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Diffusion in poro-plastic media

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SUMMARY

A model is developed for the flow of a slightly compressible fluid through a saturated inelastic porous medium. The initial-boundary-value problem is a system that consists of the diffusion equation for the fluid coupled to the momentum equation for the porous solid together with a constitutive law which includes a possibly hysteretic relation of elasto-visco-plastic type. The variational form of this problem in Hilbert space is a non-linear evolution equation for which the existence and uniqueness of a global strong solution is proved by means of monotonicity methods. Various degenerate situations are permitted, such as incompressible fluid, negligible porosity, or a quasi-static momentum equation. The essential sufficient conditions for the well-posedness of the system consist of an ellipticity condition on the term for diffusion of fluid and either a viscous or a hardening assumption in the constitutive relation for the porous solid. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: poromechanics; non-linear diffusion; elastic–plastic porous medium; monotone operators; non-linear semigroup methods

1. INTRODUCTION

1.1. The model

The present analysis is concerned with the modelling of the diffusion of a slightly compressible fluid through a saturated deforming porous medium. The simplest example of our model describes the evolution of the *fluid pressure* scalar field $p(x,t)$, the *solid displacement* vector field $\mathbf{u}(x,t)$, and the *effective stress* tensor field $\sigma(x,t)$ satisfying the coupled system of partial differential and functional equations

$$\frac{\partial}{\partial t}(c_0 p + \alpha \nabla \cdot \mathbf{u}) - \nabla \cdot k(\nabla p) = c_0^{1/2} h_0 \tag{1a}$$

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$$\rho \frac{\partial^2}{\partial t^2} \mathbf{u} - \nabla \lambda^* \frac{\partial}{\partial t}(\nabla \cdot \mathbf{u}) - \nabla \cdot \sigma + \alpha \nabla p = \rho^{1/2} \mathbf{g}_0 \tag{1b}$$

$$\sigma = \mathscr{H}(\varepsilon(\mathbf{u})) \tag{1c}$$

in the cylindrical domain $\Omega \times (0, T)$, where Ω is a non-empty bounded and open set in $\mathbb{R}^3$ with smooth boundary $\Gamma \equiv \partial\Omega$, and $(0, T)$ is the time interval of interest. Also, $h_0 : \Omega \times (0, T) \longrightarrow \mathbb{R}$ and $\mathbf{g}_0 : \Omega \times (0, T) \longrightarrow \mathbb{R}^3$ are suitably given functions. The system consists of the coupling of a diffusion equation for the pressure with the conservation of momentum and a constitutive relation for the motion of the medium, respectively. The constitutive relation (1c) is described at length below. It involves the stress σ and the *small strain* tensor $\varepsilon(\mathbf{u})$, namely, the symmetric part of the derivative of displacement,

$$\varepsilon(\mathbf{u})_{ij} \equiv (\partial_j u_i + \partial_i u_j)/2 \quad \text{for } i, j = 1, 2, 3$$

The coefficient $c_0(x) \geqslant 0$ is related to the *compressibility* of the fluid as well as the *porosity* of the medium at $x \in \Omega$. It is a measure of the amount of fluid which can be forced into the medium by a unit pressure increment with constant volume. Similarly, the coefficient $k > 0$ involves the *viscosity* of the fluid and the *permeability* of the medium as a measure of the Darcy flow corresponding to a unit pressure gradient. The parameter $\alpha \geqslant 0$ accounts for the mechanical coupling of the fluid pressure and the porous solid. Specifically, the term $\alpha \nabla \cdot \mathbf{u}(x, t)$ represents the additional fluid content due to the *dilation* of the structure, and $\alpha \nabla p(x, t)$ is the additional stress within the structure due to the fluid pressure. The coefficient $\rho(x) \geqslant 0$ is the local *density* of the porous medium, and $\lambda^* \geqslant 0$ is a physical parameter arising in connection with *secondary consolidation* effects (see below).

System (1) has to be complemented with suitable boundary and initial conditions. To this end let us introduce a pair of partitions of the boundary Γ into complementary sets $\{\Gamma_\mathrm{d}, \Gamma_\mathrm{f}\}$ and $\{\Gamma_\mathrm{c}, \Gamma_\mathrm{t}\}$. We shall assume that Γ_c has strictly positive surface measure. Moreover, set $\Gamma_\mathrm{s} \equiv \Gamma_\mathrm{t} \cap \Gamma_\mathrm{f}$ and let the measurable function $\beta : \Gamma_\mathrm{s} \longrightarrow [0, 1]$ be prescribed on this set. We seek a solution of (1) that satisfies the boundary conditions

$$p = 0 \quad \text{on } \Gamma_\mathrm{d} \tag{2a}$$

$$k(\nabla p) \cdot \mathbf{n} - \alpha\beta \frac{\partial}{\partial t}(\mathbf{u} \cdot \mathbf{n}) = 0 \quad \text{on } \Gamma_\mathrm{f} \tag{2b}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_\mathrm{c} \tag{2c}$$

$$\lambda^* \frac{\partial}{\partial t}(\nabla \cdot \mathbf{u})\mathbf{n} + \sigma\mathbf{n} - \alpha(1 - \beta)p\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_\mathrm{t} \tag{2d}$$

Here $\mathbf{n}$ denotes the unit outward normal vector to Γ, and $(\sigma\mathbf{n})_i = \sigma_{ij} n_j$ is the corresponding *normal stress*.[‡] We briefly comment on these boundary conditions. First of all, the fluid is *drained* on the portion Γ_d, and the medium is *clamped* along Γ_c. The relations (2b) and (2d) are constraints on *fluid flux* and *traction*, respectively. On the set Γ_s, where neither p nor $\mathbf{u}$ is

[‡]The convention of summation over repeated indices is assumed.

prescribed, the function β comes into play. This function specifies the fraction of the pores of the medium that are *exposed* along Γ_s. Indeed, for these pores, the motion of the solid adds their contents to the fluid flux through the term $\beta\partial/\partial t(\mathbf{u}\cdot\mathbf{n})$ in (2b). In the remaining portion, the *sealed* pores, the hydraulic pressure contributes to the total stress within the structure, and this is the origin of the normal pressure term $(1-\beta)p\mathbf{n}$ in (2d). Finally, we shall require the solution to satisfy the initial conditions

$$c_0\, p(\cdot,0) = c_0\, p_0(\cdot) \quad \text{on } \Omega \tag{3a}$$

$$\mathbf{u}(\cdot,0) = \mathbf{u}_0(\cdot), \quad \rho\mathbf{u}_t(\cdot,0) = \rho\mathbf{v}_0(\cdot) \text{ on } \Omega \tag{3b}$$

where p_0, $\mathbf{u}_0$, and $\mathbf{v}_0$ are suitably given functions.

The special linear case of the fluid mass conservation (1a) with Darcy's law for laminar flow combined with the momentum balance equation (1b) with Hooke's law for elastic deformation comprises the classical *Biot diffusion-deformation model* of linear poroelasticity. This is based on the concept of *effective stress* due to von Terzaghi [44]. For the mathematical theory of this initial-boundary-value problem for the *fully dynamic* case of (1) with $\rho>0$ in the context of thermoelasticity, see the fundamental work of Dafermos [13]. Further developments are presented in the exhaustive and complementary summaries of Carleson [8] and Kupradze [23]. In the context of strongly elliptic systems, this system was developed by Fichera [16]. There are rather few references to be found for basic theory of even the simplest linear problem for the *coupled quasi-static* case of (1) with $\rho=0$. Among these is the treatment in one spatial dimension in Day [14]. For applications to poroelasticity, see Biot [4–6], Rice and Cleary [30], Zienkiewicz *et al.* [48]. Mathematical issues of model development and well-posedness for the elastic and quasi-static case were first studied in the fundamental work of Auriault and Sanchez-Palencia [2]. This work led to a non-isotropic form of the Biot diffusion–deformation system by homogenization, and they established existence of a unique *strong* solution. See also Burridge–Keller [7] for modelling issues. In the papers of Fichera [16] and Ženišek [46] a *weak* solution is obtained. The existence, uniqueness, and regularity theory for the quasi-static Biot system together with extensions to include the possibility of viscous terms arising from secondary consolidation (see Murad-Cushman [29]) and the introduction of appropriate boundary conditions at both closed and drained interfaces were given in Showalter [36]. Extensions to the *Barenblatt–Biot double-diffusion deformation model* were developed in Showalter–Momken [38]. The model development and proof of existence of a solution with both elastic deformation and partial-saturation were given in Showalter–Su [40,41]. See Reference [37] for a summary of these and additional works, and see Charlez [10], Chen *et al.* [11], Coussy [12], Huyakorn-Pinder [19], Lewis-Sukirman [24], Minkoff *et al.* [27], Mourits-Settari [28], Selvadurai [34], and Zienkiewicz *et al.* [47] for issues of numerical simulation and applications to geomechanics.

1.2. The plan

The objective in the following is to establish a mathematical theory of well-posedness for a *mixed variational formulation* of a substantial generalization of the system (1) with boundary conditions (2) and initial conditions (3). The constitutive relation will include the linear model of poroelasticity as well as a wide class of material deformation small strain models

resulting from a parallel combination of a family of *elasto-visco-plastic* elements. In particular, our model includes *Prandtl–Ihhlinskiĭ* hysteresis relations of *stop-type*. Examples of typical deformation models are described in the following subsection. These include very general rheological materials made up of the parallel combination of elementary components of various types, elastic, viscous, and plastic, with combinations of kinematic and isotropic hardening. Such a construction may require the introduction of *internal variables*.

