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A DIFFUSION SYSTEM FOR FLUID IN FRACTURED MEDIA†

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Abstract. When modeling diffusion phenomena, equations of the form $\frac{d}{dt}A(u) + B(u) = f$ often arise. When both A and B are non-linear, many of these diffusion problems remain unsolved. In this paper a class of such problems is presented and solutions are found. Properties of these solutions are also considered.

1. Introduction. The class of problems we consider may be motivated by the physical problem associated with the flow of a fluid (liquid or gas) through fractured porous rock. In such a situation, the majority of the bulk fluid motion will be in the fissures between the blocks, and the block to block diffusion is small. However, the blocks will store and release fluid as the pressure in the fissures changes. Since the block volume will be much larger than the fissures volume, the block to fissure diffusion is a major component of the system.

If the usual continuum assumptions are made, the balance of mass in the blocks and fissures may be written as

$$\begin{aligned} &\frac{\partial}{\partial t}(m_B\rho_B) = m_B f - a\gamma \\ &\frac{\partial}{\partial t}(m_F\rho_F) + \nabla\cdot(m_F\rho_F \mathbf{U}_F) = m_F g + a\gamma \end{aligned} \tag{1.1}$$

where m_B, m_F are the ratios of block and fissure volumes to the total volume, γ is the mass flow rate of fluid exchanged between the blocks and fissures, a is a geometric quantity describing the ratio of the block-fissure surface area per unit volume, ρ_B, ρ_F are the densities of the fluid in the block and fissure, f, g are the fluid sources in the blocks and fissures respectively, $\mathbf{U}_F$ is the bulk fluid velocity in the fissures.

The porous flow assumption (d'Arcy's law) determines the velocity as

$$\mathbf{U} = -\sum_{j=0}^{n} \kappa_j\Big(\rho\frac{\partial p}{\partial x_j}\Big)\frac{\partial p}{\partial x_j}\,\mathbf{e}_j$$

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where p is the fluid pressure and the κ_j describe the permeability of the fissure network. It will be assumed that the fluid density can be described in terms of the pressure,

$$\rho \in s(p), \quad p \in \mathbf{R}_0^+.$$

It is necessary to let s be a graph, since, at $p = 0$ fluid may only partially fill the fissures or pores giving an effective density of $\xi \rho_0$ where ξ is the fraction of fluid and ρ_0 the density at $p = 0$. In order to have globally defined quantities we extend s to be zero on $\mathbf{R}^-$.

The fluid exchange, γ, between the blocks and fissures will depend upon the pressure difference of the fluid in the blocks and fissures, and upon the density of the fluid at these pressures. In the sequel it will be assumed that γ depends upon the quantity $\tilde{\rho}(p_B - p_F)$ where $\tilde{\rho}$ is the average density on the interval $[p_F; p_B]$. Defining

$$S(p) = \int_0^p s(\pi)\, d\pi, \quad u = S(p_B) \text{ and } v = S(p_F)$$

the identity

$$u - v = \int_{p_F}^{p_B} s(\pi)\, d\pi = \tilde{\rho}(p_B - p_F)$$

indicates that this is a natural choice. The balance of mass may now be written

$$\begin{aligned} &\frac{\partial}{\partial t}[m_B s \circ S^{-1}(u)] + a\gamma(u - v) \ni m_B f \\ &\frac{\partial}{\partial t}[m_F s \circ S^{-1}(v)] - a\gamma(u - v) - \sum_{j=0}^{n} \frac{\partial}{\partial x_j}[m_F \kappa_j(\frac{\partial v}{\partial x_j})\frac{\partial v}{\partial x_j}] \ni m_F g. \end{aligned} \tag{1.2}$$

The system (1.2) motivates the study of the following problem: find $\mathbf{u} = (u, v)$ such that

$$\frac{d}{dt}\mathbf{A}(\mathbf{u}) + \mathbf{B}(\mathbf{u}) \ni \mathbf{f}, \tag{1.3}$$

where $\mathbf{A}(\mathbf{u}) = [A_1(u), A_2(v)]$ and A_1 and A_2 are (maximal) monotone graphs in $\mathbf{R} \times \mathbf{R}$, $\mathbf{B}(\mathbf{u}) = [\gamma(u - v), -\gamma(u - v) + B(v)]$, γ is a monotone function on $\mathbf{R}$, and B is a (possibly degenerate) elliptic operator, $\mathbf{f} = (f, g)$ are data.

The monotonicity of A_1 and A_2 arises naturally from the physical property that the density is a non-decreasing function of the pressure, and the monotonicity of γ reflects the property that larger pressure differences will not decrease the fluid exchange rate between the blocks and fissures. The ellipticity of B is implied by the diffusivities κ_j being non-negative. Initial and boundary conditions will be added to the problem. In the sequel, Dirichlet boundary conditions will be specified for simplicity of exposition.

Both DiBenedetto and Showalter [7], and Grange and Minot [9] studied problems of the form (1.3). The difficulty in applying the results of [7, 9] to the system (1.3) is the lack of compactness due to the absence of an elliptic term in the first equation. DiBenedetto and Showalter [8] studied problem (1.3) when $\mathbf{B}$ is linear, so required all the κ_j to be constant; however, experimental evidence [2] suggests $\kappa_j(\xi) \sim \xi^{-1/2}$ for ξ large.

The Cauchy problem (1.3) will be resolved in Section (2) by combining the results of [7, 9] with the generalized solutions obtained by semigroup theory in L^1. Formally the Cauchy problem for (1.3) can be written as

$$\frac{da}{dt} + C(a) \ni f, \quad a(0) = a_0, \tag{1.4}$$

where $C = B \circ A^{-1}$. The Crandall Liggett Theorem [4] implies that the piecewise linear functions $\hat{a}^N$ constructed from the solution, $\{a_n^N\}_{n=0}^N$, of the difference scheme:

$$\begin{aligned} &a_0^N = a_0 \in Rg(A), \\ &a_{n+1}^N + \tau b_{n+1}^N = \tau f_{n+1}^N + a_n^N, \quad n = 0, 1, \cdots, (N-1), \\ &b_{n+1}^N \in C(a_{n+1}^N) \end{aligned} \tag{1.5}$$

($N \in \mathbb{N}$, $\tau = T/N$, $\{f_n^N\}_{n=1}^N$ given) converge in $C[0,T; L^1(\Omega)^2]$. This convergence result will be used to eliminate the compactness assumption in [7, 9] where corresponding direct approximation schemes for (1.3) are shown to converge. We obtain thereby additional regularity of the generalized solution of (1.3).

Since we are considering a system of equations we will work in products of Banach spaces, eg. $V = V_1 \times V_2$, elements of the product space will be identified as ordered pairs, (u, v) etc. Duals of Banach spaces will be denoted by a prime, $V' = V_1' \times V_2'$. The action of a dual element, v' upon and element v is denoted $\langle v', v \rangle$. When L^p spaces, or the associated Sobolev spaces, are involved, the dual exponent will be denoted by p'; i.e., $1/p + 1/p' = 1$. Only the duals of reflexive spaces are considered, so $1 < p < \infty$. The measure of a set $\Omega \subset \mathbb{R}^n$ is denoted $|\Omega|$. The notation $V \hookrightarrow W$ will indicate that the Banach space V is continuously embedded in W. In all instances the embeddings will be dense. A compact embedding is denoted $V \hookrightarrow\!\!\!\!\rightarrow W$. The notation $u_n \to u$ will indicate norm convergence and $u_n \rightharpoonup u$ will indicate a weakly convergent sequence.

