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PARTIALLY SATURATED FLOW IN A POROELASTIC MEDIUM

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Abstract. The formulation and existence theory is presented for a system modeling diffusion of a slightly compressible fluid through a partially saturated poroelastic medium. Nonlinear effects of density, saturation, porosity and permeability variations with pressure are included, and the seepage surface is determined by a variational inequality on the boundary.

1. **Introduction.** We consider a system modeling diffusion of a slightly compressible fluid through a partially saturated porous elastic medium $\Omega \subset \mathbb{R}^3$ for which the deformations vary sufficiently slowly that the inertia effects are negligible. This is the *quasi-static* assumption. We denote the fluid *density* by $\rho(x,t)$ and its *pressure* by $p(x,t)$ for $x \in \Omega$. Assume that the fluid is *barotropic*, *i.e.*, the density and pressure are related by the *state equation* $\rho = \rho(p)$, where the non-decreasing constitutive function $\rho(\cdot)$ characterizes the type of fluid. The (small) *displacement* from the position $x \in \Omega$ is denoted by $\mathbf{u}(x,t)$. In a homogeneous and isotropic medium the *partially saturated consolidation problem* takes the form

$$-(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\Delta\mathbf{u} + \nabla(\chi(p)\,p) = \mathbf{F}(x,t)\,, \tag{1a}$$

$$\frac{\partial}{\partial t}(\varphi(p)S(p)\rho(p) + \nabla\cdot\mathbf{u}) + \nabla\cdot(\rho(p)\,\mathbf{q}) = F(x,t)\,, \tag{1b}$$

$$\mathbf{q} = -k(p)(\nabla p + \rho(p)\mathbf{g})\,, \tag{1c}$$

consisting of the *equilibrium equation* for momentum conservation, the *storage equation* for mass conservation, and *Darcy's law* for the filtration velocity, $\mathbf{q}$. The function $\varphi(\cdot)$ is *porosity*, $S(\cdot)$ is *saturation*, and $k(\cdot)$ is the *permeability* for the laminar flow in the medium. All of these functions are non-negative and pressure dependent. The (linearized) *strain* tensor $\varepsilon_{kl}(\mathbf{u}) \equiv \frac{1}{2}(\partial_k u_l + \partial_l u_k)$ provides a measure of the local deformation of the body, and the term $\nabla\cdot\mathbf{u} = \varepsilon_{kk}(\mathbf{u})$ represents the *fluid content* due to the local volume dilation. The *total stress* σ_{ij} is the sum the

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effective stress of the of the purely elastic *isotropic* structure given by *Hooke's law* and *effective pressure* stress of the fluid on the structure, hence,

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} - \delta_{ij}\chi(p)\,p\,,$$

with positive Lamé constant λ and shear modulus μ. The *Bishop parameter* $\chi(\cdot)$ is a measure of the fraction of pore surface in contact with the fluid. Let the negative pressure $p_0 < 0$ denote the *capillary tension.* The saturation function $S(\cdot)$ is monotone with $S(p) = 1$ for $p \geq p_0$, and the Bishop parameter is well approximated in many situations by $\chi(p) \approx S(p)$.

Corresponding to a pressure $p(\cdot,\cdot)$ for a solution of the system (1) in the context of soil mechanics, the medium is fully saturated in the *groundwater region*, $\{x \in \Omega :\ p(x,t) > p_0\}$, while in the *capillary fringe*, $\{x \in \Omega :\ p(x,t) < p_0\}$, it is only partially saturated. The *phreatic surface* $\{x \in \Omega :\ p(x,t) = p_0\}$ is the unknown interface that separates these regions. The boundary of Ω is given by the disjoint union of the parts Γ_D and Γ_{fl}, and Γ_{fl} is further written as the disjoint union of Γ_N and Γ_U. The part Γ_{fl} is the *flux boundary.* On its complement, Γ_D, the value of pressure is given by the depth below the surface:

$$p(x,t) = d(x_3), \quad x = (x_1,x_2,x_3) \in \Gamma_D, \tag{2a}$$

where $d(\cdot) > 0$. On Γ_N there is no flow, so we have a null normal flux:

$$\rho(p)\,\mathbf{q}\cdot\mathbf{n} = 0, \quad x \in \Gamma_N, \tag{2b}$$

where $\mathbf{n}$ is the unit outward normal on the boundary, $\partial\Omega$. On Γ_U we have

$$p \leq 0,\ \rho(p)\mathbf{q}\cdot\mathbf{n} \geq 0,\ p\,\rho(p)\mathbf{q}\cdot\mathbf{n} = 0, \quad x \in \Gamma_U. \tag{2c}$$

Thus, the fluid pressure on the boundary cannot exceed the outside null pressure of air, and there can be no flow into Ω. Also, $p = 0$ on the *seepage surface* which is that part of Γ_U where $\mathbf{q}\cdot\mathbf{n} > 0$, and there is no flow from the boundary above that, where $p < 0$. The boundary conditions on $\partial\Omega$ will also involve the displacement or the *tractions* $\sigma_{ij}(x,t)n_j$ on $\partial\Omega$, namely,

$$u_i = 0 \text{ on } \Gamma_0\,, \quad \sigma_{ij}(x,t)n_j = t_i \text{ on } \Gamma_{tr}\,, \quad 1 \leq i \leq 3, \tag{2d}$$

where Γ_0 and Γ_{tr} are given complementary subsets of the boundary. Finally, we shall require that the initial value of the water content $\theta_0(\cdot)$ be specified,

$$\varphi(p(x,0))S(p(x,0))\rho(p(x,0)) + \nabla\cdot\mathbf{u}(x,0) = \theta_0(x)\,, \quad x \in \Omega\,, \tag{3}$$

where the initial displacement satisfies the constraint

$$-(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}(x,0)) - \mu\Delta\mathbf{u}(x,0) + \nabla(\chi(p(x,0))\,p(x,0)) = \mathbf{F}(x,0)$$

together with the boundary conditions (2d).

We have taken the model for partially saturated flow in which the saturation $S(\cdot)$ is given by a continuous monotone function which increases from near zero to unity in the vicinity of the capillary tension. The limiting case of saturated-unsaturated flow in which this function is replaced by a step function corresponds to a free boundary problem which describes large scale behavior in some sense, and this is known as the *dam problem.* Mathematical treatment in the case of a *rigid medium* began with the fundamental work of Baiocchi (1972) [5] and the extension to the non-stationary case by Torelli (1975) [25]. More general situations including the partially saturated case were treated by Gilardi (1979) [15], Visintin (1980) [26], Hornung (1982) [17], Alt-Luckhaus (1983) [2] and Alt-Luckhaus-Visintin (1984) [3] by working directly with the pressure. The case of *fully saturated flow* in an elastic medium is the Biot problem of consolidation. See Biot (1941) [7] and (1955) [8], Rice and Cleary (1976) [20], and Huyakorn-Pinder (1983) [18]. The mathematical issues of well-posedness for the linear quasi-static case were first studied in the fundamental work of J.-L Auriault and Sanchez-Palencia (1977) [4]. They derived a non-isotropic form of the Biot system by homogenization and then obtained a *strong* solution. In the later paper of Zenisek (1984) [27] the *weak* solution is obtained in the first order Sobolev space $H^1(\Omega)$, so the equations hold in the dual space, $H^{-1}(\Omega)$ (see below). The existence, uniqueness, and regularity theory for the Biot system together with extensions to include the possibility of viscous terms arising from secondary consolidation and the introduction of appropriate boundary conditions at both closed and drained interfaces were recently given in Showalter (2000) [23]. In the following we shall extend the method developed there to include both elastic deformation and partial saturation of the medium. This is the first mathematical proof of existence to include both aspects. See Zienkiewicz *et al.* (1980) [28] and (1999) [29] for additional perspectives in modeling and numerical simulation.

1.1. **The Semi-Linear Case.** Assume that there is a constant $\alpha > 0$ for which

$$(p\,\chi(p))' = \alpha\rho(p)\,k(p)\,, \quad p \in \mathbb{R}\,. \tag{4}$$

This relates the Bishop parameter $\chi(\cdot)$ to the density $\rho(\cdot)$ and relative permeability $k(\cdot)$. Since the product $\rho(\cdot)\,k(\cdot)$ is positive, this shows that $p\,\chi(p)$ is monotone. Furthermore, when $\rho(\cdot)\,k(\cdot)$ is monotone, it follows that $p\,\chi(p)$ is convex, so $\chi(\cdot)$ is monotone. Note that our assumption (4) requires that the pressure stress is given by

$$\nabla(p\,\chi(p)) = \alpha\rho(p)\,k(p)\,\nabla p\,,$$

i.e., the pressure component of the Darcy velocity. This relates the flux to the *viscous resistance* of the medium to the fluid flow.

