the phase change takes place at $u = 0$, in one spatial dimension where the free boundary is given by an explicit function, one can study the properties of solutions for which the initial temperature in the water phase is nonnegative and not identically zero. Then the global existence, finite time extinction, or blow-up of solutions is determined by an explicit energy functional [21]. Other than the few references cited above, we could not find explicit models of heat conduction with hysteresis of the type presented here. On the other hand, there is a substantial quantity of literature on the representation of hysteresis in the literature of engineering, as well as that of physics and mathematics, and some of this can be found and compared in [7] and [5].

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