We will let convex functions defined on the real line take values in $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$. Realizations of a convex function $\phi : \mathbb{R} \to \mathbb{R}_\infty$ on L^p spaces, given by $\int_\Omega \phi \circ u$ if the integral is finite and infinity otherwise, will also be denoted ϕ. The domain of a convex function $\phi : V \to \mathbb{R}_\infty$ defined on a Banach space V is the subset $D(\phi) \subset V$ on which ϕ is finite. The subgradient of a convex function $\phi : V \to \mathbb{R}_\infty$ is the mapping $\partial\phi : V \to 2^{V'}$ (the power set of V') given by $\partial\phi(u) = \{v' \in V' \mid \langle v', v - u \rangle \leq \phi(v) - \phi(u), \text{ for all } v \in V\}$. A convex function $\phi : V \to \mathbb{R}_\infty$ has an associated convex conjugate $\phi^* : V' \to \mathbb{R}_\infty$ given by $\phi^*(v') = \sup\{v'(v) - \phi(v) \mid v \in V\}$. When V is reflexive, ϕ^* inherits many properties from ϕ, in particular the subgradient $\partial\phi^*$ is the inverse graph of $\partial\phi$ (considered as subsets of the product $V \times V'$). If $\phi : V \to \mathbb{R}_\infty$ is convex, then $\partial\phi(v)$ will be a closed convex set in the dual space. If V' is strictly convex and reflexive it follows that $\partial\phi(v)$ will have a unique element of minimal norm. The mapping from V to V' which selects the element of minimal norm is the minimal section of $\partial\phi$ and is denoted $\partial\phi_0$.

2. The Cauchy problem. We first construct operators A, B, and C on the appropriate spaces. Then the semigroup generation theorem [4] is combined with the direct integrations of (1.3) by [7, 9] to get a solution.

The functions γ, κ_j and A_i appearing in equations (1.3) may all depend upon the position $\mathbf{x} \in \Omega$ as well as the pressures u and v. To indicate this dependence would be cumbersome, so the dependence of γ, κ_j and A_i upon $\mathbf{x} \in \Omega$ will be suppressed (i.e., $\gamma(x,u)$ is denoted $\gamma(u)$ etc.). Any estimates or assumptions pertaining to these quantities are to hold uniformly in $\mathbf{x} \in \Omega$. Let $\Omega \subset \mathbb{R}^n$ denote a bounded open set. The functions γ and κ_j will satisfy Caratheodory conditions, so measurablility of compositions with u, ∇u, etc., will never be in question. When making selections from A_i, only measurable selections will be considered.

Assumptions: Let $\xi_0 \in \mathbb{R}$ be fixed.

[C] Choose Banach spaces $V = L^q(\Omega) \times W_0^{1,p}(\Omega)$, $W = L^q(\Omega) \times L^p(\Omega)$, $\mathcal{V} = L^q[0,T; L^q(\Omega)] \times L^p[0,T; W_0^{1,p}(\Omega)]$ and $\mathcal{W} = L^q[0,T; L^q(\Omega)] \times L^p[0,T; L^p(\Omega)]$ where $1 < p < \infty$ and $1 < q \leq np/(n-p)$ if $p < n$ and $1 < q < \infty$ if $p \geq n$.

[A] The maximal monotone graphs A_1, $A_2 \subset \mathbb{R} \times \mathbb{R}$ satisfy

$$|\alpha| \leq a_0 |\xi|^{\beta_1} \quad \forall \alpha \in A_1(\xi), \ |\xi| \geq \xi_0;$$

$$|a| \leq a_0 |\xi|^{\beta_2} \quad \forall a \in A_2(\xi), \ |\xi| \geq \xi_0;$$

where $0 \leq \beta_1 \leq (q-1)$, $0 \leq \beta_2 \leq n(p-1)/(n-p)$ and $a_0 \in \mathbb{R}_+$ is a constant.

[G] The maximal monotone function $\gamma : \mathbb{R} \to \mathbb{R}$ satisfies $\gamma(0) = 0$ and

$$\gamma_0 |\xi|^{q-1} \leq |\gamma(\xi)| \leq \gamma_1 |\xi|^{q-1}, \quad |\xi| \geq \xi_0$$

where γ_0 and γ_1 are positive constants.

[K] $k_j : \mathbb{R} \to \mathbb{R}$, $j = 1, 2, \cdots, n$, are maximal monotone functions satisfying $k_j(0) = 0$ and

$$k_0 |\xi|^{p-1} \leq |k_j(\xi)| \leq k_1 |\xi|^{p-1}, \quad |\xi| \geq \xi_0$$

where k_0 and k_1 are positive constants $(k_j(\xi) \equiv \kappa_j(\xi)\xi)$.

Definition 1. Let A_1, $A_2 \subset \mathbb{R} \times \mathbb{R}$ be maximal monotone graphs satisfying Assumption [A] and γ, $k_j : \mathbb{R} \to \mathbb{R}$ be functions satisfying Assumptions [G] and [K] respectively. Then define ϕ_i, Γ and $K_j : \mathbb{R} \to \mathbb{R}$ by

$$\phi_i(s) = \int_0^s A_i(\xi) d\xi, \quad \Gamma(s) = \int_0^s \gamma(\xi) d\xi, \quad K_j(s) = \int_0^s k_j(\xi) d\xi.$$

Lemma 1. *Let $\phi = (\phi_1, \phi_2)$, where ϕ_1 and ϕ_2 are the realizations in $L^q(\Omega)$ and $W_0^{1,p}(\Omega)$ of the functions defined in Definition (1), and let Assumption [C] hold. Then $\phi : V \to \mathbb{R}$ is continuous and $\partial\phi : \mathcal{V} \to 2^{\mathcal{W}'}$ is bounded.*

This lemma follows from the Hölder and Sobolev inequalities for bounded domains.

Theorem 1. *Let $\psi : V \to \mathbb{R}$ ($\psi : \mathcal{V} \to \mathbb{R}$) be defined by $\psi(u,v) = \Gamma(u-v) + \sum_{j=1}^n K_j(\frac{\partial v}{\partial x_j})$, where Γ and K_j are defined in Definition (1), and let V ($\mathcal{V}$) be given in Assumption [C]. Then ψ has a coercive, bounded derivative,*

$$\psi'(u,v) = \partial\psi(u,v) = \left[\gamma(u-v), -\gamma(u-v) - \sum_{j=1}^n \frac{\partial}{\partial x_j}[k_j(\frac{\partial v}{\partial x_j})]\right].$$

The proof of this theorem follows from elementary properties of convex functions and their realizations on L^p spaces.

Lemma 2. *Let $\Omega \subset \mathbb{R}^n$ be bounded and $p > 1$. Set*

$$D(B) = \{u \in W_0^{1,p}(\Omega) \cap L^2(\Omega) \mid \sum_{j=1}^{n} \frac{\partial}{\partial x_j}[k_j(\frac{\partial u}{\partial x_j})] \in L^2(\Omega)\}$$

where the k_j satisfy Assumptions [K]. Then $B : D(B) \to L^2(\Omega)$, defined by $B(u) = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}[k_j(\frac{\partial u}{\partial x_j})]$, is maximal monotone in $L^2(\Omega)$.

The proof of this lemma follows immediately from the observation that B is the derivative of a convex function defined on the space $W_0^{1,p}(\Omega) \cap L^2(\Omega)$. The next result is a modification of Brezis and Strauss [3], and is the key to the demonstration of m-accretivity.

Theorem 2. *Let B be defined as in Lemma (2) and let $j : \mathbb{R} \to \mathbb{R}_0^+$ be a convex function satisfying $j(0) = 0$. Then*

$$(B(u) - B(v), s)_{L^2(\Omega)} \geq 0$$

where s is any selection from $\partial j(u - v)$ in $L^2(\Omega)$.

Proof: Consider the two convex functions $j_1(x) = x - t$ if $x \geq t$ and zero otherwise, $t \geq 0$ and $j_2(x) = -x + t$ if $x \leq t$ and zero otherwise, $t \leq 0$. The elementary properties of convex functions imply that $j_i(x - y) \geq j_i(x) - (\partial j_i)_0(x)y$, $i = 1$ or 2, where the zero subscript on a sub-gradient indicates the minimal section.

Next, recall the Yosida approximant of B defined by

$$B_\lambda = \frac{1}{\lambda}(I - J_\lambda), \quad J_\lambda = (I + \lambda B)^{-1}, \quad \lambda > 0.$$

The Yosida approximant is well defined for maximal monotone operators, so is well defined in this instance.