The typical form for the permeability is a monotone function $k(\cdot)$ with $k(p) = k_0$ for $p > p_0$ and $k(p) = k_1$ for $p < p_1$, where $p_1 < p_0 < 0$ and $0 \le k_1 < k_0$ are given. As a check on the consistency of the assumption (4), let's take $k(\cdot)$ to be given as above and $\rho = \rho_0$, a constant. Then choose $\alpha^{-1} = k_0\rho_0$ to get

$$\frac{d}{dp}(p\,\chi(p)) = \frac{k(p)}{k_0}.$$

We compute directly the following:
Case 1. Let $k(p) = k_0$ for $p_0 \le p$, with $p_0 < 0$. Then $\chi(p) = 1$ for $p_0 \le p$.
Case 2. If $k(p) = \dfrac{(k_0 - k_1)}{(p_0 - p_1)}(p - p_1)$, we get

$$\chi(p) = \frac{(k_0 - k_1)(p - p_1)^2}{2k_0(p_0 - p_1)p} + \frac{p_0}{p}\{1 - \frac{1}{2}(1 - \frac{k_1}{k_0})(1 - \frac{p_1}{p_0})\}$$

for $p_1 < p < p_0$.
Case 3. $k(p) = k_1$ for $p \le p_1$, with $0 \le k_1 < k_0$ and $p_1 < p_0$. Then $\chi(p) = \dfrac{k_1}{k_0} + \dfrac{const}{p}$ for $p \le p_1$. Note that $\dfrac{k_1}{k_0} < 1$.
This example shows that the Bishop parameter $\chi(\cdot)$ resulting from the assumption is quite similar to the saturation $S(\cdot)$, as expected, and its form will not be significantly changed from modest perturbations in $k(\cdot)$ and $\rho(\cdot)$.

1.2. **The Unilateral Poro-Elasticity Problem.** Let the function $K(\cdot)$ be defined by $K'(p) = \rho(p)\, k(p)\,, K(0) = 0$. We make a change of variable, $P = K(p)$, and then in the preceding notation we write our system in the form

$$-(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\Delta\mathbf{u} + \alpha\nabla P = \mathbf{F}(x,t)\,, \quad x\in\Omega\,, \tag{5a}$$

$$\frac{\partial}{\partial t}(b(P) + \nabla\cdot\mathbf{u}) - \nabla\cdot(\nabla P + \mathbf{g}(P)) = F(x,t)\,, \tag{5b}$$

$$u_i = 0 \text{ on } \Gamma_0\,, \quad \sigma_{ij}(x,t)n_j = t_i \text{ on } \Gamma_{tr}\,, \quad 1\le i\le 3, \tag{5c}$$

$$P(x,t) = d(x_3), \quad x = (x_1,x_2,x_3)\in\Gamma_D, \tag{5d}$$

$$\frac{\partial P}{\partial n} + \mathbf{g}(P)\cdot\mathbf{n} = 0, \quad x\in\Gamma_N, \tag{5e}$$

$$P\le 0,\ \frac{\partial P}{\partial n} + \mathbf{g}(P)\cdot\mathbf{n}\le 0,\ P\,(\frac{\partial P}{\partial n} + \mathbf{g}(P)\cdot\mathbf{n}) = 0, \quad x\in\Gamma_U\,, \tag{5f}$$

where $\mathbf{g}(P) \equiv ((\rho^2\, k)\circ K^{-1}(P))\mathbf{g}$ and $\sigma_{ij}n_j = (\lambda\varepsilon_{kk} - \alpha P)n_i + 2\mu\varepsilon_{ij}n_j$. Note that we have replaced $K(d(\cdot))$ by $d(\cdot)$. We shall assume that the nonlinear function $b(\cdot) \equiv (\varphi(\cdot)\, S(\cdot)\, \rho(\cdot))\circ K^{-1}(\cdot)$ is monotone and that both $\mathbf{g}(\cdot)$ and $b(\cdot)$ are Lipschitz continuous.

1.3. **The Plan.** We begin in Section 2 by introducing some notions from abstract variational calculus and related operators. Then we construct the operators used to formulate our general partially-saturated poro-elasticity problem in Section 3. This extended model includes a new boundary condition which reflects the proportion of sealed or exposed pores on the boundary. This proportion affects the fraction of pressure stress and the fluid content due to dilation on the boundary. The statement of this problem and a discussion of these more general boundary conditions are given in Section 3.4. Our goal is to prove that there exists an appropriately regular solution of this problem. This is stated as Theorem 4.1. First an abstract result of DiBenedetto and Showalter (1981) [12] on the existence of solutions of *doubly–nonlinear evolution equations* is recalled in Section 4. Then appropriate a priori estimates are obtained in Section 5 in order to treat the special case with no gravity as an application. Finally, this is extended to include gravity in the following Section 6.

2. **Preliminaries.**

2.1. **Convex Analysis.** We recall maximal monotone operators and related notions. Let V be a Hilbert space with inner product $(\cdot,\cdot)$. If V' denotes the dual of V, the Riesz representation theorem gives the isomorphism $\mathcal{R} : V \to V'$ defined by

$$(u,v) = \langle \mathcal{R}u, v\rangle \ \forall\ u,v\in V\ ,$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between V' and V. A subset $\mathcal{A}\subset V\times V'$, is called *monotone* if

$$\langle v_2 - v_1, u_2 - u_1\rangle \ge 0, \quad [u_i, v_i]\in\mathcal{A},\ i = 1,2.$$

Such an $\mathcal{A}$ is a (possibly) multivalued operator from V to V' for which $v\in\mathcal{A}(u)$ means $[u,v]\in\mathcal{A}$. The monotone $\mathcal{A}$ is *maximal monotone* if it has no monotone proper extension in $V\times V'$. This is equivalent to the condition that $(\mathcal{R}+\lambda\mathcal{A})^{-1}\equiv J_\lambda$, the resolvent of $\mathcal{A}$, is a contraction defined on all of V' for any $\lambda > 0$. The *Yosida approximation* of $\mathcal{A}$ is $\mathcal{A}_\lambda \equiv \mathcal{R}(I - J_\lambda\circ\mathcal{R})/\lambda : V\to V'$; it is Lipschitz continuous and monotone. If $u\in V$, then $\mathcal{A}_\lambda(u)\in\mathcal{A}(J_\lambda(u))$. If $\mathcal{A}$ is maximal monotone, $[u_n, v_n]\in\mathcal{A}$, $u_n \rightharpoonup u$ (*i.e.*, u_n converges weakly to u), $v_n\rightharpoonup v$, and $\liminf\langle u_n, v_n\rangle \le \langle u,v\rangle$, then $[u,v]\in\mathcal{A}$. If also $\limsup\langle u_n, v_n\rangle \le \langle u,v\rangle$, then we have $\lim \langle u_n, v_n \rangle = \langle u, v \rangle$. A maximal monotone operator $\mathcal{A}$ on V induces a maximal monotone operator (still denoted by $\mathcal{A}$) defined on $L^2(0,T;V)$ by $v \in \mathcal{A}(u)$ if $v(t) \in \mathcal{A}(u(t))$ *a.e.* on $[0,T]$. It is often convenient to interpret maximal monotone operators as maps from V to 2^V *via* the Riesz isomorphism $\mathcal{R}^{-1} : V' \to V$. We shall use these two notions interchangeably.

A special class of maximal monotone operators is the class of subgradients. If $\psi : V \to (-\infty, \infty]$ is a lower semicontinuous, proper, convex function, then the *subgradient* $\partial\psi \subset V \times V'$ is defined by

$$\partial\psi(u) = \{g \in V' : \langle g, v-u \rangle \le \psi(v) - \psi(u) \ \forall\ v \in V\} \ .$$

In this case, $\partial\psi$ is maximal monotone. The *conjugate* of ψ is the convex function $\psi^* : V' \to \mathbb{R}$ defined by

$$\psi^*(g) = \sup_{u \in V} \left(\langle g, u \rangle - \psi(u)\right) \ .$$

This function is chosen so that $\partial\psi^{-1} = \partial\psi^*$; thus $g \in \partial\psi(u)$ if and only if $u \in \partial\psi^*(g)$, and this is equivalent to $\psi(u) + \psi^*(g) = \langle u, g \rangle$. We assume throughout that $\psi(0) \le 0$ so that $\psi^*(g) \ge 0$ for all $g \in V'$. If $g(\cdot) \in H^1(0,T;V')$ and $[u(\cdot), g(\cdot)]$ belongs to the $L^2(0,T;V)$ realization of $\partial\psi$, then

$$\frac{d}{dt}\psi^*(g(t)) = \left(\frac{d}{dt}g(t), u(t)\right) \ a.e. \text{ on } \ [0,T] \ .$$

If K is a closed, convex, nonempty subset of V, then the *indicator function* $I_K(\cdot)$ of K, given by $I_K(v) = 0$ if $v \in K$ and $I_K(v) = +\infty$ otherwise, is convex, proper, and lower-semi-continuous. Its subgradient is characterized by a *variational inequality* : $f \in \partial I_K(w)$ means

$$f \in V', \ w \in K : \quad f(v - w) \le 0 \text{ for all } v \in K.$$

2.2. **Sobolev Spaces.** We describe the spaces which will be used to develop the variational formulation of the system. Let Ω be a smoothly bounded region in $\mathbb{R}^3$, and denote its boundary by $\Gamma = \partial\Omega$. Denote by $C_0^\infty(\Omega)$ the space of infinitely differentiable functions with support contained in Ω and by $L^2(\Omega)$ the Lebesgue space of (equivalence classes of) functions whose modulus squared is integrable on Ω. For any $w(\cdot) \in L^2(\Omega)$ and j, $1 \le j \le 3$, we denote by $\partial_j w$ its distributional derivative,

$$\langle \partial_j w, \varphi \rangle = -\int_\Omega w(x)\partial_j\varphi(x)\,dx\,, \quad \varphi \in C_0^\infty(\Omega)\,.$$