Let us emphasize from the very beginning the very extensive variety of models included in (1). Moreover, this analysis permits any of the parameters $c_0(\cdot)$, $\rho(\cdot)$ and λ^* to vanish! Specifically, we include in our discussion any combination of the models for *quasi-static* motion, $\rho = 0$, the cases of *incompressible* fluid or solid, $c_0 = 0$, and the *uncoupled* system, $\alpha = 0$. In addition, we shall include in our analysis the quasilinear cases that result from a non-linear permeability $k(\cdot)$ or from a non-linear and degenerate dissipation $\lambda^*(\cdot)$ in the diffusion or momentum equations, respectively.

In Section 2 we construct the operators of deformation, diffusion, dissipation, and the mixed coupling terms that will appear in our generalized system. We also address the measurability issues concerning the family of maximal monotone operators in the constitutive relation. In Section 3 we shall prove the existence and uniqueness of a *strong solution* of the initial-boundary-value problem for this non-linear and degenerate system without any coercive-type assumptions on the dissipation in (1b). Rather, the essential assumptions are restricted to the diffusion term in (1a) and the constitutive relation (1c). The coercivity assumption on the diffusion operator is standard, and a useful sufficient (hardening) condition is given for the coercivity assumption on the constitutive relation.

1.3. The constitutive relation, I

A variety of examples will be given to illustrate the material models that are included in our development, and these include cases of mixed elasto-visco-plastic type with combined kinematic and isotropic hardening and multiple yield surfaces. We assume the material response in the porous solid is determined by a family of such classical models of small-strain elasto-plasticity with hardening. The theory of each of the components is rather completely developed in solid mechanics, and they have a well established mathematical basis in convex analysis. Here we exploit the *dual formulation* which is the more common approach for the development of both the mathematical analysis and computational aspects of elasto-plasticity. In order to describe these, we denote hereafter by Σ the space of *symmetric second-order tensors*. The first and simplest example of the constitutive law (1c) is the classical case of linear elasticity, i.e. the *generalized Hooke's law*, $\mathbb{M}\sigma = \varepsilon$, in which the positive-definite and symmetric fourth-order tensor $\mathbb{M}$ is the *compliance* of the medium. Since only the time derivative of this relation will appear below, it will relate only the *variations* of the stress and strain. More generally, and to introduce a second class of examples, we mention the visco-elastic model of *fading memory type* in the form of an implicit evolution equation[§]

$$\mathbb{M}\dot{\sigma}(t) + \mathbb{L}(\sigma(t)) = \dot{\varepsilon}(t), \quad \sigma(0) = \sigma^0$$

[§]The superscript dot will be used for the derivative with respect to time.

in which the *dissipation* $\mathbb{L}(\cdot)$ is possibly non-linear and multi-valued as well as degenerate. Then the constitutive relation (1c) corresponds to the dependence of the stress $\sigma(\cdot)$ on the *strain rate* $\dot{\varepsilon}(\cdot)$.

We shall include rather general constitutive relations of *hysteresis* type in which the relation $\mathscr{H}(\cdot)$ will account for some possibly *rate-independent* memory effects. Such an example is the classic Prandtl–Reuss *elastic perfectly plastic* model, which we briefly recall from Duvaut–Lions [15]. Assume we are given a non-empty, convex and closed subset $K \subset \Sigma$ of *admissible stresses*, which determines the *yield criterion*. Then the stress–strain relationship is given by the *flow rule*

$$\mathbb{M}\dot{\sigma}(t) + \partial I_K(\sigma(t)) \ni \dot{\varepsilon}(t), \quad \sigma(0) = \sigma^0 \in K \tag{4}$$

Here the multi-valued operator $\partial I_K : \Sigma \longrightarrow 2^{\Sigma}$ corresponds to a *variational inequality*: the relation $\tau \in \partial I_K(\sigma)$ is characterized by

$$\sigma \in K \quad \text{and} \quad \tau_{ij}(\omega - \sigma)_{ij} \leqslant 0, \quad \omega \in K$$

This means that τ vanishes if σ is in the interior of K, and that τ belongs to the *normal cone* when σ is on the boundary of K. (This corresponds to the strain rate of a *rigid perfectly plastic* material.) Thus, the flow rule (4) is formally equivalent to the addition of strain rates corresponding to an elastic component and a perfectly plastic component arranged in series.

We can combine the preceding examples to obtain a model with *kinematic hardening*. Decompose the stress σ into the *back stress* associated with the translation of the yield surface in stress space and a *plastic* component by

$$\sigma = \sigma_{\mathrm{b}} + \sigma_{\mathrm{p}}$$

where σ_{b} satisfies $\mathbb{M}_{\mathrm{b}}\sigma_{\mathrm{b}} = \varepsilon$ and σ_{p} is the solution of the flow rule (4). This is formally equivalent to the addition of stresses due to an elastic component and an elastic perfectly plastic component arranged in parallel; each is independently determined by the strain.

In order to construct a model of *isotropic hardening*, we introduce a parameter $\chi \in \mathbb{R}$ to represent an internal force associated with the size of the (expanding) set of admissible stresses. Thus, let $\mathbb{K}$ be a non-empty, convex, and closed subset of the product space $\Sigma \times \mathbb{R}$, and let the pair σ_{p}, χ be the solution of the system

$$\begin{bmatrix} \mathbb{M}\dot{\sigma}_{\mathrm{p}}(t) \\ \dot{\chi}(t) \end{bmatrix} + \partial I_{\mathbb{K}}\left(\begin{bmatrix} \sigma_{\mathrm{p}}(t) \\ \chi(t) \end{bmatrix}\right) \ni \begin{bmatrix} \dot{\varepsilon}(t) \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \sigma_{\mathrm{p}}(0) \\ \chi(0) \end{bmatrix} = \begin{bmatrix} \sigma_{\mathrm{p}}^0 \\ \chi^0 \end{bmatrix} \in \mathbb{K}$$

The relation $\sigma_{\mathrm{p}} = \mathscr{H}(\varepsilon)$ corresponds to a model of isotropic hardening, and for each $\chi \in \mathbb{R}$, the set of admissible stresses is given by the projection $K(\chi) = \{\sigma \in \Sigma\colon [\sigma, \chi] \in \mathbb{K}\}$. Note that both the input to and the output from this system are only through the first component; the second component is a *hidden variable*. Similar constructions can be used to represent certain *memory functionals* of visco-elasticity.

This is a fundamental model, since each of the preceding examples can be obtained as a special case of parallel sums of this type. For example, if the convex set is a product $\mathbb{K} = K \times \mathbb{R}$, then the system decouples to the elastic perfectly plastic flow rule (4) for the component K, and if additionally we take $K = \Sigma$, then $\partial I_K = 0$ and we have a purely elastic component. Moreover, we can combine these into a model for *combined kinematic and*

isotropic hardening. For such a model, the stress–strain relationship $\sigma = \mathscr{H}(\varepsilon)$ is determined by $\sigma = \sigma_{\mathrm{b}} + \sigma_{\mathrm{p}}$, where the stress components satisfy the system

$$\begin{bmatrix} \mathbb{M}_{\mathrm{b}}\dot{\sigma}_{\mathrm{b}}(t) \\ \mathbb{M}_{\mathrm{p}}\dot{\sigma}_{\mathrm{p}}(t) \\ \dot{\chi}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \partial I_{\mathbb{K}}\left(\begin{bmatrix} \sigma_{\mathrm{p}}(t) \\ \chi(t) \end{bmatrix}\right) \end{bmatrix} \ni \begin{bmatrix} \dot{\varepsilon}(t) \\ \dot{\varepsilon}(t) \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \sigma_{\mathrm{b}}(0) \\ \sigma_{\mathrm{p}}(0) \\ \chi(0) \end{bmatrix} = \begin{bmatrix} \sigma_{\mathrm{b}}^0 \\ \sigma_{\mathrm{p}}^0 \\ \chi^0 \end{bmatrix} \in \Sigma \times \mathbb{K}$$

in which $\mathbb{M}_{\mathrm{b}}$, $\mathbb{M}_{\mathrm{p}}$ are positive-definite fourth-order symmetric tensors. In this example, the input and the output for the system involve only the first and second components. That is, the *observation* and *control* are in the first two components, and the third component is the hidden variable.

More generally, we shall permit the operator $\mathscr{H}(\cdot)$ to include all of these examples in the *Prandtl–Ihhlinskiĭ model of elasto-plasticity* with multi-yield surfaces and kinematic or isotropic strain hardening. In order to specify such a model, assume we are given a measure space $(Y, \mathscr{P}, \mu)$, where μ is a finite Borel measure, a family of fourth-order symmetric positive-definite tensors, $\mathbb{M}_y$, and a family of possibly multi-valued and non-linear operators, $\mathbb{L}_y$, on the space $\Sigma \times \mathbb{R}$, both families parameterized¶ by $y \in Y$. Then, for any tensor-valued $\varepsilon(\cdot) \in H^1(0, T; \Sigma)$ the corresponding effective stress $\sigma(\cdot) \in H^1(0, T; \Sigma)$ is the cumulative output from the family of components defined as

$$\sigma(t) \equiv \int_Y \sigma_y(t)\,\mathrm{d}\mu$$

where

$$\begin{bmatrix} \mathbb{M}_y\dot{\sigma}_y(t) \\ \dot{\chi}_y(t) \end{bmatrix} + \mathbb{L}_y\left(\begin{bmatrix} \sigma_y(t) \\ \chi_y(t) \end{bmatrix}\right) \ni \begin{bmatrix} \dot{\varepsilon}(t) \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \sigma_y(0) \\ \chi_y(0) \end{bmatrix} = \begin{bmatrix} \sigma_y^0 \\ \chi_y^0 \end{bmatrix} \quad \text{a.e. } y \in Y \tag{5}$$

with $\sigma_y(0), \chi_y(0) \in \mathrm{Dom}(\mathbb{L}_y)$ for almost every $y \in Y$. If $\mathbb{L}_y$ is a diagonal operator, the yth equation decouples, and there is no effect from the hidden variable. If $\mathbb{L}_y = 0$ then the yth-component is elastic, and visco-elastic components are realized as indicated by a bounded dissipation function $\mathbb{L}_y$. Plastic components are obtained from a collection of non-empty closed convex sets $\mathbb{K}_y \subset \Sigma \times \mathbb{R}$ by setting $\mathbb{L}_y = \partial I_{\mathbb{K}_y}$, and then system (5) contains a family of variational evolution inequalities. The unknowns in this formulation are the pairs $[\sigma_y(t), \chi_y(t)]$ which are called *generalized stress*. We refer the reader to the references below for an extensive development of such models. We just emphasize here that our hypotheses will imply that the system of evolution equations in (5) has indeed a unique solution, and thus the map $\sigma = \mathscr{H}(\varepsilon)$ is actually well defined.

