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PSEUDOPARABOLIC PARTIAL DIFFERENTIAL EQUATIONS*

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1. Introduction. Various physical phenomena have led to a study of a mixed boundary value problem for the partial differential equation

$$\frac{\partial}{\partial t}u - \eta\Delta\frac{\partial}{\partial t}u = k\Delta u, \tag{1.1}$$

where Δ denotes the Laplacian differential operator. The initial and boundary conditions for this equation are the same as those posed for solutions of the parabolic equation

$$\frac{\partial}{\partial t}u = k\Delta u \tag{1.2}$$

which is obtained from (1.1) by setting $\eta = 0$. The class of equations which are considered here will be called *pseudoparabolic*, not only because the problems which are well-posed for the parabolic equation are also well-posed for these equations, but because the generalized solution to the parabolic equation (1.2) satisfying mixed initial and boundary conditions can be obtained as the limit of a sequence of solutions to the corresponding problem for equation (1.1) corresponding to any null sequence for the coefficient η. That is, a solution of the parabolic equation can be approximated by a solution of (1.1).

More statements on the comparison of these problems will appear in the following.

A study of nonsteady flow of second order fluids [36] leads to a mixed boundary value problem for the one-dimensional case of (1.1) for the velocity of the fluid. In [36] the role of the material constant η was examined, for this constant distinguishes this theory of second order fluids from earlier ones. This principal result of interest here is that the mixed boundary value problem is mathematically well-posed.

Equations of the form (1.1) are satisfied by the hydrostatic excess pressure within a portion of clay during consolidation [35]. In this context the constant η is a composite soil property with the dimensions of viscosity. If one assumes that the resistance to compression is plastic (proportional to the rate of compression), then equation (1.1) results with $\eta > 0$. However the classical Terzaghi assumption that any increment in the hydrostatic excess pressure is proportional to an increment of the ratio of pore volume to solid volume in the clay leads to the parabolic (1.2).

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As a final example of the physical origin of (1.1) we mention the theory of seepage of homogeneous fluids through a fissured rock [4]. A fissured rock consists of blocks of porous and permeable material separated by fissures or "cracks." The liquid then flows through the blocks and also between the blocks through the fissures. In this context an analysis of the pressure in the fissures leads to (1.1), where η represents a characteristic of the fissured rock. A decrease in η corresponds to a reduction in block dimensions and an increase in the degree of fissuring, and (1.1) then tends to coincide with the classical parabolic equation (1.2) of seepage of a liquid under elastic conditions.

The equation which we shall consider here is an example of the general class of equations of Sobolev type, sometimes referred to as the Sobolev–Galpern type. These are characterized by having mixed time and space derivatives appearing in the highest order terms of the equation. Such an equation was studied by Sobolev [34], and he used a Hilbert space approach to determine that both the Cauchy problem on the whole space and the mixed boundary value problem on a bounded domain are well-posed for the equation

$$\frac{\partial^2}{\partial t^2}(\Delta u) + \frac{\partial^2}{\partial x^2}(u) = 0. \tag{1.3}$$

This equation can be handled by the methods considered here.

The methods of generalized functions [11], [16] have been used on various classes of Sobolev type equations. In particular Galpern [15] investigated the Cauchy problem for a system of equations of the form

$$M\left(t, \frac{\partial}{\partial x_k}\right)\frac{\partial \mathbf{u}}{\partial t} + L\left(t, \frac{\partial}{\partial x_k}\right)\mathbf{u} = 0, \tag{1.4}$$

where $\mathbf{u}$ is a vector of functions and M and L are quadratic polynomial matrices depending on t. An analysis by Fourier transforms was used to assert existence and regularity of a solution to this system. Kostachenko and Eskin [24] discussed correctness classes of generalized functions for (1.4) with constant coefficients.