Let $H^k(\Omega)$ be the *Sobolev space* consisting of those functions in $L^2(\Omega)$ having each of their partial derivatives through order k also in $L^2(\Omega)$. The *trace* map $\gamma : H^1(\Omega) \to L^2(\Gamma)$ is the restriction to the boundary Γ denoted by $\gamma(w) = w|_\Gamma$; we shall denote the range of this map by $\mathrm{Rg}(\gamma) = H^{\frac{1}{2}}(\Gamma)$. The space $H_0^1(\Omega)$ is the closure in $H^1(\Omega)$ of $C_0^\infty(\Omega)$, and it is characterized as the subspace of $H^1(\Omega)$ consisting of those functions whose trace is zero. The dual of $H_0^1(\Omega)$ is the space $H^{-1}(\Omega)$ of distributions on Ω which are first order derivatives of functions in $L^2(\Omega)$. Corresponding spaces of (vector) $\mathbb{R}^3$-valued functions will be denoted by bold face symbols. For example, we denote the product space $L^2(\Omega)^3$ by $\mathbf{L}^2(\Omega)$ and the corresponding triple of Sobolev spaces by $\mathbf{H}^1(\Omega) \equiv H^1(\Omega)^3$. Additional information on these spaces will be recalled from Adams (1975) [1] or Temam (1979) [24] as needed.

3. **The Initial-Boundary-Value Problem.**

3.1. **The Diffusion Operator.** We specify the appropriate spaces and operators to be used to describe the problem (5). Consider first the stationary diffusion system

$$-\nabla \cdot (\nabla p(x) + \mathbf{g}(p(x))) = F(x), \quad x \in \Omega, \tag{6a}$$

$$p = d \text{ on } \Gamma_D, \qquad \frac{\partial p}{\partial n} + \mathbf{g}(p) \cdot \mathbf{n} = g \text{ on } \Gamma_N, \tag{6b}$$

$$p \le 0, \ \frac{\partial p}{\partial n} + \mathbf{g}(p) \cdot \mathbf{n} \le g, \text{ and}$$
$$p\,(\frac{\partial p}{\partial n} + \mathbf{g}(p) \cdot \mathbf{n} - g) = 0 \text{ on } \Gamma_U. \tag{6c}$$

In order to obtain a weak formulation of this mixed unilateral boundary-value problem, we define the Sobolev spaces and convex sets

$$\begin{aligned}
V_1 =& H^1(\Omega), \qquad V_0 = \left\{p \in H^1(\Omega) : \gamma(p) = 0 \text{ on } \Gamma_D\right\},\\
C \equiv& \{\psi \in H^{\frac{1}{2}}(\Gamma) : \psi \le 0 \text{ on } \Gamma_U\},\\
K \equiv& \{p \in V_1 : \gamma(p) = d \text{ on } \Gamma_D \text{ and } \gamma(p) \le 0 \text{ on } \Gamma_U\}\\
&= \{p \in d + V_0 : \gamma(p) \in C\},
\end{aligned}$$

and operators $A : V_1 \longrightarrow V_1'$ and $\mathcal{G} : V_1 \longrightarrow V_1'$ given by

$$\begin{aligned}
Ap(q) &= \int_\Omega \nabla p(x) \cdot \nabla q(x)\, dx,\\
\mathcal{G}p(q) &= \int_\Omega \mathbf{g}(p(x)) \cdot \nabla q(x)\, dx, \qquad p,\ q \in V_1\,.
\end{aligned}$$

For each $p \in V_1$, we define $A_0 p$ and $\mathcal{G}_0 p$ in $H^{-1}(\Omega)$ to be the respective restrictions of $A\,p$ and $\mathcal{G}p$ in V_1' to $C_0^\infty(\Omega)$. The corresponding distributions are given by the operators

$$A_0\, p = -\nabla \cdot (\nabla p), \qquad \mathcal{G}_0\, p = -\nabla \cdot (\mathbf{g}(p)).$$

If $p \in V_1$ then $\mathcal{G}_0\, p \in L^2(\Omega)$, since $\mathbf{g}(\cdot)$ is Lipschitz continuous. If also $A_0 p \in L^2(\Omega)$, then the elliptic regularity theory implies that $p \in H^2_{\mathrm{loc}}(\Omega)$, and from the abstract divergence theorem we obtain

$$\begin{aligned}
Ap(q) &= (A_0 p, q)_{L^2(\Omega)} + \langle \frac{\partial p}{\partial n}, \gamma q\rangle_{\Gamma_{fl}}\,,\\
\mathcal{G}p(q) &= (\mathcal{G}_0 p, q)_{L^2(\Omega)} + \langle \mathbf{g}(p) \cdot \mathbf{n}, \gamma q\rangle_{\Gamma_{fl}}\,, \quad q \in V_0\,,
\end{aligned}$$

where $\partial p/\partial n$ and $\mathbf{g}(p) \cdot \mathbf{n}$ are meaningful in the dual $H^{\frac{1}{2}}(\Gamma_{fl})'$ of $H^{\frac{1}{2}}(\Gamma_{fl})$. These identities display the decoupling of $A\,p$ and $\mathcal{G}\,p$ into their *formal part* on Ω and *boundary part* on Γ_{fl}. Moreover, the unilateral boundary-value problem (6) is equivalent to

$$p \in K : \quad (Ap + \mathcal{G}p)(q - p) \ge f(q - p) \quad \text{for all } q \in K\,, \tag{7}$$

with the linear functional $f(\cdot)$ given by

$$f(q) = \int_\Omega F(x)\, q(x)\, dx + \int_{\Gamma_{fl}} g(s)\gamma(q)(s)\, ds \quad \text{for all } q \in V_1\,,$$

where $F \in L^2(\Omega)$ and $g \in L^2(\Gamma_{fl})$ are specified. To see this, let p be a solution of (7). Then $p \in K$, and by setting $q = p \pm \varphi$ in (7) for $\varphi \in C_0^\infty(\Omega)$, we obtain $(A_0 + \mathcal{G}_0)p = F \in L^2(\Omega)$, so $p \in H^2_{\text{loc}}(\Omega)$ and (7) gives

$$\langle \frac{\partial p}{\partial n} + \mathbf{g}(p) \cdot \mathbf{n} - g, \gamma(q) - \gamma(p) \rangle_{\Gamma_{fl}} \geq 0 \quad \text{for all } q \in K\,.$$

Since $\gamma(q)$ is arbitrary on Γ_N and can be chosen with $\gamma(q) \leq \gamma(p)$ or with $\gamma(q) = 0$ on Γ_U, we obtain (6). The converse follows even more directly. Finally, we note that the *variational inequality* (7) is equivalent to the *subgradient equation*

$$p \in d + V_0 : \qquad Ap + \mathcal{G}(p) + \partial I_K(p) \ni f \text{ in } {V_0}'\,. \tag{8}$$

This is the formulation of the unilateral boundary value problem (6) that will be used below.

3.2. **The Elasticity Operator.** The Navier system of partial differential equations describes the small displacements of a purely elastic structure. The *effective stress* σ'_{ij} is the symmetric tensor that represents the internal forces on surface elements. We have assumed this is given by *Hooke's law*,

$$\sigma'_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}\,.$$

Let Γ_0 and Γ_{tr} be the complementary subsets of the boundary as given above. The stationary elasticity system is the strongly elliptic system of partial differential equations given by

$$-\partial_j \sigma'_{ij} = -\partial_j(\lambda \delta_{ij}(\partial_k u_k) + \mu(\partial_i u_j + \partial_j u_i)) = F_i \text{ in } \Omega\,, \tag{9a}$$

$$u_i = 0 \text{ on } \Gamma_0\,, \quad \sigma'_{ij} n_j = t_i \text{ on } \Gamma_{tr}\,, \tag{9b}$$

for each $1 \leq i \leq 3$. Thus the boundary condition on Γ_0 is a constraint on displacement, and on Γ_{tr} it involves the surface density of forces or *traction* $\sigma'(\mathbf{n})$ with i-th component given by $\sigma'_{ij} n_j$ and value determined by the unit outward normal vector $\mathbf{n} = (n_1, n_2, n_3)$ on Γ_{tr}.

In order to obtain the weak formulation of this boundary-value problem, we define the Sobolev space

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \ \ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0\}$$

of *admissible displacements*. We shall assume that *measure* $(\Gamma_0) > 0$. The variational form of the elasticity system (9) is given by

$$\mathbf{u} \in \mathbf{V} : \ E(\mathbf{u})(\mathbf{v}) = \mathbf{f}(\mathbf{v}) \ \text{ for all } \mathbf{v} \in \mathbf{V}, \tag{10}$$

where the *elasticity operator* $\mathbf{E} : \mathbf{V} \longrightarrow \mathbf{V}'$ and the linear functional $\mathbf{f}(\cdot)$ in $\mathbf{V}'$ are defined by

$$\mathbf{E}(\mathbf{u})(\mathbf{v}) = \int_\Omega (\lambda(\partial_k u_k)\,(\partial_i v_i) + 2\mu \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}))\, dx$$

$$\text{and } \mathbf{f}(\mathbf{v}) = \int_\Omega F_i v_i \, d\mathbf{x} + \int_{\Gamma_{tr}} t_i v_i \, ds.$$

The variational formulation (10) is equivalent to $\mathbf{E}(\mathbf{u}) = \mathbf{f}$. It follows from the Korn's inequality and Poincare's theorem that $\mathbf{E}(\cdot)(\cdot)$ is a $\mathbf{V}$–coercive form, and hence that $\mathbf{E}(\cdot)$ is an isomorphism. (See Duvaut-Lions (1976) [13] or Ciarlet (1988) [10].)