There is a substantial mathematical literature concerning the *visco-plastic deformation problem* for the momentum equation (1b) (with $\alpha = 0$) combined with the constitutive relation (1c). This covers both the dynamic and the quasistatic cases, both dependent and independent rates.

¶Measurability issues for these operators will be addressed in Section 2.4.

The existence and uniqueness of the *weak* solution for the fundamental Prandtl–Reuss plasticity model (4) with a single yield surface was given by Duvaut–Lions [15]. For a weak solution, the strain-rate is *not* in L^2 but resides in a larger dual space. A strong solution is obtained when the subgradient is replaced by a *bounded* dissipation operator, such as in models of viscoelasticity. The extension to some general Prandtl–Ihhlinskiĭ models (5) with multi-yield surfaces was obtained by Visintin [45]. Here we extend these to allow for hidden variables. An alternative approach is taken in the work of Krejčí [22], where a large class of such general multiple component models is considered. There the dissipation properties of the hysteresis functional are developed and exploited.

The quasi-static case, in which the momentum equation (1b) is replaced by the corresponding static equation with $\rho = 0$, was developed in Johnson [20,21]. There a regularizing effect due to work-hardening of the material appeared, and both weak and strong forms of solutions were obtained. Showalter–Shi [39] obtained three classes of solutions for the case of one spatial dimension. The smoother strong solution with strain rate in L^2 resulted from a boundedness assumption on a non-trivial measurable subset of the subgradients in system (1c), and this assumption arose from a *kinematic work hardening* component in the stress or from the presence of *viscosity*. This shows that each of these characteristics has a regularizing effect. From an additional stability condition relating the convex sets of the plasticity model to the divergence operator, there was obtained a regular solution for which each component of σ is smooth. For a selection of results, see Li–Babŭska [25], Babŭska–Shi [3], Suquet [43], Han–Reddy [17], Simo–Hughes [42], and their references. See Reference [39] for elementary examples.

1.4. Preliminary material

We briefly describe some techniques of convex analysis to be used to construct appropriate operators below. For details, see [17] and [35]. A (possibly multi-valued) *operator* or *relation* $\mathscr{A}$ from a real Hilbert space H to its dual space H' is a collection of related pairs $[x, y] \in H \times H'$ denoted by $y \in \mathscr{A}(x)$; the *domain* $\mathrm{Dom}(\mathscr{A})$ is the set of all such x, and the *range* $\mathrm{Rg}(\mathscr{A})$ consists of all such y. The operator $\mathscr{A}$ is called *monotone* if for all $y_1 \in \mathscr{A}(x_1)$, $y_2 \in \mathscr{A}(x_2)$, we have $\langle y_1 - y_2, x_1 - x_2 \rangle \geqslant 0$. (We use $\langle \cdot, \cdot \rangle$ to denote any duality pairing of functionals with vectors.) If we denote the Riesz map of H onto H' by $\mathscr{R}$, then the monotonicity of $\mathscr{A}$ is equivalent to requiring that $(\mathscr{R} + h\mathscr{A})^{-1}$ be a contraction on $\mathrm{Rg}(\mathscr{R} + h\mathscr{A})$ for every $h > 0$. If, additionally, $\mathrm{Rg}(\mathscr{R} + h\mathscr{A}) = H'$ for some (equivalently, for all) $h > 0$, then we say $\mathscr{A}$ is *maximal monotone*.

Let the function $\varphi : H \to (-\infty, +\infty]$ be convex, proper, and lower-semi-continuous. Then the functional $f \in H'$ is a *subgradient* of φ at $u \in H$ if

$$u \in \mathrm{Dom}(\varphi) \quad \text{and} \quad \langle f, v - u \rangle \leqslant \varphi(v) - \varphi(u), \quad v \in H$$

where $\mathrm{Dom}(\varphi)$ stands for the effective domain. The set of all subgradients of φ at u is denoted by $\partial\varphi(u)$. The subgradient is a generalized notion of the derivative, comparable to a directional derivative. We regard $\partial\varphi$ as a multi-valued operator from H to H'; it is easily shown to be maximal monotone.

If K is a closed, convex, non-empty subset of H, then the *indicator function* $I_K(\cdot)$ of K, defined by $I_K(x) = 0$ if $x \in K$ and $I_K(x) = +\infty$ otherwise, is convex, proper, and lower-semi-

continuous. Its subgradient is characterized by a *variational inequality*: $f \in \partial I_K(x)$ means

$$f \in H', \ x \in K \quad \text{and} \quad \langle f, y - x \rangle \leqslant 0, \quad y \in K$$

The following fundamental existence theorem for degenerate semilinear evolution equations will be used below. See References [35, Theorem IV.6.1, 9, Section 3.6, 18].

Theorem 1.1
Let the linear, symmetric and monotone operator $\mathscr{B}$ be given from the real vector space $\mathbb{E}$ to its algebraic dual $\mathbb{E}^*$, and let $\mathbb{E}'_b$ be the Hilbert space which is the dual of $\mathbb{E}$ with the seminorm

$$|x|_b = \langle \mathscr{B}x, x \rangle^{1/2}, \quad x \in \mathbb{E}$$

Let $\mathscr{A} \subset \mathbb{E} \times \mathbb{E}'_b$ be a relation with domain $\mathrm{Dom}(\mathscr{A}) = \{x \in \mathbb{E}\colon \mathscr{A}(x) \neq \emptyset\}$.

(a) Assume $\mathscr{A}$ is monotone. If w_j is a solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{B}w(t)) + \mathscr{A}(w(t)) \ni f(t), \quad 0 < t < T \tag{6}$$

with data $f_j : [0, T] \to \mathbb{E}'_b$ for $j = 1, 2$, then it follows that

$$|w_1(t) - w_2(t)|_b \leqslant |w_1(0) - w_2(0)|_b + \int_0^t \|f_1(s) - f_2(s)\|_{\mathbb{E}'_b} \, \mathrm{d}s, \quad 0 \leqslant t \leqslant T$$

If $f_1 = f_2$ and if $\mathscr{B}w_1(0) = \mathscr{B}w_2(0)$, then $\mathscr{B}w_1(t) = \mathscr{B}w_2(t)$ for all $0 \leqslant t \leqslant T$. Furthermore, if $\mathscr{B} + \mathscr{A}$ is strictly monotone, then there is at most one solution of the Cauchy problem for (6).

(b) Assume $\mathscr{A}$ is monotone and $Rg(\mathscr{B} + \mathscr{A}) = \mathbb{E}'_b$. Then, for each $w_0 \in \mathrm{Dom}(\mathscr{A})$ and each $f \in W^{1,1}(0, T; \mathbb{E}'_b)$, there is a solution w of (6) with

$$\mathscr{B}w \in W^{1,\infty}(0, T; \mathbb{E}'_b), \quad w(t) \in \mathrm{Dom}(\mathscr{A}) \quad \text{for all } t \in [0, T], \text{ and } \mathscr{B}w(0) = \mathscr{B}w_0$$

(c) Let $\mathscr{A}$ be the subdifferential, $\partial\varphi$, of a convex lower-semi-continuous function $\varphi : \mathbb{E}_b \to [0, +\infty]$ with $\varphi(0) = 0$. Then for each w_0 in the $\mathbb{E}_b$-closure of $\mathrm{Dom}(\varphi)$ and each $f \in L^2(0, T; \mathbb{E}'_b)$ there is a solution w of (6) with

$$\varphi \circ w \in L^1(0, T), \quad \sqrt{t}\,\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{B}w(\cdot) \in L^2(0, T; \mathbb{E}'_b), \quad w(t) \in \mathrm{Dom}(\mathscr{A}), \text{ a.e. } t \in [0, T]$$

and $\mathscr{B}w(0) = \mathscr{B}w_0$. If in addition $w_0 \in \mathrm{Dom}(\varphi)$ then

$$\varphi \circ w \in L^\infty(0, T), \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{B}w \in L^2(0, T; \mathbb{E}'_b)$$

Part (c) will imply that the system (5) is well-posed and, hence, the constitutive relation (1c) is well-defined. Part (b) will be used to show that the initial-boundary-value problem for the full non-linear system is well-posed.