Claim: If $f, g \in L^2(\Omega)$ then

$$\int_\Omega j_i[J_\lambda(f) - J_\lambda(g)] \leq \int_\Omega j_i(f - g), \quad i = 1, 2.$$

Proof: Put $u = J_\lambda(f)$, $v = J_\lambda(g)$, and observe that $u, v \in D(B)$ and

$$\begin{aligned} j_i(f - g) &= j_i\big(u - v - \lambda \sum_{j=1}^{n} \frac{\partial}{\partial x_j}[k_j(\frac{\partial u}{\partial x_j}) - k_j(\frac{\partial v}{\partial x_j})]\big) \\ &\geq j_i(u - v) - \lambda(\partial j_i)_0(u - v) \sum_{j=1}^{n} \frac{\partial}{\partial x_j}[k_j(\frac{\partial u}{\partial x_j}) - k_j(\frac{\partial v}{\partial x_j})]. \end{aligned}$$

Integrating yields

$$\int_\Omega j_i(f - g) \geq \int_\Omega \{j_i(u - v) - \lambda(\partial j_i)_0(u - v) \sum_{j=1}^{n} \frac{\partial}{\partial x_j}[k_j(\frac{\partial u}{\partial x_j}) - k_j(\frac{\partial v}{\partial x_j})]\}.$$

Let $\phi_\epsilon : \mathbb{R} \to \mathbb{R}$ be the Yosida approximant to ∂j_i, so $\phi_\epsilon \in C^{0,1}(\mathbb{R})$ with non-negative derivative and $\lim_{\epsilon\downarrow 0} \phi_\epsilon(s) = (\partial j_i)_0(s)\ \forall s \in \mathbb{R}$. Moreover, $\phi_\epsilon(u-v) \in W_0^{1,p}(\Omega)$, since $j_i(0) = \phi_\epsilon(0) = 0$, consequently

$$\int_\Omega -\sum_{j=1}^n \frac{\partial}{\partial x_j}[k_j(\frac{\partial u}{\partial x_j}) - k_j(\frac{\partial v}{\partial x_j})](\partial j_i)_0(u-v)$$

$$= \lim_{\epsilon\downarrow 0} \int_\Omega -\sum_{j=1}^n \frac{\partial}{\partial x_j}[k_j(\frac{\partial u}{\partial x_j}) - k_j(\frac{\partial v}{\partial x_j})]\phi_\epsilon(u-v)$$

$$= \lim_{\epsilon\downarrow 0} \int_\Omega \sum_{j=1}^n [k_j(\frac{\partial u}{\partial x_j}) - k_j(\frac{\partial v}{\partial x_j})](\frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial x_j})\phi_\epsilon'(u-v) \geq 0.$$

This is extends, as in [3], to hold for the more general convex function j. To complete the proof, consider $u, v \in D(B)$. The convexity of j implies

$$j[J_\lambda(u) - J_\lambda(v)] - j(u-v) \geq s[J_\lambda(u) - J_\lambda(v) - (u-v)].$$

Integrating and recalling the definition of the Yosida approximant yields

$$0 \geq -\lambda \int_\Omega [B_\lambda(u) - B_\lambda(v)]s.$$

The Theorem now follows from the fact that $\lim_{\lambda\downarrow 0} B_\lambda(u) = B(u)$ in $L^2(\Omega)$ when $u \in D(B)$ and B is single valued.

Remarks:

- The operator B is accretive in $L^1(\Omega)$ and its closure, $\bar{B}$, in $L^1(\Omega)^2$ is m-accretive.
- Let $\Omega \subset \mathbb{R}^n$ be bounded and $p > 1$. Set

  $$D(\tilde{B}) = \{u \in W_0^{1,p}(\Omega) \mid \sum_{j=1}^n \frac{\partial}{\partial x_j}[k_j(\frac{\partial u}{\partial x_j})] \in L^1(\Omega)\}$$

  and

  $$\tilde{B}(u) = -\sum_{j=1}^n \frac{\partial}{\partial x_j}[k_j(\frac{\partial u}{\partial x_j})],$$

  then $B \subset \tilde{B} \subset \bar{B}$.

When $W_0^{1,p}(\Omega)$ is not embedded in $L^2(\Omega)$, the following technical lemma indicates when an element of $D(\tilde{B})$ may also be in $D(B)$.

Lemma 3. *Let $u \in W_0^{1,p}(\Omega)$ and $-\sum_{j=1}^n \frac{\partial}{\partial x_j}[k_j(\frac{\partial u}{\partial x_j})] = f \in L^{t'}(\Omega)$, where the k_j satisfy Assumption [K] and $t' > \frac{n}{p}$. Then $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

Proof: This is proved using Moser iteration. Assume that $u \in L^r(\Omega)$, and observe that $r = np/(n-p)$ suffices for the first iterate. Define G, $H \in C^1(\mathbb{R})$ by

$$H(s) = H_{s_0}(s) = \begin{cases} |s|^\beta & |s| \leq s_0 \\ \text{linear} & |s| > s_0, \end{cases}$$

$G(s) = \int_0^s |H'(\xi)|^p \, d\xi$, where $\beta = 1 + (r-t)/(tp)$, and $t = t'/(t'-1) < n/(n-p)$. Both G and H vanish at zero, so $G(u)$, $H(u) \in W_0^{1,p}(\Omega)$. Also $G'(s)$ is monotone on $\mathbb{R}_0^+$, so $|G(s)| \le |s|G'(s)$, similarly $|H'(s)| \le \beta |s|^{\beta-1}$. Selecting $G(u)$ as a test function yields

$$\int_\Omega \sum_{j=0}^n k_j(\frac{\partial u}{\partial x_j}) \frac{\partial u}{\partial x_j} G'(u) = \int_\Omega f G(u)$$

$$k_0 \sum_{j=1}^n \int_{|\frac{\partial u}{\partial x_j}| \ge \xi_0} |\frac{\partial u}{\partial x_j}|^p |H'(u)|^p \le \|f\|_{L^{t'}(\Omega)} [\int_\Omega (|u|\,|H'(u)|^p)^t]^{\frac{1}{t}}$$

$$k_0 \int_\Omega |\nabla u|_{l_p}^p |H'(u)|^p \le \|f\|_{L^{t'}(\Omega)} \beta^p \|u\|_{L^r(\Omega)}^{\frac{r}{t}} + n k_0 \xi_0^p \beta^p \|u\|_{L^{\frac{r-t}{t}}(\Omega)}^{\frac{r-t}{t}}$$

$$k_0 \|\nabla H(u)\|_{L^p(\Omega)^n}^p \le \|f\|_{L^{t'}(\Omega)} \beta^p \|u\|_{L^r(\Omega)}^{\frac{r}{t}} + n k_0 \xi_0^p \beta^p |\Omega|^{1-\frac{r-t}{rt}} \|u\|_{L^r(\Omega)}^{\frac{r-t}{t}}$$

$$\tilde{C} k_0 \|H(u)\|_{L^{\frac{np}{n-p}}(\Omega)}^p \le \left[\|f\|_{L^{t'}(\Omega)} + n k_0 \xi_0^p \max(1, |\Omega|)\right] \beta^p \max(1, \|u\|_{L^r(\Omega)}^{\frac{r}{t}})$$

where the Sobolev embedding Theorem was used to obtain the last line. Letting $s_0 \to \infty$ yields $\|u\|_{L^{\beta(\frac{np}{n-p})}}^{\beta p} \le C\beta^p \max(1, \|u\|_{L^r(\Omega)}^{\frac{r}{t}})$. Put $\chi = (\frac{n}{n-p})^{\frac{1}{t}} > 1$ and note that $\beta(\frac{np}{n-p}) \ge \chi r$; therefore,

$$\|u\|_{L^{\chi r}(\Omega)} \le C^{\frac{t}{tp+r-t}} \left[1 + \frac{r-t}{tp}\right]^{\frac{1}{1+\frac{r-t}{tp}}} \max(1, \|u\|_{L^r(\Omega)}).$$

Iterating this N times shows

$$\|u\|_{L^{\chi^N r}(\Omega)} \le \{ \prod_{n=0}^N C^{\frac{t}{tp+\chi^n r-t}} \left[1 + \frac{\chi^n r - t}{tp}\right]^{\frac{1}{1+\frac{\chi^n r-t}{tp}}} \} \max(1, \|u\|_{L^r(\Omega)}).$$

Since the product in the above estimate converges as $N \to \infty$, the Lemma is proved.