For $\mathbf{u} \in \mathbf{V}$ we denote the restriction of $\mathbf{E}(\mathbf{u}) \in \mathbf{V}'$ to $\mathbf{C}_0^\infty(\Omega)$ by $\mathbf{E}_0(\mathbf{u})$. This is given by the distributions $\mathbf{E}_0(\mathbf{u}) \equiv -(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) - \mu \Delta \mathbf{u}$. Then we can recover the boundary-value problem (9) from $\mathbf{E}$ as follows. If the boundary is sufficiently smooth, then the regularity theory for strongly elliptic systems shows that whenever $\mathbf{E}_0(\mathbf{u}) \in L^2(\Omega)$ we have $\mathbf{u} \in \mathbf{H}^2_{\mathrm{loc}}(\Omega)$; see Ciarlet (1988) [10] or Fichera (1972) [14]. Then from the abstract divergence theorem there follows

$$\mathbf{E}(\mathbf{u})(\mathbf{v}) = (\mathbf{E}_0(\mathbf{u}), \mathbf{v})_{L^2(\Omega)} + \langle \sigma'_{ij} n_j, v_i \rangle_{\Gamma_{tr}}, \quad \mathbf{v} \in \mathbf{V}, \tag{11}$$

as before. This shows how $\mathbf{E}(\cdot)$ decouples into the sum of its *formal* part $\mathbf{E}_0(\cdot)$ on Ω and its *boundary* part $\sigma'(\mathbf{n})$ on Γ_{tr}.

3.3. **Pressure-Dilation Operators.** Let the function $\beta(\cdot) \in L^\infty(\Gamma_{tr})$ be given; we shall assume that $0 \le \beta(s) \le 1,\ s \in \Gamma_{tr}$. Then define the corresponding *gradient* operator, $\vec{\nabla} : L^2(\Omega) \oplus L^2(\Gamma_{tr}) \to \mathbf{V}'$, by

$$\langle \vec{\nabla}[f, g],\, \mathbf{v} \rangle = -(f,\, \partial_j v_j)_{L^2(\Omega)} + (g,\, (1-\beta)\mathbf{v}\cdot\mathbf{n})_{L^2(\Gamma_{tr})}\,, \quad \mathbf{v} \in \mathbf{V}\,, \tag{12}$$

and the *divergence* operator, $\vec{\nabla}\cdot : \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_{tr}) \to V_1'$, by

$$\langle \vec{\nabla}\cdot[\mathbf{f},\, \mathbf{g}],\, p \rangle = -\int_\Omega f_j\, \partial_j p\, dx + \int_{\Gamma_{tr}} \beta\, g_j n_j p\, ds\,, \quad p \in V_1.$$

The trace map gives a natural identification $\mathbf{v} \mapsto [\mathbf{v},\, \gamma(\mathbf{v})|_{\Gamma_{tr}}]$ of $\mathbf{V} \subset \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_{tr})$, and this identification will be employed throughout the following. It also gives the identification $p \mapsto [p,\, \gamma(p)|_{\Gamma_{tr}}]$ of $V_1 \subset L^2(\Omega) \oplus L^2(\Gamma_{tr})$. Recall that $V_1 \equiv H^1(\Omega)$. We note that both of these identifications have dense range, and so the corresponding duals can be identified. That is, we have

$$\mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_{tr}) \subset \mathbf{V}', \qquad L^2(\Omega) \oplus L^2(\Gamma_{tr}) \subset V_1'.$$

For smoother functions $\mathbf{v} \in \mathbf{V} \subset \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_{tr})$ we obtain from *Stokes' Formula*

$$\begin{aligned} \langle \vec{\nabla}\cdot \mathbf{v},\, p \rangle &= -\int_\Omega v_j\, \partial_j p\, dx + \int_{\Gamma_{tr}} \beta\, v_j n_j\, p\, ds \\ &= \int_\Omega \partial_j v_j\, p\, dx - \int_{\Gamma_{tr}} (1-\beta)\mathbf{v}\cdot\mathbf{n}\, p\, ds\,, \quad p \in V_1. \end{aligned}$$

This shows that the restriction maps

$$\vec{\nabla}\cdot :\ \mathbf{V} \to L^2(\Omega) \oplus L^2(\Gamma_{tr}) \tag{13}$$

and that the divergence has a *formal* part in Ω as well as a *boundary* part on Γ_{tr}. We denote the part in $L^2(\Omega)$ by $\nabla\cdot$, that is, $\nabla\cdot\mathbf{v} = \partial_j v_j$, and the identity above is indicated by

$$\vec{\nabla}\cdot\mathbf{v} = [\nabla\cdot\mathbf{v},\, -(1-\beta)\mathbf{v}\cdot\mathbf{n}] \in L^2(\Omega) \oplus L^2(\Gamma_{tr}), \quad \mathbf{v} \in \mathbf{V}. \tag{14}$$

Moreover, this shows the *dual* of the restricted divergence (13) is the negative of the gradient (12). Similarly, we find that the restriction of the gradient to V_1 satisfies

$$\langle \vec{\nabla} p,\ \mathbf{v} \rangle \equiv \int_\Omega \partial_j p\, v_j\, dx - \int_{\Gamma_{tr}} \beta\, p n_j\, v_j\, ds\,, \quad p \in V_1,\ \mathbf{v} \in \mathbf{V}\,.$$

This consists of the *formal* part ∇p in Ω and the *boundary* part $-\beta\, p\, \mathbf{n}$ on Γ_{tr}, and we denote this representation by

$$\vec{\nabla} p = [\nabla p,\, -\beta\, p\, \mathbf{n}] \in \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_{tr}), \quad p \in V_1. \tag{15}$$

The preceding constructions are summarized in the following diagram.

$$
\begin{array}{ccccc}
\mathbf{L^2}(\Omega) \oplus \mathbf{L^2}(\Gamma_{tr}) & \xrightarrow{\vec{\nabla}\cdot\,=-\vec{\nabla}'} & V_1' & & \\
\bigcup & & \bigcup & & \\
\mathbf{V} & \xrightarrow{\vec{\nabla}\cdot} & L^2(\Omega) \oplus L^2(\Gamma_{tr}) & \xrightarrow{\vec{\nabla}=-(\vec{\nabla}\cdot)'} & \mathbf{V}' \\
 & & \bigcup & & \bigcup \\
 & & V_1 & \xrightarrow{\vec{\nabla}} & \mathbf{L^2}(\Omega) \oplus \mathbf{L^2}(\Gamma_{tr})
\end{array}
$$

3.4. **The Evolution system.** Let $I_K(\cdot)$ be the indicator function of the closed convex set K. With the preceding notation, we can write our system in the form

$$\mathbf{u}(t) \in \mathbf{V} : \ \mathbf{E}(\mathbf{u}(t)) + \alpha\vec{\nabla}(p(t)) = \mathbf{f}(t) \text{ in } \mathbf{V'}\,, \tag{16a}$$

$$\frac{d}{dt}(b(p(t)) + \vec{\nabla}\cdot\mathbf{u}(t)) + Ap(t) + \mathcal{G}(p(t)) + \partial I_K(p(t)) \ni f(t) \text{ in } V_0'\,, \tag{16b}$$

with the linear functionals $\mathbf{f}(\cdot)$ and $f(\cdot)$ given by

$$\mathbf{f}(t)(\mathbf{v}) = \int_\Omega \mathbf{F}(x,t)\cdot\mathbf{v}(x)\,dx + \int_{\Gamma_{tr}} \mathbf{t}(s,t)\cdot\gamma(\mathbf{v})(s)\,ds \quad \text{for all } \mathbf{v}\in\mathbf{V}\,,$$

$$f(t)(q) = \int_\Omega F(x,t)\,q(x)\,dx + \int_{\Gamma_{fl}} g(s,t)\gamma(q)(s)\,ds \quad \text{for all } q\in V_0\,,$$

where $\mathbf{F}(t) \in \mathbf{L}^2(\Omega)$, $\mathbf{t}(t) \in \mathbf{L}^2(\Gamma_{fl})$, $F(t) \in L^2(\Omega)$, and $g(t) \in L^2(\Gamma_{fl})$ are specified for each $t > 0$. Of course, it is implicit in (16b) that $p(t) \in K$.