2. VARIATIONAL FORMULATION

We shall start by fixing some notation. As usual we will use bold letters to indicate vectors in $\mathbb{R}^3$ and Greek letters to indicate (symmetric second-order) tensors in Σ. Let Ω be a smoothly

bounded region in $\mathbb{R}^3$, and denote its boundary by $\Gamma = \partial\Omega$. We will make use of the Sobolev spaces $L^2(\Omega), H^1(\Omega), H_0^1(\Omega)$, etc., and the reader is referred to Adams [1], Lions-Magenes [26], or Showalter [35] for definitions and properties. We shall denote the corresponding spaces of vector-valued functions by $\mathbf{L}^2(\Omega) = (L^2(\Omega))^3, \mathbf{H}^1(\Omega) \equiv (H^1(\Omega))^3$, etc. In particular, we will use the notation $(\cdot,\cdot)$ for the scalar product in any of the L^2-type spaces and $\langle\cdot,\cdot\rangle$ for the duality pairing on a space and its dual, possibly including subscripts. Finally, we will indicate topological dual with a prime.

2.1. The diffusion

For the representation of pore fluid pressure, let us introduce the Hilbert space

$$V \equiv \{p \in H^1(\Omega):\ p = 0 \text{ on } \Gamma_\mathrm{d}\}$$

For the construction of the classical linear diffusion operator appearing in the pressure equation (1a), let the coefficient function $k(\cdot) \in L^\infty(\Omega)$ be given with $k(x) \geqslant k_0 > 0$, and define the symmetric monotone linear operator $A : V \longrightarrow V'$ by

$$\langle A p, q\rangle \equiv \int_\Omega k(x)\nabla p(x) \cdot \nabla q(x)\,\mathrm{d}x, \quad p, q \in V$$

The *formal* part is defined to be the restriction of $A(p)$ to $C_0^\infty(\Omega)$, and it is given in $H^{-1}(\Omega)$ by the *elliptic* operator, $A_0 p = -\partial_j(k\partial_j p)$ (in the distributional sense) for $p \in V$. If additionally $A_0 p \in L^2(\Omega)$, and if $k(\cdot)$ is smooth, then the elliptic regularity theory implies that $p \in V \cap H^2(\Omega)$, and then Stokes' theorem yields

$$\langle A p, q\rangle = (A_0 p,\, q)_{L^2(\Omega)} + (k\partial p/\partial n,\, q)_{L^2(\Gamma_\mathrm{f})}, \quad q \in V$$

This provides the decoupling of A into a *formal* part on Ω and a boundary operator corresponding to *flux* on Γ_f, and we denote this representation by

$$A(p) = [A_0(p),\, k\partial p/\partial n] \in L^2(\Omega) \times L^2(\Gamma_\mathrm{f})$$

More generally, we can expect a *non-linear* Darcy law which gives rise to a quasi-linear operator defined as follows. Assume given the functions $A_j : \Omega \times \mathbb{R}^n \to \mathbb{R}$ for $j = 1, \ldots, n$ which satisfy

(i) $A_j(x,\xi)$ is measurable in x for all ξ and continuous in ξ for a.e. $x \in \Omega$,
(ii) $|A_j(x,\xi)| \leqslant C\|\xi\|_{\mathbb{R}^n} + K(x)$, $\xi \in \mathbb{R}^n$, for a.e. $x \in \Omega$,
(iii) $(A_j(x,\xi) - A_j(x,\eta))(\xi_j - \eta_j) \geqslant 0$, $\xi, \eta \in \mathbb{R}^n$, for a.e. $x \in \Omega$,

where $K(\cdot) \in L^2(\Omega)$. The special case of a linear Darcy law for non-homogeneous and non-isotropic material is given in the form $A_j(x,\xi) = k_{ij}(x)\xi_i$ with a positive-definite symmetric matrix $k_{ij}(x)$ at almost every $x \in \Omega$. From (i) and (ii) we define $A : V \to V'$ by

$$\langle A(p), q\rangle = \int_\Omega A_j(x, \nabla p(x))\partial_j q(x)\,\mathrm{d}x, \quad p, q \in V \tag{7}$$

and it is continuous and bounded by $\|A(p)\|_{V'} \leqslant C\|p\|_V + \|K\|_{L^2}\sqrt{n}$; also A is monotone by (iii), hence, it is maximal monotone. The formal part of this operator in $H^{-1}(\Omega)$ is given by the *quasi-linear elliptic operator*

$$A_0(p) = -\partial_j A_j(\cdot, \nabla p(\cdot)), \quad p \in V$$

and the corresponding boundary part on Γ_f is

$$\partial_A(p) = A_j(\cdot, \nabla p(\cdot))n_j$$

This is made precise by the *abstract Green's theorem*, and we recall this construction from [35, Proposition II.5.3, p. 65].

Let $\gamma : V \to B$ denote the *trace operator* that assigns boundary values in $B = \gamma(V) \subset L^2(\Gamma_f)$ to functions from $V \subset H^1(\Omega)$. The kernel of γ is $V_0 = H_0^1(\Omega)$, and this is dense in $L^2(\Omega)$, so we have the continuous inclusions $V_0 \hookrightarrow L^2(\Omega) \hookrightarrow V_0'$. The *annihilator* $V_0^\perp$ of V_0 in V' is isomorphic to the dual space B', and we have continuous and dense inclusions $B \hookrightarrow L^2(\Gamma_f) \hookrightarrow B'$. Thus, if $p \in V$ and if the formal part satisfies $A_0(p) \in L^2(\Omega)$, then $A(p) - A_0(p) \in V_0^\perp$, so this difference determines a unique functional $\partial_A(p) \in B'$ for which we have

$$\langle A(p), q\rangle = (A_0(p), q)_{L^2(\Omega)} + \langle \partial_A(p), \gamma(q)\rangle, \quad p, q \in V$$

Of course, when p is smooth, this boundary operator is given as above in $L^2(\Gamma_f)$.

Also, we introduce the functional $h \in L^2(0, T; L^2(\Omega))$ as

$$\langle h(t), q\rangle \equiv \int_\Omega c_0^{1/2} h_0(x,t)\, q(x)\, \mathrm{d}x, \quad q \in V$$

The latter functional is well defined for any $h_0 \in L^2(0, T; L^2(\Omega))$.

2.2. The deformation

For the representation of displacements and velocities of the porous structure, we introduce the Hilbert space

$$\mathbf{V} \equiv \{\mathbf{v} \in \mathbf{H}^1(\Omega)\colon\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_c\}$$

endowed with the scalar product given by the bilinear form

$$e(\mathbf{u}, \mathbf{v}) \equiv \int_\Omega \partial_j u_i \partial_j v_i \, \mathrm{d}x, \quad \mathbf{u}, \mathbf{v} \in \mathbf{V}$$

It will be assumed that Γ_c has a strictly positive measure, so it follows from *Korn's inequality* that this form is $\mathbf{H}^1(\Omega)$-coercive (see, e.g. Reference [15, Theorem 3.1, p. 110]). The vector-valued trace operator is likewise denoted by γ, and its kernel is $\mathbf{V}_0 = \mathbf{H}_0^1(\Omega)$. The annihilator $\mathbf{V}_0^\perp$ in $\mathbf{V}'$ is isomorphic to the dual $\mathbf{B}'$ of the space of boundary values $\mathbf{B} = \gamma(\mathbf{V})$, i.e. the range of the trace operator, and we have the continuous and dense inclusions $\mathbf{B} \hookrightarrow \mathbf{L}^2(\Gamma_t) \hookrightarrow \mathbf{B}'$ as before.

Next, we introduce the Lebesgue space of square-summable symmetric second-order tensors $L^2(\Omega, \Sigma)$ with the usual scalar product,

$$(\sigma, \tau) \equiv \int_\Omega \sigma(x) : \tau(x)\, \mathrm{d}x = \int_\Omega \sigma_{ij}(x)\tau_{ij}(x)\, \mathrm{d}x, \quad \sigma, \tau \in L^2(\Omega, \Sigma)$$

and identify it with its dual. The linearized *strain operator* $\varepsilon : \mathbf{V} \longrightarrow L^2(\Omega, \Sigma)$ was defined above, and it is straightforward to check that, for any pair $\mathbf{u}, \mathbf{v} \in \mathbf{V}$, one has

$$e(\mathbf{u}, \mathbf{v}) = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))$$

Denote by $\varepsilon' : L^2(\Omega, \Sigma) \longrightarrow \mathbf{V}'$ the indicated dual operator,

$$\langle \varepsilon' \sigma, \mathbf{v} \rangle \equiv \int_\Omega \sigma : \varepsilon(\mathbf{v})\, \mathrm{d}x = \int_\Omega \sigma_{ij} \partial_j v_i \, \mathrm{d}x, \quad \sigma \in L^2(\Omega, \Sigma), \ \mathbf{v} \in \mathbf{V}$$

This operator has a *formal* part that is defined as above by the restriction to $(C_0^\infty(\Omega))^3$, and it is given in $\mathbf{H}^{-1}(\Omega)$ by the *vector divergence*, $\varepsilon_0' \sigma = -\nabla \cdot \sigma$ for $\sigma \in L^2(\Omega, \Sigma)$. If we have $\sigma \in L^2(\Omega, \Sigma)$ and additionally $\varepsilon_0' \sigma \in \mathbf{L}^2(\Omega)$, then as above there is a functional $\sigma(\mathbf{n}) \in \mathbf{B}'$ for which

$$\langle \varepsilon' \sigma, \mathbf{v} \rangle = (-\nabla \cdot \sigma, \mathbf{v})_{\mathbf{L}^2(\Omega)} + \langle \sigma(\mathbf{n}), \gamma(\mathbf{v}) \rangle, \quad \mathbf{v} \in \mathbf{V}$$

When σ is smooth, we have from Stokes' theorem that this functional is given by $\sigma(\mathbf{n}) = \sigma \cdot \mathbf{n} \in \mathbf{L}^2(\Gamma_\mathrm{t})$. This displays the decoupling of ε' into a *formal* part on Ω and a boundary operator on Γ_t.