Definition 2. Let

$$\begin{aligned} D(C) = \big\{(\alpha, a) \in W' \,|\, \exists\, (u,v) \in L^q(\Omega) \times D(B), \text{ such that} \\ (\alpha, a) \in \partial\phi(u,v) \text{ and } B(v) \in L^{p'}(\Omega)\big\}, \end{aligned}$$

then $C : D(C) \to 2^{W'}$ is defined by

$$C(\alpha, a) = \{[\gamma(u-v), -\gamma(u-v) + B(v)] \,|\, (u,v) \in D(C),\ (\alpha, a) \in \partial\phi(u,v)\},$$

where γ and W satisfy Assumptions [G] and [C] respectively, and ϕ and B are defined in Definition 1 and Lemma 2 respectively.

Lemma 4. *The operator C, defined in Definition 2, is accretive in $L^1(\Omega)^2$.*

Proof: Let (α, a), $(\tilde{\alpha}, \tilde{a}) \in D(C)$ and let $(y, z) = [\gamma(u-v), -\gamma(u-v) + B(v)] \in C(\alpha, a)$, $(\tilde{y}, \tilde{z}) = [\gamma(\tilde{u}-\tilde{v}), -\gamma(\tilde{u}-\tilde{v}) + B(\tilde{v})] \in C(\tilde{\alpha}, \tilde{a})$.

$$\|(\alpha - \tilde{\alpha}) + \lambda(y - \tilde{y})\|_{L^1(\Omega)} + \|(a - \tilde{a}) + \lambda(z - \tilde{z})\|_{L^1(\Omega)}$$
$$\geq \int_\Omega |\alpha - \tilde{\alpha}| + |a - \tilde{a}| + \lambda[\gamma(u-v) - \gamma(\tilde{u}-\tilde{v})](s_1 - s_2) + \lambda[B(v) - B(\tilde{v})]s_2$$

where

$$s_1 = \begin{cases} 1 & \text{if } \alpha > \tilde{\alpha} \\ \mathrm{sgn}_0(u - \tilde{u}) & \text{if } \alpha = \tilde{\alpha}, \\ -1 & \text{if } \alpha < \tilde{\alpha} \end{cases} \qquad s_2 = \begin{cases} 1 & \text{if } a > \tilde{a} \\ \mathrm{sgn}_0(v - \tilde{v}) & \text{if } a = \tilde{a} \\ -1 & \text{if } a < \tilde{a}. \end{cases}$$

Since $s_1 \in \mathrm{sgn}\,(u - \tilde{u})$, $s_2 \in \mathrm{sgn}\,(v - \tilde{v})$, the latter two terms in the integral are non-negative $\forall\, \lambda > 0$.

Theorem 3. *Let $(f, g) \in L^\infty(\Omega)^2$. Then the problem $(\alpha, a) \in D(C)$ $(I + C)(\alpha, a) \ni (f, g)$, has a solution, where C is the operator defined in Definition 2. Consequently, the closure of C in $L^1(\Omega)^4$ is m-accretive. Moreover, (α, a) satisfy the estimates*

$$\|\alpha^+\|_{L^\infty(\Omega)} + \|a^+\|_{L^\infty(\Omega)} \leq \|f^+\|_{L^\infty(\Omega)} + \|g^+\|_{L^\infty(\Omega)},$$
$$\|\alpha^-\|_{L^\infty(\Omega)} + \|a^-\|_{L^\infty(\Omega)} \leq \|f^-\|_{L^\infty(\Omega)} + \|g^-\|_{L^\infty(\Omega)}.$$

Proof: Let $\tilde{A}_1(s) = A_1(s)$ if $|A_1(s)| < N$, $\tilde{A}_1(s) = N$ if $A_1(s) \geq N$ and $\tilde{A}_1(s) = -N$ if $A_1(s) \leq -N$, where $A_1(\cdot)$ is the monotone graph defined in Section 2 and $N > \|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\Omega)}$. Define $\tilde{A}_2$ similarly. Let (u, v) be a solution of the problem $(u, v) \in L^q(\Omega) \times W_0^{1,p}(\Omega)$,

$$\alpha + \gamma(u - v) = f, \quad a - \gamma(u - v) - \sum_{j=1}^n \frac{\partial}{\partial x_j}[k_j(\frac{\partial v}{\partial x_j})] = g,$$

$\alpha \in \tilde{A}_1(u)$, $a \in \tilde{A}_2(v)$. A solution can be found as the minimum of a convex function. The construction of $\tilde{A}_1$ and $\tilde{A}_2$ guarantees $(\alpha, a) \in L^\infty(\Omega)^2$. It is then clear that $\sum_{j=1}^n \frac{\partial}{\partial x_j}[k_j(\frac{\partial v}{\partial x_j})] \in L^\infty(\Omega)$, so, by Lemma 3, $v \in D(B)$.

Let $k \in \mathbf{R}_0^+$ be chosen so large that $\exists\, k_1 \in \tilde{A}_1(k)$ and $k_2 \in \tilde{A}_2(k)$ where $k_1 + k_2 \geq \|f^+\|_{L^\infty(\Omega)} + \|g^+\|_{L^\infty(\Omega)}$. If no such k exists A_1 and A_2 are upper bounded, and the estimate on (α^+, a^+) is trivially satisfied. Define

$$s_1 = \mathrm{sgn}_0^+(u + \alpha - k - k_1) \in \mathrm{sgn}^+(u - k) \cap \mathrm{sgn}^+(\alpha - k_1),$$
$$s_2 = \mathrm{sgn}_0^+(v + a - k - k_2) \in \mathrm{sgn}^+(v - k) \cap \mathrm{sgn}^+(a - k_2).$$

Subtracting k_1 and k_2 from both sides of the equations for (α, a) and multiplying them by s_1 and s_2 respectively results in the following estimate

$$\int_\Omega \{(\alpha - k_1)^+ + (a - k_2)^+ + \gamma(u - v)(s_1 - s_2) + B(v)s_2\} \leq \int_\Omega (f^+ + g^+ - k_1 - k_2).$$

Since the right hand side is non-positive and the latter two terms in the first integral are non-negative, it is clear that $\|\alpha^+\|_{L^\infty(\Omega)}+\|a^+\|_{L^\infty(\Omega)} \le \|f^+\|_{L^\infty(\Omega)}+\|g^+\|_{L^\infty(\Omega)}$.

Similarly $\|\alpha^-\|_{L^\infty(\Omega)} + \|a^-\|_{L^\infty(\Omega)} \le \|f^-\|_{L^\infty(\Omega)} + \|g^-\|_{L^\infty(\Omega)}$. These estimates imply that $\alpha \in A_1(u)$ and $a \in A_2(v)$. The Theorem follows. ∎

The results in Grange and Mignot [9] and DiBenedetto and Showalter [7] may be combined to give the following theorem, showing that the differencing scheme converges weakly in $\mathcal{V}'$, giving a distribution solution to the partial differential equations.

Theorem 4. [Grange-Mignot]. *Let $V \hookrightarrow W$ be a dense embedding of reflexive Banach spaces and let $\mathcal{V} = L^p[0,T;V]$, $\mathcal{W} = L^p[0,T;W]$. Suppose that*

1. *$\phi : W \to \mathbb{R}_\infty$ is a proper, convex, lower semi-continuous function, $A \equiv \partial\phi$,*
2. *$\phi : V \to \mathbb{R}_\infty$ is continuous at some point and $\phi(0) = 0$,*
3. *$A : \mathcal{V} \to 2^{\mathcal{W}'}$ is bounded, and*
4. *$B : \mathcal{V} \to 2^{\mathcal{V}'}$ is bounded, coercive and maximal monotone.*

If $N \in \mathbb{N}$, $\tau = \frac{T}{N}$, $\{f_n^N\}_{n=1}^N \subset V'$, then the differencing scheme

$$\{u_n^N\}_{n=1}^N \subset V, \quad a_0^N = a_0 \in Rg(A), \quad a_{n+1}^N + \tau b_{n+1}^N = \tau f_{n+1}^N + a_n^N,$$
$$a_{n+1}^N \in A(u_{n+1}^N), \quad b_{n+1}^N \in B(u_{n+1}^N), \quad n = 0, 1, \cdots, (N-1),$$

has a solution.