We shall display the system (16) explicitly in its parts as an initial-boundary-value problem for the system of partial differential equations and boundary conditions. This follows by splitting each of the operators in this system into its respective formal part on Ω and boundary part on $\partial\Omega$. The calculation is accomplished as above, and the equivalent system (16) takes the form

$$-(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}(t)) - \mu\Delta\mathbf{u}(t) + \alpha\nabla p(t) = \mathbf{F}(t) \quad \text{and} \tag{17a}$$

$$\frac{\partial}{\partial t}(b(p(t)) + \nabla\cdot\mathbf{u}(t)) - \nabla\cdot(\nabla p(t) + \mathbf{g}(p(t))) = F(t) \text{ in } \Omega\,, \tag{17b}$$

$$\mathbf{u}(t) = \mathbf{0} \text{ on } \Gamma_0, \qquad \sigma'(\mathbf{n}) - \alpha\,\beta p(t)\mathbf{n} = \mathbf{t}(t) \text{ on } \Gamma_{tr}\,, \tag{17c}$$

$$p(t) = d \text{ on } \Gamma_D,\ -(1-\beta)\dot{\mathbf{u}}(t)\cdot\mathbf{n} + \frac{\partial p(t)}{\partial n} + \mathbf{g}(p)\cdot\mathbf{n} = g(t) \text{ on } \Gamma_N, \tag{17d}$$

$$-(1-\beta)\dot{\mathbf{u}}(t)\cdot\mathbf{n} + \frac{\partial p(t)}{\partial n} + \mathbf{g}(p)\cdot\mathbf{n} + \partial I_C(p(t)) \ni\ g(t) \text{ on } \Gamma_U\,, \tag{17e}$$

for each $t > 0$. The given functions $\mathbf{F}(\cdot)$ and $\mathbf{t}(\cdot)$ are the distributed forces in $\mathbf{L^2}(\Omega)$ and $\mathbf{L}^2(\Gamma_{tr})$, and $F(\cdot)$ and $g(\cdot)$ are distributed fluid sources in $L^2(\Omega)$ and $L^2(\Gamma_{fl})$, respectively. Note that equation (16a) is equivalent to the pair (17a) and (17c), because $p(t)$ belongs to V_1. Furthermore, for the strong solution, we have sufficient additional regularity to guarantee that $A_0(p(t)) \in L^2(\Omega)$, and then (16b) is equivalent to (17b), (17d), and (17e). The system (17) contains the original problem (5) as a special case with $\beta = 1$ and $g(\cdot) = 0$.

Let's consider the meaning of the boundary conditions in the context of this poroelasticity model. The equations (17c) consist of the complementary pair requiring null displacement on the *clamped boundary*, Γ_0, and a balance of forces on the *traction boundary*, Γ_{tr}. The boundary conditions (17d) require a specified pressure on Γ_D and a balance of fluid mass flux on Γ_N. Finally, the subgradient inclusion (17e) is equivalent to the variational inequality

$$p(t) \le 0, \quad -(1-\beta)\dot{\mathbf{u}}(t)\cdot\mathbf{n} + \frac{\partial p(t)}{\partial n} + \mathbf{g}(p(t))\cdot\mathbf{n} \le g(t), \text{ and}$$
$$p(t)\,(-(1-\beta)\dot{\mathbf{u}}(t)\cdot\mathbf{n} + \frac{\partial p(t)}{\partial n} + \mathbf{g}(p(t))\cdot\mathbf{n} - g(t)) = 0 \text{ on } \Gamma_U\,,$$

and this determines the *seepage surface* as described in the Introduction. The function $\beta(\cdot)$ is defined on the traction boundary Γ_{tr}, and it specifies the surface fraction of the pores which are *sealed*. For these the effective pressure contributes to the traction along Γ_{tr}. The remaining portion $1-\beta(\cdot)$ of the pores are *exposed* along Γ_{tr}, and these contribute to the flux. On any portion of Γ_{tr} which is completely exposed, that is, where $\beta = 0$, only the *effective* or elastic component of stress is specified, since there the fluid pressures do not contribute to the support of the matrix. On the flux boundary Γ_{fl} there is a transverse flow that is given by the input $g(\cdot)$ and the relative normal velocity of the structure. This input could be specified in the form $g(t) = -(1-\beta)\mathbf{v}(t)\cdot\mathbf{n}$, where $\mathbf{v}(t)$ is the given velocity of fluid on Γ_{fl}. In this case (17d) shows that the flux $\mathbf{q}\cdot\mathbf{n} = -\partial p(t)/\partial n - \mathbf{g}(p(t))\cdot\mathbf{n}$ is proportional to the exposed fraction of pores, $1-\beta$, so a completely sealed portion of Γ_N is *impermeable*.

4. **The Cauchy Problem.** In order to resolve the system (16), we invert $\mathbf{E}$ and substitute

$$\mathbf{u}(t) = -\mathbf{E}^{-1}(\alpha\vec{\nabla}p - \mathbf{f}(t))\,,$$

to obtain the equivalent single equation

$$\frac{d}{dt}(b(p(t)) - \vec{\nabla}\cdot\mathbf{E}^{-1}(\alpha\vec{\nabla}p(t) - \mathbf{f}(t))) \\ + A(p(t)) + \mathcal{G}(p(t)) + \partial I_K(p(t)) \ni f(t) \text{ in } V_0'\,. \quad (18)$$

We can simplify the form of this equation. Recall that the convex set is given by $K = \{q \in d + V_0 : \ \gamma(q) \in C\}$. By introducing the translate of this set, namely,

$$K_0 \equiv \{q \in V_0 : \ \gamma(q+d) \in C\}\,,$$

and by making the corresponding change of variable, *i.e.*, by replacing the solution $p(\cdot)$ in the above by its translate, $p(\cdot) + d$, one obtains the equivalent equation

$$\frac{d}{dt}(b(p(t)+d) - \vec{\nabla}\cdot\mathbf{E}^{-1}(\alpha\vec{\nabla}(p(t)+d) - \mathbf{f}(t))) \\ + A(p(t)+d) + \mathcal{G}(p(t)+d) + \partial I_{K_0}(p(t)) \ni f(t) \text{ in } V_0'\,.$$

By adjusting the term $f(\cdot)$ appropriately, it is clear that we may assume without loss of generality that $\mathbf{f}'(\cdot) = \mathbf{0}$ and eliminate the term $A(d)$ in the above. This gives the abstract evolution equation

$$\frac{d}{dt}(\mathcal{B}(p(t))) + \mathcal{A}(p(t)) + \mathcal{G}(p(t)+d) \ni f(t) \text{ in } V_0' \quad (19)$$

in which the operators $\mathcal{B}(\cdot) \equiv b(\cdot + d) - \vec{\nabla}\cdot\mathbf{E}^{-1}\,\alpha\vec{\nabla}(\cdot)$ on $L^2(\Omega)\oplus L^2(\Gamma_{tr})$ and $\mathcal{A}(\cdot) \equiv A(\cdot) + \partial I_{K_0}(\cdot)$ from V_0 to its dual V_0' are monotone.

In the remaining sections we shall prove the following existence result for the system (16).

Theorem 4.1. *Assume that the data in the system (16) satisfies the following:*

(B) *The function $b(\cdot):\mathbb{R}\to\mathbb{R}$ is non-decreasing and Lipschitz continuous. (Hence, the operator $\mathcal{B}$ is monotone and Lipschitz continuous on $L^2(\Omega)\oplus L^2(\Gamma_{tr})$.)*

(G) *The gravitation term $\mathbf{g}(\cdot):\mathbb{R}\to\mathbb{R}^3$ is Lipschitz continuous.*

(F) *The forcing term satisfies $f(\cdot)\in H^1(0,T;V_0')$, and the boundary pressure is determined by a $d\in H^2(\Omega)$ which satisfies $\gamma(d)\in C$. (The first of these requires that $F(\cdot)\in H^1(0,T;V_0')$, $\gamma^*g(\cdot)=g(\cdot)\circ\gamma\in H^1(0,T;V_0')$, and $\mathbf{E}^{-1}(\mathbf{f}(\cdot)) \in H^2(0,T;\mathbf{L}^2(\Omega)\oplus\mathbf{L}^2(\Gamma_{tr}))$. The second means that the boundary data can be extended to a pressure function on Ω which satisfies the unilateral constraint on Γ_U.)*

(I) *There is a $p_0\in K$ satisfying $\mathcal{B}(p_0)=\theta_0$.*

Then the Cauchy problem for (19) *has a solution $p(\cdot)$, $w(\cdot)$ which satisfies*

$$p(\cdot)\in L^\infty(0,T;V_0)\ ,\ \ \mathcal{B}(p(\cdot))\in H^1(0,T;L^2(\Omega)\oplus L^2(\Gamma_{tr})))$$
$$w(\cdot)\in L^2(0,T;V_0')\ ,\quad w(\cdot)\in\partial I_{K_0}(p(\cdot))\ \textit{a.e. on}\ [0,T]\ ,$$
$$\frac{d}{dt}\mathcal{B}(p(t))+A(p(t))+\mathcal{G}(p(t)+d)+w(t)=f(t)\ \textit{a.e. on}\ [0,T]\ ,\ \textit{and}$$
$$\mathcal{B}(p(0))=\theta_0\,.$$

Furthermore, $p(\cdot)\in H^1(0,T;L^2(\Omega))$, so $A_0(p(\cdot))\in L^2(0,T;L^2(\Omega))$ and the solution is strong*. That is, the translate $p(\cdot)-d$ is a solution of the evolution equation* (18)*, and this is equivalent to the system* (17)*.*