Assume that we are given a function $\lambda^* : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

(i) $\lambda^*(x, \xi)$ is measurable in x for all ξ and continuous in ξ for a.e. $x \in \Omega$,
(ii) $|\lambda^*(x, \xi)| \leqslant C|\xi| + K(s)$, $\xi \in \mathbb{R}$, for a.e. $x \in \Omega$,
(iii) $(\lambda^*(x, \xi) - \lambda^*(x, \eta))(\xi - \eta) \geqslant 0$, $\xi, \eta \in \mathbb{R}$, for a.e. $x \in \Omega$.

From (i) and (ii) we can define the *dilation operator* $D : \mathbf{V} \longrightarrow \mathbf{V}'$ by

$$\langle D\mathbf{u}, \mathbf{v} \rangle \equiv \int_\Omega \lambda^*(x, \partial_i u_i) \partial_j v_j \, \mathrm{d}x = \int_\Omega \lambda^*(x, \varepsilon(\mathbf{u})_{ii}) \varepsilon(\mathbf{v})_{jj} \, \mathrm{d}x, \quad \mathbf{u}, \mathbf{v} \in \mathbf{V} \tag{8}$$

and note that it can likewise be decomposed into formal and boundary parts if $\nabla \cdot \mathbf{u}$ is appropriately smooth. The linear dilation operator appearing in (1b) is obtained by specializing λ^* to be a non-negative constant. We stress that the operator D is monotone and continuous, hence, maximal monotone, but it degenerates on the subspace $\{\mathbf{u} \in \mathbf{V} : \varepsilon(\mathbf{u}) : 1 = 0\} \subset \mathbf{V}$.

Next, let us define the functional $\mathbf{g}(\cdot) \in L^2(0, T; \mathbf{L}^2(\Omega))$ as

$$\langle \mathbf{g}(t), \mathbf{v} \rangle \equiv \int_\Omega \rho^{1/2} \mathbf{g}_0(x, t) \cdot \mathbf{v}(x) \, \mathrm{d}x, \quad \mathbf{v} \in \mathbf{V}$$

Once again, it suffices to ask for $\mathbf{g}_0 \in L^2(0, T; \mathbf{L}^2(\Omega))$ in order to ensure that $\mathbf{g} \in L^2(0, T; \mathbf{L}^2(\Omega))$.

2.3. Coupling terms

We construct the operators corresponding to the coupling between (1a) and (1b). Denote by Γ_s that portion of the boundary on which neither pressure nor displacement is specified, i.e. $\Gamma_\mathrm{s} = \Gamma_\mathrm{t} \cap \Gamma_\mathrm{f}$. Let the function $\beta(\cdot) \in L^\infty(\Gamma_\mathrm{s})$ be given; we shall assume that $0 \leqslant \beta(\cdot) \leqslant 1$. The trace map gives a natural identification $\mathbf{v} \mapsto [\mathbf{v}, \gamma(\mathbf{v})|_{\Gamma_\mathrm{s}}]$ of

$$\mathbf{V} \subset \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Gamma_\mathrm{s})$$

and this identification will be employed throughout the following. It also gives the identification $p \mapsto [p, \gamma(p)|_{\Gamma_\mathrm{s}}]$ of

$$V \subset L^2(\Omega) \times L^2(\Gamma_\mathrm{s})$$

Note that both of these identifications have dense range, and so the corresponding duals can be identified. That is, we have

$$\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Gamma_s)\subset\mathbf{V}',\quad L^2(\Omega)\times L^2(\Gamma_s)\subset V'$$

These density conditions result from the respective requirements $\Gamma_s\subset\Gamma_t$ and $\Gamma_s\subset\Gamma_f$.

We define the divergence operator

$$\vec{\nabla}\cdot\,:\mathbf{V}\to L^2(\Omega)\times L^2(\Gamma_s)$$

consisting of a *formal* part in Ω as well as a *boundary* part on Γ_s. The part in $L^2(\Omega)$ is the usual divergence given by $\nabla\cdot\mathbf{v}=\partial_j v_j$, and the full operator is indicated by

$$\vec{\nabla}\cdot\mathbf{v}=[\nabla\cdot\mathbf{v},-\beta\mathbf{v}\cdot\mathbf{n}]\in L^2(\Omega)\times L^2(\Gamma_s),\quad \mathbf{v}\in\mathbf{V} \tag{9}$$

Then define the *gradient* operator

$$\vec{\nabla}:L^2(\Omega)\times L^2(\Gamma_s)\to\mathbf{V}'$$

to be the negative of the corresponding dual operator. This is given by

$$\langle\vec{\nabla}[p,q],\mathbf{v}\rangle\equiv-\langle[p,q],\vec{\nabla}\cdot\mathbf{v}\rangle=-\int_\Omega p\,\nabla\cdot\mathbf{v}\,\mathrm{d}x+\int_{\Gamma_s}\beta\,q\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}s$$

$$[p,q]\in L^2(\Omega)\times L^2(\Gamma_s),\ \mathbf{v}\in\mathbf{V}$$

For the smoother functions $p\in V\subset L^2(\Omega)\times L^2(\Gamma_s)$, we obtain from *Stokes' Formula*

$$\langle\vec{\nabla}p,\mathbf{v}\rangle=\int_\Omega\partial_j p\,v_j\,\mathrm{d}x-\int_{\Gamma_s}(1-\beta)pn_jv_j\,\mathrm{d}s\quad p\in V,\ \mathbf{v}\in\mathbf{V}$$

This consists of the usual gradient ∇p in Ω and the boundary part $-(1-\beta)p\mathbf{n}$ on Γ_s, and we denote this representation by

$$\vec{\nabla}p=[\nabla p,\,-(1-\beta)p\mathbf{n}]\in\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Gamma_s),\quad p\in V \tag{10}$$

2.4. The constitutive relation, II

We shall realize system (5) in a form distributed over the measure space Y. Thus, let $(Y,\mathscr{P},\mu)$ be a measure space with finite Borel measure μ. Assume that $\{\mathbb{M}_y\}_{y\in Y}$ is a family of symmetric fourth-order tensors which is uniformly bounded and uniformly positive-definite on Σ. Furthermore, we assume that this family is μ-measurable. By this we mean that for each pair, $\sigma_1,\sigma_2\in\Sigma$ the map $y\mapsto\langle\mathbb{M}_y\sigma_1,\sigma_2\rangle$ is measurable from Y to $\mathbb{R}$. It follows that for each measurable function $\sigma(\cdot)$ from Y to Σ the composite function $y\mapsto\mathbb{M}_y\sigma(y)$ is likewise measurable. See Reference [35, pp. 103–108] for details. Define a bounded invertible symmetric and monotone operator $\mathbb{M}$ on $L^2(Y,\Sigma\times\mathbb{R}^m)$ by means of

$$\mathbb{M}(\tau)(y)=[\mathbb{M}_y\sigma(y),\chi(y)],\ \text{for a.e. } y\in Y,\quad \tau=[\sigma(\cdot),\ \chi(\cdot)]\in L^2(Y,\Sigma\times\mathbb{R}^m)$$

We denote by Π the *product space* $\Sigma\times\mathbb{R}^m$ for integer $m\geqslant 0$, with the understanding that $\mathbb{R}^0=\{0\}$. Let $\{\mathbb{L}_y\}_{y\in Y}$ be a family of maximal monotone operators on the product space Π

with $L_y(0) \ni 0$ for almost every $y \in Y$. Assume additionally that for each $h > 0$ and $\tau \in \Pi$ the map $y \mapsto (I + h\mathbb{L}_y)^{-1}(\tau)$ is measurable from Y to Π. Then, for each $\tau(\cdot) \in L^2(Y, \Pi)$, it follows that the composite map $y \mapsto (I + h\mathbb{L}_y)^{-1}(\tau(y))$ is measurable and belongs to $L^2(Y, \Pi)$, and we have the estimate $\|(I + h\mathbb{L}_{(\cdot)})^{-1}(\tau(\cdot))\|_{L^2(Y,\Pi)} \leqslant \|\tau(\cdot)\|_{L^2(Y,\Pi)}$. Now define similarly the corresponding distributed operator $\mathbb{L}$ on $L^2(Y, \Pi)$, $\Pi = \Sigma \times \mathbb{R}^m$, by

$$\mathbb{L}(\tau)(y) = \mathbb{L}_y([\sigma(y), \chi(y)]), \text{ for a.e. } y \in Y, \quad \tau(\cdot) = [\sigma(\cdot), \chi(\cdot)] \in L^2(Y, \Pi)$$

That is, $f \in \mathbb{L}(\tau)$ means that $f, \tau \in L^2(Y, \Pi)$ and they satisfy $f(y) \in \mathbb{L}_y(\tau(y))$ for a.e. $y \in Y$. It follows that $\mathbb{L}(\cdot)$ is maximal monotone on $L^2(Y, \Pi)$.