Put $f^N(t) = \sum_{n=0}^{N-1} \chi_{[n\tau;(n+1)\tau)}(t) f_{n+1}^N$ where $\chi_{(\cdot)}$ is the characteristic function of the indicated set, and define u^N, a^N and b^N similarly. Let $\hat{a}^N$ be the linear interpolant of $\{a_n^N\}_{n=0}^N$. If $\lim_{N\to\infty} f^N = f$ in $\mathcal{V}'$, there are subsequences (also indexed by N) such that

$$\begin{aligned} &u^N \rightharpoonup u && \text{in } \mathcal{V}, \quad b^N \rightharpoonup b \quad \text{in } \mathcal{V}', \\ &a^N \rightharpoonup a && \text{in } \mathcal{W}', \quad \hat{a}^N \rightharpoonup a \quad \text{in } \mathcal{W}' \\ &\frac{d\hat{a}^N}{dt} \rightharpoonup \frac{da}{dt} && \text{in } \mathcal{V}', \end{aligned}$$

and

$$\frac{da}{dt} + b = f \quad \text{in } \mathcal{V}', \quad a(0) = a_0.$$

If, in addition, $\hat{f}^N$ is the linear interpolant of $\{f_n^N\}_{n=0}^N$, $\hat{f}^N \to f$ in $L^p[0,T;V']$ and $B = \partial\psi$, where $\psi \to \mathbb{R}_\infty$ is proper, convex and lower semi-continuous, then $u^N \in L^\infty[0,T;V']$, $a^N \in L^\infty[0,T;W']$ and $b^N \in L^\infty[0,T;V']$ are all bounded independently of N.

Suppose, additionally, that $a \in A(u)$. Then $b \in B(u)$, so that (u,a,b) constitute a solution to the problem

$$u \in \mathcal{V}, \quad a \in \mathcal{W}', \quad \frac{da}{dt} \in \mathcal{V}', \quad b \in \mathcal{V}',$$
$$\frac{da}{dt} + b = f \quad \text{in } \mathcal{V}', \quad a(0) = a_0, \quad a \in A(u), \quad b \in B(u).$$

In particular, if $a^N \to a$ in $\mathcal{V}'$ then $a \in A(u)$ so the above holds.

Grange and Mignot [9] obtain solutions by assuming that $\mathcal{V} \hookrightarrow\!\!\!\!\rightarrow \mathcal{W}$. In this situation, there is a convergent sub-sequence, $a^N \to a$ in $\mathcal{V}'$. Clearly this is not the case for the problem considered here; however, the following theorem shows that the strong L^1 convergence given by the Crandall Liggett Theorem can be used in place of compactness.

Theorem 5. *Let $\Omega \subset \mathbf{R}^n$ be bounded and suppose $A_1, A_2 \subset \mathbf{R}\times\mathbf{R}$ and γ, $k_j : \mathbf{R} \to \mathbf{R}$ satisfy Assumptions* [A], [G] and [K] *respectively. If $(f,g) \in L^1[0,T; L^1(\Omega)^2]$ the problem*

$$\frac{d\alpha}{dt} + \gamma(u-v) = f,$$
$$\frac{da}{dt} - \gamma(u-v) - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}[k_j(\frac{\partial v}{\partial x_j})] = g,$$
$$\alpha \in A_1(u), \quad a \in A_2(v),$$

$(\alpha,a)(0) = (\alpha_0, a_0)$, has a unique generalized solution $(\alpha,a) \in C[0,T; L^1(\Omega)^2]$ provided $(\alpha_0,a_0) \in \overline{D(C)}$ where $D(C)$ is defined in Definition 2.

If V, W, $\mathcal{V}$ and $\mathcal{W}$ satisfy Assumption [C], and $(f,g) \in L^1[0,T; L^1(\Omega)^2] \cap \mathcal{V}'$, then the generalized solution also satisfies

$$(\alpha,a) \in \mathcal{W}', \quad \frac{d}{dt}(\alpha,a) \in \mathcal{V}', \quad (u,v) \in \mathcal{V},$$

$$\left.\begin{aligned} &\frac{d\alpha}{dt} + \gamma(u-v) = f \\ &\frac{da}{dt} - \gamma(u-v) - \textstyle\sum_{j=1}^{n} \frac{\partial}{\partial x_j}[k_j(\frac{\partial v}{\partial x_j})] = g \end{aligned}\right\} \quad \textit{in } \mathcal{V}',$$
$$\alpha \in A_1(u), \quad a \in A_2(v), \quad (\alpha,a)(0) = (\alpha_0,a_0),$$

provided $(\alpha_0,a_0) \in Rg(A_1 \times A_2)$.

If, additionally, $(f,g) \in L^p[0,T; V']$, then $(u,v) \in L^\infty[0,T; V]$, $\gamma(u-v)$, $(\alpha,a) \in L^\infty[0,T; W']$ and $\sum_{j=1}^{n} \frac{\partial}{\partial x_j}[k_j(\frac{\partial v}{\partial x_j})] \in L^\infty[0,T; V']$, and if $(f,g) \in L^1[0,T; L^\infty(\Omega)^2]$ and $(\alpha_0,a_0) \in L^\infty(\Omega)^2$ then $(\alpha,a) \in L^\infty[0,T; L^\infty(\Omega)^2]$.

Proof: The existence of the generalized solution follows immediately from Theorem 3, which demonstrates that the operator C is m-accretive in $L^1(\Omega)^2$, and the Crandall-Liggett Theorem [4].

The existence of a solution in $\mathcal{V}'$ will follow from Theorem 4. Lemma 1 and Theorem 1 show that all of the hypotheses for Theorem 4 are in place except $(\alpha,a) \in A(u,v)$. This is demonstrated as follows (see also [1]).

Let $(\tilde{\alpha},\tilde{a}) \in A(\tilde{u},\tilde{v})$ where $(\tilde{u},\tilde{v}) \in D(A)$ is arbitrary. If $(\tilde{\alpha}-\alpha^N, \tilde{a}-a^N)_b$ denotes the pair formed by truncating each component above and below by $\pm b$ respectively ($b > 0$), then the monotonicity of A implies

$$0 \le \int_0^T \int_\Omega (\tilde{\alpha}-\alpha^N, \tilde{a}-a^N)_b \cdot (\tilde{u}-u^N, \tilde{v}-v^N).$$

The convergence of (α^N, a^N) to (α, a) in $C[0,T;L^1(\Omega)^2]$ and the uniform bound on $(\tilde{\alpha}-\alpha^N, \tilde{a}-a^N)_b$ in $L^\infty[0,T;L^\infty(\Omega)^2]$ implies that $(\tilde{\alpha}-\alpha^N, \tilde{a}-a^N)_b \to (\tilde{\alpha}-\alpha, \tilde{a}-a)_b$ in $\mathcal{W}'$. Passing to the limit in the above equation implies

$$0 \le \int_0^T \int_\Omega (\tilde{\alpha}-\alpha, \tilde{a}-a)_b \cdot (\tilde{u}-u, \tilde{v}-v)$$

(recall that $(u^N, v^N) \rightharpoonup (u,v)$ in $\mathcal{V} \hookrightarrow \mathcal{W}$). Letting $b \to \infty$ in the above implies

$$0 \le \langle (\tilde{\alpha}, \tilde{a}) - (\alpha, a), (\tilde{u}, \tilde{v}) - (u,v) \rangle_{\mathcal{W}'-\mathcal{W}} \quad \text{for all } (\tilde{\alpha}, \tilde{a})t \in A(\tilde{u}, \tilde{v}).$$

The maximal monotonicity of A then guarantees that $(\alpha, a) \in A(u,v)$.

If $(f,g) \in L^1[0,T;L^\infty(\Omega)^2]$, the estimate in Theorem 3 guarantees

$$\|(\alpha,a)^N\|_{L^\infty[0,T;L^\infty(\Omega)^2]} \le \|(\alpha_0, a_0)\|_{L^\infty(\Omega)^2} + \|(f,g)\|_{L^1[0,T;L^\infty(\Omega)^2]}$$

Remarks:

- In each of the differencing schemes, a discretization $\{(f,g)_n^N\}_{n=1}^N$ of the right hand side (f,g) is required. In all instances $(f,g)_{n+1}^N = \frac{1}{\tau}\int_{n\tau}^{(n+1)\tau}(f_N, g_N)(\xi)d\xi$ suffices where $\tau = T/N$ and the subscript N indicates the function is to be truncated at $\pm N$ to render it bounded.
- In the construction of the solution in $\mathcal{V}'$, subsequences of $(\alpha,a)^N$ were chosen to obtain weakly convergent subsequences. However, this is unnecessary since the whole sequence converged in $C[0,T;L^1(\Omega)^2]$, so could not have two weak limits in $\mathcal{W}'$. This implies that the sequences $\{(g^N, -g^N + b^N)\}_{N=1}^\infty$ also converged weakly in $\mathcal{V}'$, where $g^N = \gamma(u^N - v^N)$ and $b^N = -\sum_{j=0}^n \frac{\partial}{\partial x_j}[k_j(\frac{\partial v^N}{\partial x_j})]$.