Zalenyak [41] obtained a class of solution of (1.3) satisfying a homogeneous initial condition and then [42] exhibited a class of solutions for the more general equation

$$\sum_{i=1}^{N} \frac{\partial^i}{\partial t^i}\left(a_{i1}\frac{\partial^2 u}{\partial x^2} + a_{i2}\frac{\partial^2 u}{\partial y^2} + b_i(x, y)\frac{\partial u}{\partial x} + c_i(x, y)\frac{\partial u}{\partial y} + d_i(x, y)u\right) = 0$$

in which the a_{ij} are constants.

In the following we shall consider equations of the form

$$M\frac{\partial u}{\partial t} + Lu = f$$

for which M and L are second order differential operators in the space variable and M is elliptic. These operators are independent of t but contain variable coefficients.

This class of equations contains (1.1), and the original Sobolev equation (1.3) can be handled similarly. A generalized mixed boundary value problem for this equation will be solved in the Hilbert space H_0^1 which is the Sobolev space of functions having square integrable first order derivatives and which vanish on the boundary in a generalized sense. The Sobolev spaces are introduced in § 2 along with other information that will be used in the following development. The statement of the generalized form of the problem and of the existence and uniqueness of the solution are the content of § 3.

The proof of the existence-uniqueness theorem comprises § 4, and the regularity of the solution is demonstrated in § 5. In particular it is shown that the solution is just as smooth as the initial function and the coefficients of the equation allow it to be. These results depend on the well-developed theory of the Dirichlet problem by means of L^2 estimates.

The asymptotic behavior of solutions is discussed in § 6 where it is shown that the solution decays exponentially along with all first order space derivatives. Section 7 extends the existence, uniqueness and regularity results to the non-homogeneous equation with a time-varying boundary condition.

The results contained in § 8 account for the name *pseudo-parabolic* which we have given to the equation under consideration. In particular it is shown that the solution of (1.1) depends continuously on the coefficient η, and that if η is close to zero then the corresponding solution of (1.1) is arbitrarily close to the solution of (1.2) which satisfies the same initial and boundary data.

Finally in § 9, a similar problem is posed and solved in the Schauder space of functions with uniformly Hölder-continuous derivatives. It is shown that the problem is well-posed in this Banach space, and the same method of constructing a solution as used in the Hilbert space development is applicable here. This section is independent of the previous material, but it depends on the solution of the Dirichlet problem by means of the estimates of Schauder.

2. Preliminary material. In this section we shall recall some standard definitions and notations for various spaces of functions. In particular we shall discuss the domain G associated with the problem we are to consider as well as the Sobolev spaces of functions defined on G.

R^n will denote the n-dimensional real Euclidean space with points specified by coordinates of the form

$$x = (x_1, x_2, \cdots, x_n).$$

For any open set Ω in R^n we shall denote by $C^m(\Omega)$ the set of all functions defined on Ω which have continuous derivatives of all orders up through the integer m. By $C^m(\hat{\Omega})$ we shall mean those elements of $C^m(\Omega)$ for which all the indicated derivatives are uniformly continuous and hence have unique continuous extensions to the boundary of Ω, and we set

$$C^\infty(\hat{\Omega}) = \bigcap_{m=1}^{\infty} C^m(\hat{\Omega}).$$

The support of a function on Ω is the closure of the set of points for which the function is nonzero. The set consisting of those functions in $C^\infty(\hat{\Omega})$ with compact support contained in Ω is denoted by $C_0^\infty(\Omega)$. Each of the sets defined above is a linear space under pointwise addition and scalar multiplication of the elements. The αth derivative of a function φ in $C^m(\Omega)$ is denoted by

$$D^\alpha \varphi = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} \varphi,$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is an n-tuple of nonnegative integers and the order of this derivative is denoted by

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

The domain G associated with the problem is a bounded open point set in R^n whose boundary ∂G is an $(n-1)$-dimensional manifold with G all on one side of ∂G. With regard to the degree of smoothness of the boundary we shall say that ∂G is of the class C^m for a positive integer m if at each point of ∂G there is a neighborhood Ω in which ∂G has a representation of the form

$$x_i = g(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n),$$

where g is in $C^m(\Omega)$.