Finally, we denote by $\iota : \Sigma \to L^2(Y, \Sigma \times \mathbb{R}^m)$ the realization of the tensors in Σ as constant Π-valued functions on Y, i.e. $\iota(\sigma)(y) = [\sigma, 0]$, $y \in Y$. The dual operator is given by the integral,

$$\iota'(\tau) = \int_Y \sigma(y)\, \mathrm{d}\mu_y \in \Sigma, \quad \tau = [\sigma(\cdot), \chi(\cdot)] \in L^2(Y, \Pi)$$

With these operators, the constitutive relation (5) may be rewritten and made precise as

$$\begin{gathered} \sigma(t) = \iota'(\tau(t)) \text{ in } \Sigma, \text{ for a.e. } t \in (0, T), \quad \tau(0) = \tau^0 \text{ in } L^2(Y, \Pi) \\ \mathbb{M}\dot{\tau}(t) + \mathbb{L}(\tau(t)) \ni \iota(\dot{\varepsilon}(t)) \text{ in } L^2(Y, \Pi), \quad \text{for a.e. } t \in (0, T) \end{gathered} \tag{11}$$

Note also that $\mathbb{M}(\cdot)$ is diagonal, so any coupling between the components can come only from $\mathbb{L}(\cdot)$.

We shall need to restrict attention to the case in which $\mathbb{L}(\cdot)$ is a subgradient.

Definition 2.1
For each $y \in Y$, let the function $\varphi(y, \cdot) : \Pi \to [0, +\infty]$ be convex and lower-semi-continuous, with $\varphi(y, 0) = 0$. The function $\varphi : Y \times \Pi \to [0, +\infty]$ is called a *normal integrand* if there exists a countable collection M of measurable functions from Y to Π such that $y \mapsto \varphi(y, m(y))$ is measurable for every $m \in M$ and the set $M(y) = \{m(y) : m \in M\}$ satisfies that $M(y) \cap \mathrm{Dom}(\varphi(y, \cdot))$ is dense in the set $\mathrm{Dom}(\varphi(y, \cdot)) = \{\tau \in \Pi : \varphi(y, \tau) < +\infty\}$ for every $y \in Y$.

It follows, that, whenever φ is a normal integrand, $y \mapsto \varphi(y, \tau(y))$ is a measurable function for *every* measurable function $\tau : Y \to \Pi$, and so we can define

$$\hat{\varphi}(\tau) = \int_Y \varphi(y, \tau(y))\, \mathrm{d}\mu, \quad \tau \in L^2(Y, \Pi) \tag{12}$$

which is again a convex and lower-semi-continuous function. When the *interior* of $\mathrm{Dom}(\varphi(y, \cdot))$ is non-empty for every $y \in Y$, $\varphi(\cdot, \cdot)$ is a normal integrand if and only if the function $y \mapsto \varphi(y, \tau)$ is measurable for each $\tau \in \Pi$. Also note that $\varphi(\cdot, \cdot)$ is a normal integrand if it is independent of $y \in Y$. This holds more generally in the *discrete* situation where the measure $\mathrm{d}\mu$ assigns the mass $\mu_y > 0$ to each point $y \in Y$, and then the integral in (12) is a series

$$\hat{\varphi}(\tau) = \sum_{y \in Y} \varphi(y, \tau(y))\mu_y, \quad \tau \in L^2(Y, \Pi)$$

See Rockafellar [31–33] for these and additional issues of measurability.

Assume that the normal integrand $\varphi(\cdot,\cdot)$ has been given. For each $y \in Y$, we denote the subgradient of $\varphi(y,\cdot)$ at the point $\tau \in \Pi$ by $\mathbb{L}_y(\tau) = \partial\varphi(y,\tau)$.

Lemma 2.2
For each $h > 0$ and $\tau \in \Pi$, the map $y \mapsto (I + h\mathbb{L}_y)^{-1}(\tau)$ is measurable from Y to Π.

Proof
For each $y \in Y$ and $h > 0$, consider the *Yosida* approximation

$$\varphi^h(y,\tau) = \inf_{\xi \in \Pi} \left\{ \frac{1}{2h}\|\xi - \tau\|^2 + \varphi(y,\xi) \right\}$$

Each of these convex functions $\varphi^h(y,\cdot)$ is *Fréchet differentiable*, and we denote the derivative by $\mathbb{L}_y^h(\cdot) = \partial\varphi^h(y,\cdot)$. Each of the maps $y \mapsto \mathbb{L}_y^h(\tau)$ is measurable, since it is the limit of measurable functions, and these are related to the resolvents of $\mathbb{L}_y(\cdot)$ by

$$\mathbb{L}_y^h(\tau) = 1/h(I - (I + h\mathbb{L}_y)^{-1})(\tau)$$

Thus, each resolvent map $y \mapsto (I + h\mathbb{L}_y)^{-1}(\tau)$ is measurable. □

Corollary 2.3
The subgradient $\mathbb{L} = \partial\hat{\varphi}$ of the convex function (12) in $L^2(Y,\Pi)$ is given by the distributed operator

$$\mathbb{L}(\tau)(y) = \mathbb{L}_y(\tau(y)), \quad \text{a.e. } y \in Y$$

Proof
This is now a straightforward exercise, since we have established the measurability of the various operators. See Reference [35, Section II.8], for example. □

Now it follows from Theorem 1.1 that for each $\tau_0 \in \mathrm{Dom}(\mathbb{L})$ and each $\varepsilon \in H^1(0,T;\Sigma)$, the implicit subgradient evolution equation (11) has a unique solution with

$$\tau \in H^1(0,T;L^2(Y,\Pi)), \quad \tau(t) \in \mathrm{Dom}(\mathbb{L}) \text{ for a.e. } t \in [0,T], \text{ and } \tau(0) = \tau^0$$

and so we have $\sigma = \mathscr{H}(\varepsilon) \in H^1(0,T;\Sigma)$. Note that we needed for $\mathbb{L}$ to be a subgradient because we only have the right side $\imath(\dot{\varepsilon}) \in L^2(0,T;L^2(Y,\Pi))$. Moreover, we needed to introduce the *degenerate* operator $\imath$ in order to include *hidden variables* χ_y for the case of *isotropic strain hardening* in plasticity models or the representation of *memory functionals* in viscosity models.

3. THE MAIN RESULT

3.1. The system

It is now straightforward to check that the variational formulation of the system of partial differential equations (1) and boundary conditions (2) takes the form

$$p(t) \in V: \quad c_0\dot{p}(t) + \alpha\vec{\nabla}\cdot\mathbf{v}(t) + A(p(t)) = h(t) \text{ in } V' \tag{13a}$$

$$\mathbf{v}(t)\in\mathbf{V}: \quad \rho\dot{\mathbf{v}}(t)+D(\mathbf{v}(t))+\varepsilon'\iota'\tau(t)+\alpha\vec{\nabla}p(t)=\mathbf{g}(t) \text{ in } \mathbf{V}' \tag{13b}$$

$$\tau(t)\in L^2(\Omega\times Y,\Pi): \quad \mathbb{M}\dot{\tau}(t)+\mathbb{L}(\tau(t))\ni\iota\varepsilon(\mathbf{v}(t)) \text{ in } L^2(\Omega\times Y,\Pi) \tag{13c}$$

and then we recover displacement $\mathbf{u}(t)$ from $\dot{\mathbf{u}}(t)\equiv\mathbf{v}(t)$ with $\mathbf{u}(0)=\mathbf{u}_0$ and stress $\sigma(t)$ from $\sigma(t)=\iota'\tau(t)$. We intend to reformulate system (13) as an *implicit evolution equation*. Indeed, letting $w(t)\equiv[p(t),\mathbf{v}(t),\tau(t)]$, $f(t)\equiv[h(t),\mathbf{g}(t),0]$, it is a standard matter to check that the system may be rewritten in the form (6), where the operators $\mathscr{A},\mathscr{B}:\mathbb{E}\equiv V\times\mathbf{V}\times L^2(\Omega\times Y,\Sigma\times\mathbb{R}^m)\longrightarrow\mathbb{E}'=V'\times\mathbf{V}'\times L^2(\Omega\times Y,\Sigma\times\mathbb{R}^m)$ are given by

$$\mathscr{B}=\begin{pmatrix} c_0 & 0 & 0\\ 0 & \rho & 0\\ 0 & 0 & \mathbb{M}\end{pmatrix}, \quad \mathscr{A}=\begin{pmatrix} A & \alpha\vec{\nabla}\cdot & 0\\ \alpha\vec{\nabla} & D & \varepsilon'\iota'\\ 0 & -\iota\varepsilon & \mathbb{L}\end{pmatrix}$$

Moreover, we find that the dual of the space $\mathbb{E}$ with the seminorm determined by $\mathscr{B}$ is given by

$$\mathbb{E}'_b=c_0^{1/2}L^2(\Omega)\times\rho^{1/2}\mathbf{L}^2(\Omega)\times L^2(\Omega\times Y,\Sigma\times\mathbb{R}^m)$$

Of course, the initial conditions (3) are rewritten as $w(0)=w_0$ in $\mathbb{E}_b$, i.e.

$$\begin{gathered}\lim_{t\to 0^+}c_0^{1/2}p(t)=c_0^{1/2}p_0 \text{ in } L^2(\Omega), \quad \lim_{t\to 0^+}\rho^{1/2}\mathbf{v}(t)=\rho^{1/2}\mathbf{v}_0 \text{ in } \mathbf{L}^2(\Omega)\\ \text{and} \quad \tau(0)=\tau_0 \text{ in } L^2(\Omega\times Y,\Sigma\times\mathbb{R}^m)\end{gathered} \tag{14}$$

We are now in position to state our main result.