3. Properties of the solutions.

3.1. Non-negativity of the solutions.

Lemma 1. *Let (α, a) and $(\tilde{\alpha}, \tilde{a})$ be two solutions to Problem 1.3 corresponding to data (α_0, a_0), (f,g), and $(\tilde{\alpha}_0, \tilde{a}_0)$, $(\tilde{f}, \tilde{g})$ respectively. If*

$$(f,g) \le (\tilde{f}, \tilde{g}) \quad \text{and} \quad (\alpha_0, a_0) \le (\tilde{\alpha}_0, \tilde{a}_0) \text{ a.e.}$$

then $(\alpha, a) \le (\tilde{\alpha}, \tilde{a})$ almost everywhere (where comparison of a pair means component wise comparison).

Proof: The differencing scheme (1.5) for the approximate solutions yields

$$\frac{1}{\tau}(\alpha_{n+1}^N - \tilde{\alpha}_{n+1}^N) + [\gamma(u_{n+1}^N - \tilde{v}_{n+1}^N) - \gamma(\tilde{u}_{n+1}^N - \tilde{v}_{n+1}^N)] = f_{n+1}^N - \tilde{f}_{n+1}^N$$

$$\frac{1}{\tau}(a_{n+1}^N - \tilde{a}_{n+1}^N) - [\gamma(u_{n+1}^N - \tilde{v}_{n+1}^N) - \gamma(\tilde{u}_{n+1}^N - \tilde{v}_{n+1}^N)] + B(v_{n+1}^N) - B(\tilde{v}_{n+1}^N)$$
$$= g_{n+1}^N - \tilde{g}_{n+1}^N$$

where v_{n+1}^N, $\tilde{v}_{n+1}^N \in D(B)$. Multiplying the above equations by

$$\begin{aligned} s_1 &= \operatorname{sgn}_0^+[(\alpha_{n+1}^N - \tilde{\alpha}_{n+1}^N) + (u_{n+1}^N - \tilde{u}_{n+1}^N)] \\ &\in \operatorname{sgn}^+(\alpha_{n+1}^N - \tilde{\alpha}_{n+1}^N) \cap \operatorname{sgn}^+(u_{n+1}^N - \tilde{u}_{n+1}^N) \end{aligned}$$

and

$$\begin{aligned} s_2 &= \operatorname{sgn}_0^+[(a_{n+1}^N - \tilde{a}_{n+1}^N) + (v_{n+1}^N - \tilde{v}_{n+1}^N)] \\ &\in \operatorname{sgn}^+(a_{n+1}^N - \tilde{a}_{n+1}^N) \cap \operatorname{sgn}^+(v_{n+1}^N - \tilde{v}_{n+1}^N) \end{aligned}$$

respectively, and integrating over Ω yield

$$\int_\Omega \Big\{\frac{1}{\tau}(\alpha_{n+1}^N - \tilde{\alpha}_{n+1}^N)^+ + \frac{1}{\tau}(a_{n+1}^N - \tilde{a}_{n+1}^N)^+$$
$$+ [\gamma(u_{n+1}^N - v_{n+1}^N) - \gamma(\tilde{u}_{n+1}^N - \tilde{v}_{n+1}^N)](s_1 - s_2) + [B(v_{n+1}^n) - B(\tilde{v}_{n+1}^N)]s_2\Big\} \leq 0.$$

Since the latter two terms in the integral are non-negative, $(\alpha_{n+1}^N - \tilde{\alpha}_{n+1}^N)^+ = 0$ and $(a_{n+1}^N - \tilde{a}_{n+1}^N)^+ = 0$ almost everywhere, implying $(\alpha^N, a^N) \leq (\tilde{\alpha}^N, \tilde{a}^N)$ almost everywhere. Since each of these pairs converge in $L^1(\Omega)^2$ it follows that their limits preserve this ordering.

Corollary 1. *If $0 \in A_1(0)$ and $0 \in A_2(0)$ and (α, a) is a solution to Problem 1.3 with non-negative data (f, g) and (α_0, a_0), then (α, a) is non-negative.*

Proof: The assumptions $\gamma(0) = 0$ and $k_j(0) = 0$ imply that $(\tilde{u}, \tilde{v}) = (\tilde{\alpha}, \tilde{a}) = 0$ is a solution to Problem 1.3 with zero data. The Lemma then yields $(\alpha, a) \geq (\tilde{\alpha}, \tilde{a}) = 0$. ∎

The following example shows that the pressures (u, v) need not be non-negative unless additional hypotheses are specified.

Example: Let γ be linear on $(-\infty; -s_0)$ and $(s_0; \infty)$ where $s_0 > 0$, and let $\gamma(s) = 0$ on $[-s_0; s_0]$. Set $A_1 = A_2$ to be the Heaviside graph. Then $(\alpha, a) = 0$, $(u, v) = (-s, 0)$, $s \in [0; s_0]$, is a solution to Problem 1.3 with data $(f, g) = (\alpha_0, a_0) = 0$.

In the above example, both the 'spatial' (γ and B) and 'temporal' (A_1 and A_2) operators were degenerate near zero. If either one of them is strictly monotone at zero the following Lemma demonstrates that the pressures, (u, v), will be non-negative when the data (f, g) and (α_0, a_0) are non-negative.

Lemma 2. *Let $0 \in A_1(0)$, $0 \in A_2(0)$ and let (f, g) and (α_0, a_0) be non-negative data satisfying the assumptions of Theorem 2.5. Then Problem 1.3 has a corresponding solution with non-negative pressures. If either*

(1) *$A_1(-\infty; 0) \subset (-\infty; 0)$ and $A_2(-\infty; 0) \subset (-\infty; 0)$, or*

(2) *The (spatial) operator $\psi' : V \to V'$ defined by*

$$\psi'(u, v) = \Big[\gamma(u - v), -\gamma(u - v) - \sum_{j=1}^n \frac{\partial}{\partial x_j}[k_j(\frac{\partial v}{\partial x_j})]\Big]$$

is strictly monotone at zero,

then all of the solutions constructed in Theorem 2.5 have non-negative pressures.

Proof: The first statement will follow from (1) of the second, since if $(\tilde{u}, \tilde{v})$, $(\tilde{\alpha}, \tilde{a})$ is a solution to

$$\frac{d}{dt}\tilde{\alpha} + \gamma(\tilde{u} - \tilde{v}) = f$$
$$\frac{d}{dt}\tilde{a} - \gamma(\tilde{u} - \tilde{v}) + B(\tilde{v}) = g$$

$\tilde{\alpha} \in \tilde{A}_1(\tilde{u})$, $\tilde{a} \in \tilde{A}_2(\tilde{v})$ where $A_1 = \tilde{A}_1$ and $A_2 = \tilde{A}_2$ on $[0, \infty)$ and are perturbed to be strictly monotone on $(-\infty; 0)$, then (1) will imply that $(\tilde{\alpha}, \tilde{a})$ and $(\tilde{u}, \tilde{v})$ are non-negative, so constitute a solution to the original problem.

Proof of (1): The non-negativity of the data implies, by Corollary 1, that (α, a) is non-negative. But $\alpha \in A_1(u)$ and $a \in A_2(v)$ almost everywhere and (1) imply (u, v) are non-negative.