4.1. **Implicit Evolution Equations.** We first recall some existence results from [12] which will be extended in order to apply to the gravity-free case of (19). (Also see [16].) Let W and V be Hilbert spaces for which the embedding $\iota:V\hookrightarrow W$ is compact. Denote the dual restriction operator by $\iota':W'\to V'$. Let $\varphi:W\to\mathbb{R}$ be a proper, convex, and lower semicontinuous function, and suppose $\mathcal{B}$ is given by $\mathcal{B}\equiv\iota'\circ\partial\varphi\circ\iota$. We also assume that $\partial\varphi\circ\iota:V\to W'$ is bounded. Let $\mathcal{A}:V\to V'$ be maximal monotone and bounded. Denote by $\mathcal{R}$ the Riesz map $V\to V'$. Fix $f\in L^2(0,T;V')$ and $[u_0,v_0]\in\mathcal{B}$. Then for each $\lambda>0$, there is a pair $u_\lambda\in H^1(0,T;V)$, $v_\lambda\in H^1(0,T;V')$ such that

$$v_\lambda(t)\in\mathcal{B}(u(t))\ \text{for all}\ t\in[0,T]\ ,$$
$$\frac{d}{dt}(\mathcal{R}u_\lambda(t)+v_\lambda(t))+\mathcal{A}_\lambda(u_\lambda(t))=f(t),$$
$$\mathcal{R}u_\lambda(0)+v_\lambda(0)=\mathcal{R}u_0+v_0\,.$$

By standard techniques, one obtains *a priori* estimates that show the norms

$$\|u_\lambda\|_{L^\infty(0,T;V)}\quad,\quad\|v_\lambda\|_{L^\infty(0,T;V')}\quad,\quad\|J_\lambda(\mathcal{R}u_\lambda)\|_{L^\infty(0,T;V)}\ ,$$
$$\|\mathcal{A}_\lambda(u_\lambda)\|_{L^\infty(0,T;V')}\quad,\quad\|\dot u_\lambda\|_{L^2(0,T;V)}\quad,\quad\|\mathcal{R}\dot u_\lambda\|_{L^2(0,T;V')}$$

are bounded independent of $\lambda>0$. Choose a subsequence (still denoted by subscript λ) for which

$$u_\lambda\rightharpoonup u,\ \dot u_\lambda\rightharpoonup\dot u\ \text{in}\ L^2(0,T;V)\ ,$$
$$v_\lambda\rightharpoonup v,\ \dot v_\lambda\rightharpoonup\dot v\ \text{in}\ L^2(0,T;V')\ ,\ \text{and}$$
$$\mathcal{A}_\lambda(u_\lambda)\rightharpoonup w\ \text{in}\ L^2(0,T;V')\,.$$

Note that, since $\{v_\lambda\}$ and $\{u_\lambda\}$ are uniformly equicontinuous functions, it follows that

$$u_\lambda(t)\to u(t)\ \text{and}\ v_\lambda(t)\to v(t)\ \text{for all}\ t\in[0,T]\,.$$

These limits are shown to be a solution of the following *regularized* problem.

Theorem 4.2. *The triple* $[u, v, w]$ *satisfies*

$$
\begin{aligned}
&u \in H^1(0,T;V)\ ,\ \ v \in H^1(0,T;V')\ ,\ \ w \in L^2(0,T;V')\ ,\\
&v \in \mathcal{B}(u)\ ,\ \ w \in \mathcal{A}(u)\ \textit{a.e. on}\ [0,T]\ ,\\
&\frac{d}{dt}(\mathcal{R}u(t) + v(t)) + w(t) = f(t)\ \textit{a.e. on}\ [0,T]\ ,\ \textit{and}\\
&\mathcal{R}(u(0)) + v(0) = \mathcal{R}(u_0) + v_0\,.
\end{aligned}
$$

The second existence result of [12] concerns the corresponding (possibly) degenerate Cauchy problem. With the additional hypotheses that the realizations $\mathcal{B} : L^2(0,T;V) \to L^2(0,T;V')$ and $\mathcal{A} : L^2(0,T;V) \to L^2(0,T;V')$ are bounded, and that the solutions to the λ-*regularizations*

$$
\begin{aligned}
&v_\lambda \in \mathcal{B}(u_\lambda), w_\lambda \in \mathcal{A}(u_\lambda)\ \textit{a.e.}\ \text{on}\ [0,T]\ ,\\
&\frac{d}{dt}\big(\lambda\mathcal{R}u_\lambda(t) + v_\lambda(t)\big) + w_\lambda(t) = f(t)\ \textit{a.e.}\ \text{on}\ [0,T],\ \text{and}\\
&\lambda\mathcal{R}(u_\lambda(0)) + v_\lambda(0) = \lambda\mathcal{R}(u_0) + v_0\ ,
\end{aligned}
$$

satisfy $\|u_\lambda\|_{L^2(0,T;V)} \le M$ for some M independent of λ, additional *a priori* bounds are derived, from which it follows that some subsequence (still denoted by subscript λ) satisfies

$$
\begin{aligned}
&u_\lambda \rightharpoonup u \text{ in } L^2(0,T;V)\\
&v_\lambda \rightharpoonup v \text{ and } \dot v_\lambda \rightharpoonup \dot v \text{ in } L^2(0,T;V')\\
&w_\lambda \rightharpoonup w \text{ in } L^2(0,T;V')\,.
\end{aligned}
$$

Again these limits are shown to be a solution of the following problem.

Theorem 4.3. *The triple* $[u, v, w]$ *is a solution to*

$$
\begin{aligned}
&u \in L^2(0,T;V)\ ,\ \ v \in H^1(0,T;V')\ ,\ \ w \in L^2(0,T;V')\ ,\\
&v \in \mathcal{B}(u)\ ,\ \ w \in \mathcal{A}(u)\ \textit{a.e. on}\ [0,T]\ ,\\
&\frac{d}{dt}v(t) + w(t) = f(t)\ \textit{a.e. on}\ [0,T]\ ,\ \textit{and}\\
&v(0) = v_0\,.
\end{aligned}
$$

4.2. **A-priori estimates.** We would like to apply Theorem 4.3 to the *monotone* case of our system (19), that is, the special case of $\mathcal{G}(\cdot) = 0$. For this we set $W \equiv L^2(\Omega) \oplus L^2(\Gamma_{tr})$ and $V \equiv V_0$. But this fails to meet the hypotheses of Theorem 4.3 because the operator $\mathcal{A}(\cdot)$ is *not bounded.* However, we shall obtain directly in Section 5 an a priori bound on $\mathcal{A}(p_\lambda(\cdot))$ for any solution $p_\lambda(\cdot)$ of the λ-regularization of (19). Thereby, we obtain a weak solution for our problem with $\mathcal{G}(\cdot) = 0$ from the existence result of Theorem 4.3. Moreover, we also get an estimate on $\|\frac{d}{dt}\mathcal{B}(p)\|_{L^2(0,T;W)}$, and this shows that the solution is *strong.* Then in Section 6 we shall extend this to the full equation (19) with gravity.

5. **The Monotone Case.** We consider first the case of $\mathcal{G} \equiv \mathbf{0}$. The initial-value problem is given by

$$
\begin{cases}
\dfrac{d}{dt}\mathcal{B}(p(t)) + \mathcal{A}(p(t)) \ni f(t) \quad \text{in } V',\\
\quad\mathcal{B}(p(0)) = \theta,
\end{cases}
\tag{20}
$$

where

$$\begin{aligned}\langle \mathcal{B}(p), q\rangle &= (b(p+d), q)_{L^2(\Omega)} + \alpha\langle \mathbf{E}^{-1}(\vec{\nabla} p), \vec{\nabla} q\rangle, \quad p,\ q \in W,\\ \langle \mathcal{A}(p), q\rangle &= (\nabla p, \nabla q)_{L^2(\Omega)} + (\partial I_{K_0}(p), q)_{L^2(\Gamma_U)}, \quad p,\ q \in V,\\ f(t) &= (F(t), g(t)) \in W\,.\end{aligned}$$

5.1. **Preliminaries.** Let $\varphi_B : W \to \mathcal{R}$ be the convex functional

$$\varphi_B(q) = \int_\Omega [b^*(q+d) - b^*(d)]dx - \frac{\alpha}{2}\langle \vec{\nabla}\cdot \mathbf{E}^{-1}\vec{\nabla} q, q\rangle, \quad q \in W,$$

where $b^*(z) = \int_0^z b(s)ds$. Then we have $\partial\varphi_B = \mathcal{B}$. Let $\varphi_B^* : W \to \mathcal{R}$ be the convex conjugate of φ_B. Then $\varphi_B^*(q) \geq 0$ for all $q \in W$, since $\varphi_B(0) = 0$.

Define the convex functional $\varphi_A : V \to [0,\infty]$ by

$$\varphi_A(q) = \frac{1}{2}(\nabla q, \nabla q)_{L^2(\Omega)} + I_{K_0}(q), \quad q \in V.$$

Then $\partial\varphi_A = \mathcal{A} : V \to V'$ is monotone but unbounded.

The following properties of $\mathcal{A}$ and $\mathcal{B}$ will be used below.