Theorem 3.1

Assume that we are given the spaces V and $\mathbf{V}$, with Γ_c having strictly positive surface measure, and that the operators of strain $\varepsilon:\mathbf{V}\longrightarrow L^2(\Omega,\Sigma)$, divergence $\vec{\nabla}\cdot:\mathbf{V}\to L^2(\Omega)\times L^2(\Gamma_s)$, gradient $\vec{\nabla}:L^2(\Omega)\times L^2(\Gamma_s)\to\mathbf{V}'$, the maximal monotone *diffusion operator* $A:V\longrightarrow V'$, and the maximal monotone *dilation operator* $D:\mathbf{V}\longrightarrow\mathbf{V}'$ are given as in Section 2. In addition, let $(Y,\mathscr{P},\mu)$ be a finite Borel measure space, let $\{\mathbb{M}_y\}_{y\in Y}$ be a measurable uniformly bounded family of symmetric positive-definite fourth-order tensors, and define $\mathbb{M}$ on $L^2(Y,\Pi)$ pointwise as above, where $\Pi=\Sigma\times\mathbb{R}^m$. Let $\varphi:Y\times\Pi\to[0,+\infty]$ be a *normal integrand* with $\varphi(y,0)=0$, and set $\mathbb{L}_y=\partial\varphi(y,\cdot)$ for a.e. $y\in Y$, the indicated subgradients on Π. The corresponding subgradient $\partial\hat{\varphi}(\cdot)$ of (12) on $L^2(Y,\Pi)$ is given by $\mathbb{L}(\tau)(y)=\mathbb{L}_y(\tau(y))$ for a.e. $y\in Y$. Denote by $\iota:\Sigma\to L^2(Y,\Pi)$ the realization of a tensor $\sigma\in\Sigma$ as a constant function $[\sigma,\mathbf{0}]\in\Pi$ on Y.

Assume the following:

A1: The coefficient functions $c_0(\cdot)$ and $\rho(\cdot)$ are bounded, measurable, and non-negative on Ω.

A2: The tensors $\{\mathbb{M}_y\}_{y\in Y}$ are uniformly positive-definite, i.e. for some $m_0>0$, we have

$$(\mathbb{M}_y\xi,\xi)\geqslant m_0|\xi|^2 \quad \text{for all } \xi\in\Sigma, \text{ for a.e. } y\in Y$$

A3: The operator $c_0I+A:V\to V'$ is V-coercive.

A4: The Ω-distributed operator $\iota'(\mathbb{M}+\mathbb{L})^{-1}\iota$ is $L^2(\Omega,\Sigma)$-coercive.

In addition to these structural assumptions, we assume the given data satisfies

$$p_0 \in V, \quad \mathbf{v}_0 \in \mathbf{V}, \quad \tau_0 \in L^2(\Omega \times Y, \Pi) \text{ such that} \tag{15a}$$

$$\alpha \vec{\nabla} \cdot \mathbf{v}_0 + A(p_0) \in c_0^{1/2} L^2(\Omega), \quad D(\mathbf{v}_0) + \varepsilon' \imath' \tau_0 + \alpha \vec{\nabla} p_0 \in \rho^{1/2} \mathbf{L}^2(\Omega) \tag{15b}$$

$$\mathbb{L}(\tau_0) - \imath \varepsilon(\mathbf{v}_0) \in L^2(\Omega \times Y, \Pi), \quad \text{and that} \tag{15c}$$

$$h(\cdot) \in W^{1,1}(0, T; c_0^{1/2} L^2(\Omega)), \quad \mathbf{g}(\cdot) \in W^{1,1}(0, T; \rho^{1/2} \mathbf{L}^2(\Omega)) \tag{15d}$$

Then there exists a triplet of functions

$$p(\cdot) : [0, T] \to V, \quad \mathbf{v}(\cdot) : [0, T] \to \mathbf{V} \quad \text{and} \quad \tau(\cdot) : [0, T] \to L^2(\Omega \times Y, \Pi)$$

such that

$$c_0^{1/2} p(\cdot) \in W^{1,\infty}(0, T; L^2(\Omega)), \quad \rho^{1/2} \mathbf{v}(\cdot) \in W^{1,\infty}(0, T; \mathbf{L}^2(\Omega))$$
$$\tau(\cdot) \in W^{1,\infty}(0, T; L^2(\Omega \times Y, \Pi))$$

the system of evolution equations (13) is satisfied almost everywhere in $(0, T)$, and the initial conditions (14) hold.

Assume A1, A2, and the following:

A5: The operator $c_0 I + A : V \to V'$ is strictly monotone.
A6: The Ω-distributed operator $\imath'(\mathbb{M} + \mathbb{L})^{-1} \imath$ is strictly monotone on $L^2(\Omega, \Sigma)$.

Then there is at most one such solution of system (13) subject to (14).

Remark 3.1
The assumption A1 permits important special degenerate cases. Specifically, if $c_0(\cdot) = 0$, the fluid and solid are *incompressible*. If $\rho(\cdot) = 0$, the momentum equation is *quasi-static*. Also, the operators D and $\{\mathbb{L}_y\}_{y \in Y}$ may be degenerate, and if $\alpha = 0$ the fluid flow equation is decoupled from the deformation system.

Remark 3.2
The assumption A2 implies that $\mathbb{M} + \mathbb{L}$ is *strongly monotone*, hence *coercive*. Since $\mathbb{M} + \mathbb{L}$ is maximal monotone on $L^2(Y, \Pi)$, it is necessarily onto $L^2(Y, \Pi)$, and $(\mathbb{M} + \mathbb{L})^{-1}$ is Lipschitz continuous. The same holds for the corresponding pair of Ω-distributed operators on $L^2(\Omega \times Y, \Pi)$.

For the combined kinematic and isotropic hardening model, the assumption A4 means the yield surface in $\Sigma \times \mathbb{R}^m$ is *not* perpendicular to the Σ-axis. In this case, this basic plasticity example also satisfies the *safe load condition* of Johnson [20,21] and that of Han–Reddy [17]. A direct comparison of their assumptions with A4 is not obvious in the general situation. However, in the special case of a single element in one spatial dimension, A4 requires less than the condition of Han–Reddy. In particular, we do *not* require that the stress operator be strongly monotone. A convenient condition sufficient for A4 is given in the following Proposition, and it shows that the more general assumption A4 is fulfilled in the situation of Reference [39]. This last assumption means that there is a non-trivial stress component which is either viscous or elastic.

Proposition 3.2
Assume that there is a subset $Y_0 \subset Y$ with $\mu(Y_0) > 0$ on which the operators $\{\mathbb{L}_y\}_{y \in Y_0}$ are *uniformly linearly bounded*, i.e. they satisfy

$$\int_{Y_0} \|[\sigma, \psi]\|_{\Sigma \times \mathbb{R}^m} \, \mathrm{d}\mu \leqslant C_0 \int_{Y_0} \|[\tau, \chi]\|_{\Sigma \times \mathbb{R}^m} \, \mathrm{d}\mu \quad [\sigma, \psi] \in \mathbb{L}([\tau, \chi])$$

Then the assumption A4 holds, i.e. the Ω-distributed operator $\imath'(\mathbb{M} + \mathbb{L})^{-1}\imath$ is coercive on $L^2(\Omega, \Sigma)$.

Proof
To see this, let $(\mathbb{M} + \mathbb{L})([\tau, \chi]) \ni \imath\xi, \ \xi \in \Sigma$. Since $\mathbb{L}$ is monotone, we have

$$\begin{aligned}(\imath'([\tau, \chi]), \xi)_\Sigma &= ([\tau, \chi], \imath(\xi))_{L^2(Y, \Sigma \times \mathbb{R}^m)} \\ &\geqslant ([\tau, \chi], \mathbb{M}([\tau, \chi]))_{L^2(Y, \Sigma \times \mathbb{R}^m)} \geqslant m_0 \|[\tau, \chi]\|^2_{L^2(Y_0, \Sigma \times \mathbb{R}^m)}\end{aligned}$$

Let $[\sigma, \psi] = \imath\xi - \mathbb{M}([\tau, \chi]) \in \mathbb{L}([\tau, \chi])$, so we have

$$\begin{aligned}\mu(Y_0)|\xi|^2_\Sigma &= \int_{Y_0} \|(\mathbb{M}([\tau, \chi])(y) + [\sigma(y), \psi(y)])\|^2_{\Sigma \times \mathbb{R}^m} \, \mathrm{d}\mu \\ &\leqslant (M_0 + C_0)^2 \|[\tau, \chi]\|^2_{L^2(Y_0, \Sigma \times \mathbb{R}^m)}\end{aligned}$$

where $M_0 \geqslant 1$ is the uniform bound on the $\mathbb{M}_y$, and consequently

$$(\imath'([\tau, \chi]), \xi)_\Sigma \geqslant m_0 \|[\tau, \chi]\|^2_{L^2(Y_0, \Sigma \times \mathbb{R}^m)} \geqslant \frac{m_0 \mu(Y_0)}{(M_0 + C_0)^2} |\xi|^2_\Sigma$$

This shows that $\imath'(\mathbb{M} + \mathbb{L})^{-1}\imath$ is coercive. □

By the same calculations on differences, we obtain the following result, which provides a useful condition for uniqueness.

Proposition 3.3
Assume that there is a subset $Y_0 \subset Y$ with $\mu(Y_0) > 0$ on which the operators $\{\mathbb{L}_y\}_{y \in Y_0}$ are *single valued*. Then the assumption A6 holds, i.e. the Ω-distributed operator $\imath'(\mathbb{M} + \mathbb{L})^{-1}\imath$ is strictly monotone on $L^2(\Omega, \Sigma)$.