Proof of (2): Consider the approximate equations for (u^N, v^N) and (α^N, a^N) and write $\alpha^N_{n+1} = \alpha_{n+1}$, etc. Multiplying the equations by (u^-_{n+1}, v^-_{n+1}) and integrating yields

$$\int_\Omega \{\frac{1}{\tau}(\alpha_{n+1}u^-_{n+1} + a_{n+1}v^-_{n+1}) + \gamma(u_{n+1} - v_{n+1})(u^-_{n+1} - v^-_{n+1})$$
$$+ \sum_{j=1}^n k_j(\frac{\partial v_{n+1}}{\partial x_j})\frac{\partial v^-_{n+1}}{\partial x_j}\} \leq 0.$$

The hypotheses $0 \in A_1(0)$ and $0 \in A_2(0)$ imply $\alpha_{n+1}u^-_{n+1} = a_{n+1}v^-_{n+1} = 0$ almost everywhere. Since

$$\sum_{j=1}^n k_j(\frac{\partial v_{n+1}}{\partial x_j})\frac{\partial v^-_{n+1}}{\partial x_j} = \sum_{j=1}^n k_j(\frac{\partial v_{n+1}}{\partial x_j})\frac{\partial v^-_{n+1}}{\partial x_j}$$

and a case by case comparison shows

$$\gamma(u^-_{n+1} - v^-_{n+1})(u^-_{n+1} - v^-_{n+1}) \leq \gamma(u_{n+1} - v_{n+1})(u^-_{n+1} - v^-_{n+1}),$$

it follows that

$$\int_\Omega \{\gamma(u^-_{n+1} - v^-_{n+1})(u^-_{n+1} - v^-_{n+1}) + \sum_{j=1}^n k_j(\frac{\partial v_{n+1}}{\partial x_j})\frac{\partial v^-_{n+1}}{\partial x_j}\} \leq 0.$$

The above is equivalent to $\psi'(u^-_{n+1}, v^-_{n+1})(u^-_{n+1}, v^-_{n+1}) \leq 0$ and the strict monotonicity at zero implies $(u^-_{n+1}, v^-_{n+1}) = 0$.

3.2. Bounds on the pressures. Theorem 2.5 gives explicit bounds on the densities, (α, a) when the data are bounded. If A_1 and A_2 are unbounded (above or below) then a bound on (α, a) will imply a corresponding bound on the pressures (above or below). This is the situation encountered when the fluid under consideration is a gas. When the fluid under consideration is a liquid A_1 and A_2 will normally be bounded, so no bounds on the pressures are implied. For a purely incompressible fluid, the next lemma demonstrates that the pressures are bounded above when the data are bounded. Of course the previous section identifies when the pressures are bounded below by zero.

Lemma 3. *Let A_1 and A_2 be Heaviside graphs and suppose that (α, a), (u, v) is a solution to Problem 1.3 with data $(\alpha_0, a_0) \in L^\infty(\Omega)^2$ and $(f, g) \in L^\infty[0, T; L^\infty(\Omega)] \times L^\infty[0, T; L^{t'}(\Omega)]$ where $t' > n/p$. Then the pressures (u, v) are bounded above.*

Proof: This is proved by showing that the approximating sequences from the finite differencing scheme (1.5) are bounded above. The superscript N will be omitted (i.e., $(u_{n+1}^N, v_{n+1}^N) = (u_{n+1}, v_{n+1})$, etc.).

Let $E \subset \Omega$ be a measurable set and $\|f\|_{L^\infty(\Omega)} \leq k$. Subtract k from both sides of the first finite difference equation and multiply by $\chi_E(u_{n+1} - k)^+$ (χ_E is the characteristic function for E) to get

$$\int_E \{\frac{1}{\tau}(\alpha_{n+1} - \alpha_n) + \big[\gamma(u_{n+1} - v_{n+1}) - k\big]\}(u_{n+1} - k)^+ \leq 0.$$

$(\alpha_{n+1} - \alpha_n)(u_{n+1} - k)^+$ is non-negative, since it is clearly zero when $u_{n+1} \leq k$, and when u_{n+1} is positive, $\alpha_{n+1} = 1$ while $0 \leq \alpha_n \leq 1$. This implies

$$\int_E \big[\gamma(u_{n+1} - v_{n+1}) - k\big](u_{n+1} - k)^+ \leq 0$$

for every measurable $E \subset \Omega$, so the integrand must be non-positive almost everywhere, in particular $u_{n+1} > k$ implies $\gamma(u_{n+1} - v_{n+1}) \leq k$.

Let $l \geq k$ and multiply the second finite difference equation by $w = (v_{n+1} - l)^+ \in W_0^{1,p}(\Omega)$ to get

$$\int_\Omega \{\frac{1}{\tau}(a_{n+1} - a_n)w - \gamma(u_{n+1} - v_{n+1})w + \sum_{j=1}^n k_j(\frac{\partial v_{n+1}}{\partial x_j})\frac{\partial w}{\partial x_j}\} = \int_\Omega g_{n+1} w.$$

The argument of the previous paragraph shows that the first term in the integral is non-negative, and a case by case check, using the result of the above paragraph, shows that when $l \geq k$

$$-\gamma(u_{n+1} - v_{n+1})w \geq -kw.$$

This yields the estimate

$$\int_\Omega \sum_{j=1}^n k_j(\frac{\partial v_{n+1}}{\partial x_j})\frac{\partial w}{\partial x_j} \leq \int_\Omega (g_{n+1} + k)w.$$

Since $\nabla w = \chi_{E_l} \nabla v_{n+1}$, where $E_l = \{\mathbf{x} \in \Omega \,|\, v_{n+1} > l\}$, it follows that

$$\int_{E_l} \sum_{j=1}^n k_j(\frac{\partial w}{\partial x_j})\frac{\partial w}{\partial x_j} \leq \int_{E_l} \tilde{g} w,$$

where $\tilde{g} = g_{n+1} + k$.

The remainder of the proof is a standard application of the Di Gorgi technique for bounding solutions to elliptic equations, which also satisfy an estimate similar to the one obtained above. Since the results are standard, only a terse outline will be presented.

Assumption [K] implies

$$k_0\|\nabla w\|^p_{L^p(E_l)} \le \|\tilde g\|_{L^{t'}(E_l)}\|w\|_{L^t(E_l)} + nk_0\xi_0^p|E_l|.$$

Using the Sobolev and Young inequalities to eliminate w from the right hand side yields

$$\|\nabla w\|^p_{L^p(E_l)} \le \{C|E_l|^{\frac{1}{t}-\frac{1}{p^*}}\|\tilde g\|_{L^{t'}(E_l)}\}^{p'} + 2n\xi_0^p|E_l|,$$

where $p^* = np/(n-p)$ if $p<n$ and if $p=n$ any $p^* > tp$ suffices. The estimate

$$(h-l)^p|E_h|^{p/p^*} \le \|w\|^p_{L^{p^*}(E_l)} \le C\|\nabla w\|^p_{L^{p^*}(E_l)}, \quad h \ge l$$

yields

$$|E_h| \le C\big(\|\tilde g\|^{p'/p}_{L^{t'}(\Omega)} + 1\big)^{p^*}\frac{|E_l|^\beta}{(h-l)^{p^*}}$$

provided $h \ge l$ and l is chosen larger than k. The constant β is given by

$$\beta = \frac{p^*}{p}\min\big[1, p'(\frac{1}{t}-\frac{1}{p^*})\big] > 1 \ \text{ when } \ t' > \frac{n}{p}.$$

This estimate yields $|E_L| = 0$ for some large L depending only upon the data, C, $\tilde g$ and β, (see Kinderlehrer and Stampacchia [10]), implying $v_{n+1} \le L$. The bound on u follows from $\gamma(u_{n+1} - v_{n+1}) < k$ if $u_{n+1} > k$.

3.3. Existence of free surfaces. Non-linear diffusion equations often have solutions which exhibit free surfaces. In the current context a free surface indicates an interface where the pressure goes to zero. Such surfaces are observed physically; however, linear equations fail to predict this phenomenon. For the gas and liquid models discussed in Section 1, a free surface corresponds to the fact that the domain, Ω, fills with fluid slowly. In this section it is demonstrated that free surfaces may exist for the solutions developed in Section 2.

The proof of the existence of a free surface for an incompressible fluid is particularly simple, so is presented first. It is interesting to note that the more involved proof required for compressible fluids fails in this case.