Lemma 1. *Assume $\gamma(d) \in C$. Then we have* (cf. [12], [22])

$$\varphi_B^*(\mathcal{B}(p)) = \langle \mathcal{B}(p), p\rangle - \varphi_B(p), \quad p \in W; \tag{21a}$$

$$\langle w, p\rangle \geq \varphi_A(p) \geq c\|p\|_V^2, \quad w \in \mathcal{A}(p),\ p \in V; \tag{21b}$$

$$\langle \frac{d}{dt}\mathcal{B}(p), p\rangle = \frac{d}{dt}\varphi_B^*(\mathcal{B}(p)) \ \textit{if}\ \mathcal{B}(p) \in H^1(0,T;W); \tag{21c}$$

$$\langle w, \frac{dp}{dt}\rangle = \frac{d}{dt}\varphi_A(p) \ \textit{if}\ w \in \mathcal{A}(p),\ p \in H^1(0,T;V); \tag{21d}$$

$$\|p\|^2_{L^2(\Omega)} \leq C\varphi_B^*(\mathcal{B}(p)), \quad p \in W; \tag{21e}$$

$$\|p_1 - p_2\|_{L^2(\Omega)} \leq C\|\mathcal{B}(p_1) - \mathcal{B}(p_2)\|_W, \quad p_1, p_2 \in W. \tag{21f}$$

Proof. We let $c > 0$ denote a generic constant. First of all, we claim that

$$-\langle \vec{\nabla}\cdot \mathbf{E}^{-1}(\vec{\nabla} p), p\rangle \geq c\|p\|^2_{L^2(\Omega)}, \quad p \in W. \tag{22}$$

In fact, we have

$$-\langle \vec{\nabla}\cdot \mathbf{E}^{-1}(\vec{\nabla} p), p\rangle = \langle \mathbf{E}^{-1}(\vec{\nabla} p), \vec{\nabla} p\rangle \geq c\|\vec{\nabla} p\|^2_{V'} \geq c\|p\|^2_{L^2(\Omega)},$$

since $\mathbf{E} : \mathbf{V} \to \mathbf{V}'$ is isomorphism. Next, according to (21a) and (22),

$$\begin{aligned}\varphi_B^*(\mathcal{B}(p)) &= (b(p+d), p)_{L^2(\Omega)} - \int_\Omega [b^*(p+d) - b^*(d)]dx\\ &\quad - \frac{\alpha}{2}\langle \vec{\nabla}\cdot \mathbf{E}^{-1}(\vec{\nabla} p), p\rangle\\ &\geq -\frac{\alpha}{2}\langle \vec{\nabla}\cdot \mathbf{E}^{-1}(\vec{\nabla} p), p\rangle \geq c\|p\|^2_{L^2(\Omega)},\end{aligned}$$

since $b(\cdot) = b^{*\prime}(\cdot)$ is non-decreasing. Also, from the monotonicity of $b(\cdot)$ and (22), we obtain

$$\begin{aligned}&\langle \mathcal{B}(p_1) - \mathcal{B}(p_2), p_1 - p_2\rangle\\ &\qquad \geq -\langle \vec{\nabla}\cdot \mathbf{E}^{-1}\vec{\nabla}(p_1 - p_2), p_1 - p_2\rangle\\ &\qquad \geq c\|p_1 - p_2\|^2_{L^2(\Omega)}, \quad p_1, p_2 \in W,\end{aligned}$$

which yields the estimate

$$\|p_1 - p_2\|_{L^2(\Omega)} \leq C\|\mathcal{B}(p_1) - \mathcal{B}(p_2)\|_W,$$

and then (21f). The remaining identities are standard from convex analysis. $\square$

5.2. **Uniform Estimates.** Let $p(\cdot)$ be a regular solution of (20). Then for some $w(t) \in \mathcal{A}(p(t))$ we have

$$\frac{d}{dt}\mathcal{B}(p(t)) + w(t) = f(t) \quad \text{in } V',\ t \in (0,T]. \tag{23}$$

Applying (23) to $p(t) \in V$ and integrating over $[0,\tau]$, $\tau \in (0,T]$, lead to

$$\begin{aligned}\int_0^\tau (\langle \frac{d}{dt}\mathcal{B}(p(t)), p(t)\rangle + \langle w(t), p(t)\rangle)dt &= \int_0^\tau \langle f(t), p(t)\rangle dt \\ &\le \varepsilon \int_0^\tau \|p(t)\|_V^2 dt + C(\varepsilon)\int_0^\tau \|f(t)\|_{V'}^2 dt, \end{aligned} \tag{24}$$

and then from (21b) and (21c) we obtain

$$\varphi_B^*(\mathcal{B}(p(\tau))) + \int_0^\tau \|p(t)\|_V^2 dt \le C\left(\varphi_B^*(\mathcal{B}(p_0)) + \int_0^\tau \|f(t)\|_{V'}^2 dt\right).$$

By (21e) in Lemma 1 this implies

$$\|p(\tau)\|_{L^2(\Omega)}^2 + \int_0^\tau \|p(t)\|_V^2 dt \le C\left(\varphi_B^*(\mathcal{B}(p_0)) + \int_0^\tau \|f(t)\|_{V'}^2 dt\right). \tag{25}$$

For the next estimates we begin with the equality

$$\int_0^\tau \left(\langle \frac{d}{dt}\mathcal{B}(p), \frac{dp}{dt}\rangle + \langle w, \frac{dp}{dt}\rangle\right) dt = \int_0^\tau \langle f, \frac{dp}{dt}\rangle dt. \tag{26}$$

Since $\mathcal{B} : W \to W$ is monotone and Lipschitz continuous, we have

$$\int_0^\tau \langle \frac{d}{dt}\mathcal{B}(p), \frac{dp}{dt}\rangle dt \ge c\int_0^\tau \|\frac{d}{dt}\mathcal{B}(p)\|_W^2 dt,$$

with $c > 0$. By (21d),

$$\int_0^\tau \langle w, \frac{dp}{dt}\rangle dt = \varphi_A(p(\tau)) - \varphi_A(p_0) \ge c\|p(\tau)\|_V^2 - \varphi_A(p_0).$$

Also we have

$$\begin{aligned}\int_0^\tau \langle f, \frac{dp}{dt}\rangle dt &= \langle f(\tau), p(\tau)\rangle - \langle f(0), p_0\rangle - \int_0^\tau \langle \frac{df}{dt}, p\rangle dt \\ &\le C(\varepsilon_1)\left(\|p_0\|_V^2 + \|f(0)\|_{V'}^2 + \|f(\tau)\|_{V'}^2\right) \\ &\quad + \varepsilon_1\|p(\tau)\|_V^2 + |\int_0^\tau \langle \frac{df}{dt}, p\rangle dt|,\end{aligned}$$

so it follows that

$$\begin{aligned}&\int_0^\tau \|\frac{d}{dt}\mathcal{B}(p)\|_W^2 dt + \|p(\tau)\|_V^2 \\ &\quad \le C\left(\|p_0\|_V^2 + \varphi(p_0) + \|f(0)\|_{V'}^2 + \|f(\tau)\|_{V'}^2 + |\int_0^\tau \langle \frac{df}{dt}, p\rangle dt|\right).\end{aligned} \tag{27}$$

Thus, if $f \in H^1(0,T;V')$, then in view of (25) we obtain from (27),

$$\begin{aligned}\int_0^\tau \|\frac{d}{dt}\mathcal{B}(p)\|_W^2 dt + \|p(\tau)\|_V^2 \\ \le C\big(\|p_0\|_V^2 + \varphi(p_0) + \|f(0)\|_{V'}^2 + \|f(\tau)\|_{V'}^2\big) \\ + \varepsilon \int_0^\tau \|\frac{df}{dt}\|_{V'}^2 dt + C(\varepsilon)\Big(\varphi_B^*(\mathcal{B}(p_0)) + \int_0^\tau \|f\|_{V'}^2 dt\Big).\end{aligned} \tag{28}$$

These estimates hold likewise for the corresponding regularized equations, so we see that it is unnecessary to assume separately that the operator $\mathcal{A}(\cdot)$ is bounded.

6. **Gravity-driven Flow.** Consider the Cauchy problem

$$\frac{d}{dt}\mathcal{B}(p(t)) + \mathcal{A}(p(t)) + \mathcal{G}(p(t)) \ni f(t) \quad \text{in } V', \tag{29a}$$

$$\mathcal{B}(p(0)) = \mathcal{B}(p_0) \quad \text{in } V'. \tag{29b}$$

To deal with the gravity term, we view it as a perturbation to the gravity-free equation and then use a "delay" approximation to establish the existence of solutions. More precisely, we shall construct a sequence of approximate solutions inductively as follows:

Let N be a positive integer, and $h = T/N$. Consider the following problem with h-delay:

$$\frac{d}{dt}\mathcal{B}(p(t)) + \mathcal{A}(p(t)) \ni f(t) - \mathcal{G}(p(t-h)), \quad t \in (0,T], \tag{30a}$$

$$\mathcal{B}(p(t)) = \mathcal{B}(p_0), \quad t \in (-h, 0]. \tag{30b}$$

It can be solved inductively for $t \in [(k-1)h, kh]$, $k = 1, 2, \cdots, N$. Denote by $p_h(\cdot)$ the solution, and set $p_h(t) = p_0$ for $t \in (-h, 0]$. Supposing $p_h(t)$, $t \in ((k-2)h, (k-1)h]$, to be given, we shall find a pair $p_h(t)$, $w_h(t)$, satisfying $w_h(t) \in \mathcal{A}(p_h(t))$ and

$$\frac{d}{dt}\mathcal{B}(p_h(t)) + w_h(t) = f(t) - \mathcal{G}(p_h(t-h)) \quad \text{a.e. } t \in ((k-1)h, kh], \tag{31a}$$

$$\lim_{t\to(k-1)h^{+0}} \mathcal{B}(p_h(t)) = \mathcal{B}(p_h((k-1)h)), \tag{31b}$$

for $k = 1, 2, \cdots, N$. Then we shall show that the sequence $\{p_h\}$ has a convergent subsequence and obtain a solution to (29).