3.2. Proof of Theorem 3.1

Proof
In order to prove the existence claim of Theorem 3.1, it suffices to show that the abstract Cauchy problem for (6) has a solution $w : [0, T] \to \mathbb{E}$ with $\mathscr{B}w(\cdot) \in W^{1,\infty}(0, T; \mathbb{E}'_b)$ for any given $f \in W^{1,1}(0, T; \mathbb{E}'_b)$, where the operators and spaces are given as in Section 3.1. For this, we apply Theorem 1.1 to the semilinear Cauchy problem for system (13). Of course it is straightforward to check that $\mathscr{B}$ is linear symmetric and monotone and that $\mathscr{A}$ is monotone.

In order to establish the existence of a solution, it suffices to prove that $\mathrm{Rg}(\mathscr{B}+\mathscr{A})=\mathbb{E}'_b$. This range condition requires that, for each $f\in\mathbb{E}'_b$, there exists a solution $w=[p,\mathbf{v},\tau]\in\mathbb{E}$ to the *resolvent equation*

$$(\mathscr{B}+\mathscr{A})w\ni f \tag{16}$$

That is, we want to solve the stationary system

$$p\in V:\quad c_0 p+A(p)+\alpha\vec{\nabla}\cdot\mathbf{v}=c_0^{1/2}h_0 \text{ in } V' \tag{17a}$$

$$\mathbf{v}\in\mathbf{V}:\quad \rho\mathbf{v}+D(\mathbf{v})+\varepsilon'\imath'\tau+\alpha\vec{\nabla}p=\rho^{1/2}\mathbf{g}_0 \text{ in } \mathbf{V}' \tag{17b}$$

$$\tau\in L^2(\Omega\times Y,\Pi):\quad \mathbb{M}\tau+\mathbb{L}(\tau)-\imath\varepsilon(\mathbf{v})\ni\sigma_0 \text{ in } L^2(\Omega\times Y,\Pi) \tag{17c}$$

with $h_0\in L^2(\Omega)$, $\mathbf{g}_0\in\mathbf{L}^2(\Omega)$ and $\sigma_0\in L^2(\Omega\times Y,\Pi)$ given. Let us now exploit Remark 3.2 and write the last line (17c) as

$$\tau=(\mathbb{M}+\mathbb{L})^{-1}(\imath\varepsilon(\mathbf{v})+\sigma_0)$$

Substitute this into (17b) to obtain the equivalent system

$$p\in V:\quad c_0 p+Ap+\alpha\vec{\nabla}\cdot\mathbf{v}=c_0^{1/2}h_0 \text{ in } V' \tag{18a}$$

$$\mathbf{v}\in\mathbf{V}:\quad \rho\mathbf{v}+D\mathbf{v}+\varepsilon'\imath'(\mathbb{M}+\mathbb{L})^{-1}(\imath\varepsilon(\mathbf{v})+\sigma_0)+\alpha\vec{\nabla}p=\rho^{1/2}\mathbf{g}_0 \text{ in } \mathbf{V}' \tag{18b}$$

Now from assumption A4 we see that the operator $\mathbf{v}\mapsto\varepsilon'\imath'(\mathbb{M}+\mathbb{L})^{-1}(\imath\varepsilon(\mathbf{v})+\sigma_0)\equiv S(\mathbf{v})$ is monotone, coercive and Lipschitz continuous from $\mathbf{V}$ to $\mathbf{V}'$, and so the sum $\rho+D+S$ is maximal monotone and coercive. The same holds for the system (18) by A3. The existence of a solution of the stationary system (17) now follows by solving (18) for $p\in V$, $\mathbf{v}\in\mathbf{V}$, and then defining τ as above to get the required solution of (17).

As for the uniqueness for the Cauchy problem, it suffices to show that $(\mathscr{B}+\mathscr{A})^{-1}$ is single-valued. But this can be shown from the equivalence of (17) and (18) and the strict monotonicity conditions A5 and A6. This finishes the proof of Theorem 3.1. □

Remark 3.3
For the existence, it is clearly sufficient for the sum $\rho+D+S$ to be coercive, hence, surjective. This corresponds to a combination of viscous or plastic hardening assumptions in the model.

Example
We now aim to give an easy example of a model in this class. To this end, let Y be the semi-line $(0,+\infty)$ and take $\mathrm{d}\mu\equiv\varphi\mathrm{d}y$ where $\mathrm{d}y$ is the standard Lebesgue measure on Y and $\varphi\in L^1(Y)$. For the sake of simplicity let us set $\mathbb{M}\equiv 1$ (that is $\mathbb{M}_{ijlk}=(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})/2$ where δ is the Kronecker symbol) and $K_y=\{\sigma\in\Sigma:(\sigma:\sigma)\leqslant y^2\}$. Then, it is straightforward to compute that the coercivity of the stress operator is indeed equivalent to the condition

$$\int_Y y\,\varphi(y)\,\mathrm{d}y=+\infty$$

4. CONCLUDING REMARKS

We have shown existence and uniqueness of a *strong solution* of system (13) on $(0,T)$ subject to the initial conditions (14). Consequently, we have each term of (13a) is in $L^2(\Omega)\times L^2(\Gamma_f)$ at a.e. $t\in[0,T]$, and this implies in particular that $A_0(p(t))\in L^2(\Omega)$, so the diffusion operator can be decoupled into its formal and boundary parts. That is, Equation (13a) is equivalent to the pair

$$c_0\dot{p}(t)+\alpha\nabla\cdot\mathbf{v}(t)+A_0(p(t))=c_0^{1/2}h_0(t)\quad\text{in } L^2(\Omega)$$

$$-\alpha\beta\mathbf{v}(t)\cdot\mathbf{n}+\partial_A(p(t))=0\quad\text{in } L^2(\Gamma_f)$$

Similarly, the left side of the momentum equation (13b) is in $\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Gamma_t)$ at a.e. $t\in[0,T]$, but this implies only that the *sum* of the dilation and stress terms belongs to $\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Gamma_t)$ at a.e. $t\in[0,T]$. Thus, we need consider the decoupling of this sum into formal and boundary parts. Proceeding as in Section 2, suppose that we have a pair $\sigma\in L^2(\Omega,\Sigma)$, $\mathbf{v}\in\mathbf{V}$ so that $D(\mathbf{v})+\varepsilon'\sigma\in\mathbf{V}'$. Assume furthermore, that the formal part, namely, the restriction of the sum

$$(D(\mathbf{v})+\varepsilon'\sigma)|_{(C_0^\infty(\Omega))^3}=-\nabla\lambda^*(\nabla\cdot\mathbf{v})-\nabla\cdot\sigma$$

belongs to $\mathbf{L}^2(\Omega)$. Then the *abstract Green's theorem* [35, Proposition II.5.3] shows that there is a functional $b\in\mathbf{B}'$ on the space of boundary values for which

$$\langle D\mathbf{v}+\varepsilon'\sigma,\mathbf{w}\rangle=-(\nabla\lambda^*(\nabla\cdot\mathbf{v})-\nabla\cdot\sigma,\mathbf{w})_{\mathbf{L}^2(\Omega)}+\langle b,\gamma(\mathbf{w})\rangle,\quad \mathbf{w}\in\mathbf{V}$$

When $\nabla\cdot\mathbf{v}$ and σ are sufficiently smooth, the Stokes' theorem shows that this boundary functional is given by

$$b=\lambda^*(\nabla\cdot\mathbf{v})\mathbf{n}+\sigma(\mathbf{n})$$

Thus, for our strong solution we have

$$D(\mathbf{v}(t))+\varepsilon'\iota'\tau(t)\in\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Gamma_t)$$

and the abstract Green's theorem shows that we can write this as

$$[-\nabla\cdot(\lambda^*(\nabla\cdot\mathbf{v}(t))\delta+\sigma(t)),(\lambda^*(\nabla\cdot\mathbf{v})\delta+\sigma)\mathbf{n}]\in\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Gamma_t)$$

where $(\delta)_{ij}=\delta_{ij}$ is the unit tensor. The solution of (13) satisfies the system

$$p(t)\in V:\quad c_0\dot{p}(t)+\alpha\nabla\cdot\mathbf{v}(t)-\partial_jA_j(\cdot,\nabla p(t))=c_0^{1/2}h_0(t)\ \text{in } L^2(\Omega)\tag{19a}$$

$$\mathbf{v}(t)\in\mathbf{V}:\quad \rho\dot{\mathbf{v}}(t)-\nabla\cdot(\lambda^*(\nabla\cdot\mathbf{v}(t))\delta+\sigma(t))+\alpha\nabla p(t)=\rho^{1/2}\mathbf{g}_0(t)\ \text{in } \mathbf{L}^2(\Omega)\tag{19b}$$

$$\sigma=\mathscr{H}(\varepsilon(\mathbf{v}))\ \text{in } L^2(\Omega,\Sigma)\tag{19c}$$

and the boundary conditions

$$p(t)=0\quad\text{in } B\subset L^2(\Gamma_d)\tag{20a}$$

$$\partial_A(p(t)) - \alpha\beta\,\mathbf{v}(t)\cdot\mathbf{n} = 0 \quad \text{in } L^2(\Gamma_f) \subset B' \tag{20b}$$

$$\mathbf{v}(t) = \mathbf{0} \quad \text{in } \mathbf{B} \subset \mathbf{L}^2(\Gamma_c) \tag{20c}$$

$$b(t) - \alpha(1-\beta)p(t)\mathbf{n} = \mathbf{0} \quad \text{in } \mathbf{L}^2(\Gamma_t) \subset \mathbf{B}' \tag{20d}$$

where the boundary operators $\partial_A(p(t))$ and $b(t)$ are extensions of $A_j(\cdot, \nabla p(t))n_j$ and $(\lambda^*(\nabla\cdot\mathbf{v}(t))\delta + \sigma(t))\mathbf{n}$ taking values in B' and $\mathbf{B}'$, respectively.

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