Lemma 4. *Let (u,v), (α,a) be a solution to Problem 1.3 with*

$$A_1(\mathbf{x},s) = r_1(\mathbf{x})H(s), \quad A_2(\mathbf{x},s) = r_2(\mathbf{x})H(s), \quad \mathbf{x}\in\Omega, \ s \in \mathbf{R}$$

where H is the Heaviside graph, r_1, $r_2 \in L^\infty(\Omega)$ and r_1, $r_2 \ge r_0 > 0$ almost everywhere. If

$$\frac{1}{r_0}\|\alpha_0\|_{L^1(\Omega)} < |\Omega| \quad\text{and}\quad \frac{1}{r_0}\|a_0\|_{L^1(\Omega)} < |\Omega|,$$

then there exists $t_0 > 0$ such that the measure of the sets

$$E_1(t) = \{\mathbf{x}\in\Omega \,|\, u(\mathbf{x},t) > 0\} \quad\text{and}\quad E_2(t) = \{\mathbf{x}\in\Omega \,|\, v(\mathbf{x},t) > 0\}$$

is strictly less than $|\Omega|$ for $t \in [0;t_0)$.

Proof: Theorem (2.5) guarantees that $(\alpha, a) \in C[0,T;L^1(\Omega)^2]$. Then

$$|E_1(t)| \le \frac{1}{r_0}\int_{E_1(t)} \alpha(t) \le \frac{1}{r_0}\int_\Omega \alpha(t) \le \frac{1}{r_0}\|\alpha(t)\|_{L^1(\Omega)}.$$

The continuity of $\|\alpha(t)\|_{L^1(\Omega)}$ implies that if the right hand side of the above expression is smaller than $|\Omega|$ at some point in time, then it will remain so for small neighboring times.

The proof that $|E_2(t)| \le |\Omega|$ for small times is identical. ∎

This result can be extended to cover compressible fluids using the ideas of Diaz and Veron [6]. We omit the proof of the following result since it is identical in spirit to [6], and rather long.

Theorem 1. *Let (α, a), (u, v) be a solution to Problem 1.3 with data $(\alpha_0, a_0) \in W'$ and $(f, g) \in \mathcal{V}' \cap L^1[0,T;L^1(\Omega)^2]$. Assume*

- *[A′] A_1 and A_2 satisfy Assumption [A] and in addition $0 \in A_1(0)$, $0 \in A_2(0)$ and for some $r \in (1,p)$, $a \in A_2(v) \Rightarrow \phi_2^*(a) \ge c|v|^r$ where ϕ_2 is the convex function defined in Definition 2.1 ($A_2 = \partial\phi_2$).*
- *[K′] The functions k_j satisfy Assumption [K] and the strengthened estimate*

$$k_0|\xi|^{p-1} \le |k_j(\xi)| \le k_1|\xi|^{p-1}, \quad \xi \in \mathbb{R}.$$

If there is a $0 < t_1 \le T$ such that for $t \in [0, t_1)$

$$B_{\rho_0}(\mathbf{x}_0) \subset \Omega - \{\mathrm{supp}[f(t)] \cup \mathrm{supp}[g(t)] \cup \mathrm{supp}(\alpha_0) \cup \mathrm{supp}(a_0)\},$$

then there exists $t_0 > 0$ such that $v = 0$ on the cylinder $B_{\rho_0/2}(\mathbf{x}_0) \times [0, t_0)$. If either γ is strictly monotone at zero or $\phi_1^*(\alpha) = 0 \Rightarrow u = 0$ when $\alpha \in A_1(u)$, then $u = 0$ on this cylinder.

Since the behavior of ϕ^* is not readily observable from its inverse sub-gradient, A, some monotone graphs are listed with their convex conjugates below.

$A(s)$	$\phi^*(t)$, $t \in D(\phi^*)$	$\phi^* \circ A(s)$, $s \in D(A)$	$\phi^* \circ A(s) \ge c\|s\|^r$
$H(s)$	0	0	never
$1 - e^{-\beta s}$, $s \ge 0$	$\frac{1}{\beta}\frac{t}{t-1}$	$\frac{1}{\beta}[e^{\beta s} - 1]$	any $r > 1$
s^{r-1}	$\frac{1}{r'}t^{r'}$	$\frac{1}{r'}s^r$	any $r > 1$

3.4. Support of solutions. The solutions obtained for Problem 1.3 are sufficiently smooth to guarantee that the derivative $\frac{d}{dt}\alpha$ is a function, so all of the terms in the first equation of Problem 1.3 are (real valued) functions. The following lemma indicates that elementary calculus may be used to determine the properties of solutions to this equation.

Lemma 5. *If* $\alpha \in W^{1,q'}[0,T; L^{q'}(\Omega)]$, *then there exists* $A \in L^{q'}[(0,T) \times \Omega]$ *such that*

1) $\alpha(t) = A(t, \cdot)$ *almost everywhere,*
2) $A(\cdot, \mathbf{x})$ *is absolutely continuous on* $[0,T]$ *almost everywhere* $\mathbf{x}$, *and*
3) $\frac{\partial}{\partial t} A(t, \cdot) = \alpha'(t)$ *almost everywhere.*

This is proved in Showalter [11]. DiBenedetto and Showalter [8] proved the following lemma when γ and B are linear.

Lemma 6. *Let* (α, a), (u, v) *be a solution to Problem 1.3 with data* (f, g), (α_0, a_0) *and assume* α_0, $f \geq 0$ *and* $(u, v) \geq 0$. *If* $(A_1)_0(s) \geq c\gamma(s)^{\frac{1}{r}}$, $r \geq 1$, $s \in [0, \|u\|_{L^\infty(\Omega)}]$, *where* $(A_1)_0$ *is the minimal section, then* $\alpha_0(\mathbf{x}) > 0$ *implies* $\alpha(\mathbf{x}, t) > 0$ *for* $t \geq 0$ *and it is bounded below by*

$$\alpha(\mathbf{x}, t) \geq \begin{cases} [\alpha_0(\mathbf{x})^{1-r} + c(r-1)t]^{\frac{1}{1-r}} & r > 1 \\ e^{-\frac{t}{c}} \alpha_0(\mathbf{x}) & r = 1. \end{cases}$$

Proof: $\alpha \in A_1(u)$ implies $\alpha \geq c\gamma(u)^{\frac{1}{r}}$. The equation for α may be written as

$$\frac{d\alpha}{dt} + \gamma(u) = f + \gamma(u) - \gamma(u - v)$$

where the right hand side is non-negative since $v \geq 0$ and γ is monotone. It follows that

$$\frac{d\alpha}{dt} + (\frac{\alpha}{c})^r \geq 0. \quad \text{a.e. } \mathbf{x} \in \Omega.$$

The lemma now follows from an elementary calculus exercise with the above equation.

Remark: The Heaviside graph satisfies the hypothesis $H_0(s) \geq (\frac{s}{s_0})^{1/r}$ for any $r \geq 1$ and $0 \leq s \leq s_0$, so this lemma is applicable to purely incompressible fluids provided the pressures are bounded. Bounds on the pressures were found in Lemma 4.

Corollary 2. *If, in addition to the hypothesis for Lemma* 6, $A_1(0) \subset (-\infty; 0]$, *then* $\{\mathbf{x} \in \Omega \,|\, u(t) > 0\}$ *is non-decreasing.*

If, in addition, $\gamma(-\infty; 0) \subset (-\infty; 0)$ *(i.e.,* γ *is strictly monotone at* 0^-*), then*

$$\{\mathbf{x} \in \Omega \,|\, v(t) > 0\} \subset \{\mathbf{x} \in \Omega \,|\, u(t) > 0\}.$$

Proof: The first statement follows immediately from the fact $\alpha \in A_1(u)$. To prove the second statement suppose $\mathbf{x} \in \Omega - \{\mathbf{x} \in \Omega \,|\, u(t) > 0\}$. It follows that $u(\mathbf{x}, \tau) = 0$ and $\alpha(\mathbf{x}, \tau) = 0$ for $0 \leq \tau \leq t$. Then $\frac{d\alpha}{dt} + \gamma(u - v) = f$ implies $\gamma(0 - v) = f \geq 0$ at $(\mathbf{x}, \tau)$, $0 \leq \tau \leq t$, and the assumption on γ yields $v(\mathbf{x}, \tau) = 0$.

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