To achieve this, we shall show successively that

(a) there exists such a sequence $p_h(\cdot) \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)$;
(b) $\{p_h(\cdot)\}$ is bounded in $H^1(0,T;L^2(\Omega))$ and $L^\infty(0,T;V)$; and
(c) there exists a limit function $p(\cdot)$ which is a solution of (29).

6.1. **Existence for the delay equation.** As $\mathbf{g}(\cdot)$ is Lipschitz continuous, we note that $\mathcal{G} : L^2(\Omega) \to V'$ is Lipschitz continuous, and for any $a, b \in \mathcal{R}$ satisfying $a < b$, $\mathcal{G} : H^1(a,b;L^2(\Omega)) \to H^1(a,b;V')$ is bounded. In fact, we have, for any $q \in V$,

$$\begin{aligned}|\langle \mathcal{G}(p), q\rangle| &= |(\mathbf{g}(p+d), \nabla q)_{L^2(\Omega)}| \le \|\mathbf{g}(p+d)\|_{L^2(\Omega)}\|q\|_V \\ &\le C\left(1 + \|p\|_{L^2(\Omega)}\right)\|q\|_V,\end{aligned}$$

$$\begin{aligned}|\langle \mathcal{G}(p_1) - \mathcal{G}(p_2), q\rangle| &= |(\mathbf{g}(p_1+d) - \mathbf{g}(p_2+d), \nabla q)_{L^2(\Omega)}| \\ &\le C\|p_1 - p_2\|_{L^2(\Omega)}\|q\|_V,\end{aligned}$$

which gives

$$\|\mathcal{G}(p)\|_{V'} \le C(1 + \|p\|_{L^2(\Omega)}), \tag{32}$$

$$\|\mathcal{G}(p_1) - \mathcal{G}(p_2)\|_{V'} \le C\|p_1 - p_2\|_{L^2(\Omega)}, \tag{33}$$

$$\|\frac{d}{dt}\mathcal{G}(p)\|_{L^2(a,b;V')} \le C\|\frac{dp}{dt}\|_{L^2(a,b;L^2(\Omega))}. \tag{34}$$

In particular, if $p_h \in H^1((k-2)h, (k-1)h; L^2(\Omega))$, then $\mathcal{G}(p_h) \in H^1((k-2)h, (k-1)h; V')$, and hence, by the preceding result the problem (31a)–(31b) indeed has at least one solution pair

$$p_h \in H^1(0,T; L^2(\Omega)) \cap L^\infty((k-1)h, kh; V),$$
$$w_h \in L^2((k-1)h, kh; V').$$

Accordingly, the problem (31) has a solution $p_h \in H^1(0,T; L^2(\Omega)) \cap L^\infty(0,T;V)$. Furthermore, from the strong monotonicity estimate (21f) for $\mathcal{B}|_{L^2(\Omega)} : L^2(\Omega) \to L^2(\Omega)$ there follows

$$\|\frac{d}{dt}\mathcal{G}(p)\|_{L^2(a,b;V')} \le C\|\frac{d}{dt}\mathcal{B}(p)\|_{L^2(a,b;W)}. \tag{35}$$

This will be used in the boundedness estimates of $\{p_h(\cdot)\}$.

6.2. **Estimates on $\{p_h\}$.** We recall the estimates (25) and (28) in the case when $\mathcal{G}(p) = 0$, that is, for any $\tau \in (0,T]$,

$$\|p(\tau)\|^2_{L^2(\Omega)} + \int_0^\tau \|p\|^2_V dt \le C\left(1 + \int_0^\tau \|f\|^2_{V'} dt\right), \tag{36}$$

$$\int_0^\tau \|\frac{d}{dt}\mathcal{B}(p)\|^2_W dt + \|p(\tau)\|^2_V \le \varepsilon \int_0^\tau \|\frac{df}{dt}\|^2_{V'} dt + C(\varepsilon)\left(1 + \|f(\tau)\|^2_{V'} + \int_0^\tau \|f\|^2_{V'} dt\right). \tag{37}$$

Now replacing $f(t)$ by $f(t) - \mathcal{G}(p_h(t-h))$ in (36) and (37), and using (32) and (35), we obtain

$$\|p_h(\tau)\|^2_{L^2(\Omega)} + \int_0^\tau \|p_h\|^2_V dt \le C\left(1 + \int_0^\tau \|p_h(t-h)\|^2_{L^2(\Omega)} dt\right), \tag{38}$$

$$\int_0^\tau \|\frac{d}{dt}\mathcal{B}(p_h)\|^2_W dt + \|p_h(\tau)\|^2_V \le \varepsilon \int_0^\tau \|\frac{d}{dt}\mathcal{B}(p_h(t-h))\|^2_W dt + C(\varepsilon)\left(1 + \|p_h(\tau-h)\|^2_{L^2(\Omega)} + \int_0^\tau \|p_h(t-h)\|^2_{L^2(\Omega)} dt\right). \tag{39}$$

Note that $p_h(t) = p_0$ for $t \in (-h, 0]$, so

$$\int_0^\tau \|p_h(t-h)\|^2_{L^2(\Omega)} dt = h\|p_0\|^2_{L^2(\Omega)} + \int_0^{\tau-h} \|p_h(t)\|^2_{L^2(\Omega)} dt \le C\left(1 + \int_0^\tau \|p_h(t)\|^2_{L^2(\Omega)} dt\right).$$

Thus, applying Gronwall's lemma to (38) yields

$$\|p_h(\tau)\|^2_{L^2(\Omega)} \le C, \quad \tau \in (0,T]. \tag{40}$$

Again the fact that $p_h(t) = p_0$ for $t \in (-h, 0]$ leads to

$$\int_0^\tau \| \frac{d}{dt} \mathcal{B}(p_h(t-h)) \|_W^2 dt \le \int_0^\tau \| \frac{d}{dt} \mathcal{B}(p_h(t)) \|_W^2 dt.$$

Substituting into (39), and taking (40) into account, we obtain

$$\int_0^\tau \| \frac{d}{dt} \mathcal{B}(p_h) \|_W^2 dt + \| p_h(\tau) \|_V^2 \le C. \tag{41}$$

That is, $\{\mathcal{B}(p_h)\}$ and $\{p_h\}$ are bounded, respectively in $H^1(0,T;W)$ and $L^\infty(0,T;V)$. Furthermore, $\{\mathcal{G}(p_h)\}$ is bounded in $L^2(0,T;V')$, and hence, in terms of the equation (31a), $\{w_h)\}$ is also bounded in $L^2(0,T;V')$. In addition, $\{p_h\}$ is bounded in $H^1(0,T;L^2(\Omega))$, since $\mathcal{B}|_{L^2(\Omega)} : L^2(\Omega) \to L^2(\Omega)$ is strongly monotone.

6.3. **The Limit.** Now we may select subsequences of $\{p_h\}$, $\{\mathcal{B}(p_h)\}$, and $\{w_h\}$, denoted by $\{p_h\}$, $\{\mathcal{B}(p_h)\}$, and $\{w_h\}$ again, such that for some $p(\cdot) \in L^\infty(0,T;V) \cap H^1(0,T;L^2(\Omega))$, $v(\cdot) \in H^1(0,T;W)$, and $w(\cdot) \in L^2(0,T;V')$,

$$\begin{aligned} p_h &\longrightarrow p \quad \text{weakly}^* \text{ in } L^\infty(0,T;V), \\ p_h &\longrightarrow p \quad \text{strongly in } C([0,T];L^2(\Omega)), \\ \mathcal{B}(p_h) &\longrightarrow v \quad \text{weakly in } H^1(0,T;W), \\ w_h &\longrightarrow w \quad \text{weakly in } L^2(0,T;V'). \end{aligned}$$

Furthermore, since $\mathcal{G} : L^2(\Omega) \to V'$ is continuous, we have strong convergence $\mathcal{G}(p_h) \longrightarrow \mathcal{G}(p)$ in $L^2(0,T;V')$. From (30a) we get

$$\begin{aligned} &\int_0^T (\langle \frac{d}{dt} \mathcal{B}(p_h(t)), q \rangle + \langle w_h(t), q \rangle) dt \\ &\qquad = \int_0^T (\langle f(t), q \rangle - \langle \mathcal{G}(p_h(t-h)), q \rangle) dt, \quad q \in V. \end{aligned}$$

Letting $h \to 0$ gives

$$\int_0^T (\langle \frac{d}{dt} v(t), q \rangle + \langle w(t), q \rangle) dt = \int_0^T (\langle f(t), q \rangle - \langle \mathcal{G}(p(t)), q \rangle) dt.$$

Namely,

$$\frac{d}{dt} v(t) + w(t) + \mathcal{G}(p(t)) = f(t) \quad \text{in } V', \text{ a.e. } t \in [0,T].$$

Now exactly as before we verify $v(t) = \mathcal{B}(p(t))$, $w(t) \in \mathcal{A}(p(t))$ for almost all $t \in [0,T]$, and therefore, $p(\cdot)$ is a solution of the Cauchy problem (29).

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