We shall make use of a generalization of the concept of differentiation in order to obtain a large class of differentiable functions. Let $L^2(G)$ denote the space of (equivalence classes of) square-summable functions on G.

Definition 2.1. For each integer $k \geqq 0$, $H^k(G)$ is the set of (equivalence classes of) real-valued measurable functions f on G for which the αth derivative $D^\alpha f$ belongs to $L^2(G)$ whenever $|\alpha| \leqq k$.

The linear space $H^k(G)$ has a norm and scalar product defined on it by

$$\|f\|_k = \left(\sum_{|\alpha| \leqq k} \int_G |D^\alpha f|^2 \right)^{1/2}$$

and

$$(f, g)_k = \sum_{|\alpha| \leqq k} \int_G (D^\alpha f \cdot D^\alpha g),$$

respectively. From the definition of $H^k(G)$ and the completeness of $L^2(G)$ it follows easily that $H^k(G)$ is complete with respect to the indicated norm and is hence a Hilbert space.

We shall want to distinguish those elements of $H^k(G)$ which vanish on ∂G in some generalized sense. This is accomplished as follows.

Definition 2.2. For each integer $k \geqq 0$, $H_0^k(G)$ is the closure of $C_0^\infty(G)$ in $H^k(G)$.

Thus $H_0^k(G)$ is a closed subspace of $H^k(G)$. It can be shown that if ∂G is of the class C^k and if φ belongs to $C^{k-1}(\mathrm{cl}(G))$, then φ is in $H_0^k(G)$ if and only if φ is in $H^k(G)$ and $D^\alpha \varphi = 0$ on ∂G whenever $|\alpha| \leqq k - 1$. Furthermore it can be shown that an element f in $H^k(G)$ is in $H_0^k(G)$ if and only if $D^\alpha f$ belongs to $H_0^1(G)$ for all α with $|\alpha| \leqq k - 1$.

It is worthwhile to note that $C_0^\infty(G)$ is not in general a dense subset of $H^k(G)$, although it is true that $H_0^0(G) = H^0(G) = L^2(G)$ since $C_0^\infty(G)$ is dense in $L^2(G)$.

Also, we note that most rules of the calculus can be extended to generalized derivatives, [1], [12].

The following result is known as Poincare's inequality and relates the L^2-norm of a function to that of its derivatives.

PROPOSITION 2.1. *There is a constant* $K \geqq 1$ *depending only on* G *such that for all* φ *in* $H_0^1(G)$

$$\int_G \varphi^2 \leqq K \int_G \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \varphi \right)^2 .$$

The proof of this proposition [17, pp. 181–182] depends only on integration by parts.

Another useful result for domains with smooth boundaries is the Sobolev lemma. Letting $[\gamma]$ denote the greatest integer less than or equal to the real number γ, we have the following uniform bound on functions in $H^k(G)$ when k is sufficiently large, [12, pp. 282–284].

PROPOSITION 2.2. *Let* ∂G *be of class* C^1 *and* $k = [n/2] + 1$. *There is a constant* C_s *(depending on* G*) such that for any* u *in* $H^k(G)$ *and almost all* x *in* G *we have*

$$|u(x)| \leqq C_s \|u\|_k .$$

COROLLARY. *If* u *is in* $H^k(G)$, $k = [n/2] + 1$, *then* u *can be identified with a uniformly continuous function* $u(x)$ *on* G *for which the above inequality is true.*

3. The boundary value problem. In the following we shall let M and L denote differential operators of second order of the form

$$M = - \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_i} m_{ij}(x) \frac{\partial}{\partial x_j} + m(x) \tag{3.1}$$

$$L = - \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_i} l_{ij}(x) \frac{\partial}{\partial x_j} + \sum_{i=1}^{n} l_i(x) \frac{\partial}{\partial x_i} + l(x). \tag{3.2}$$

The (classical) problem under consideration is that of finding a function $u(x, t)$ of the space and time variables x and t which satisfies the partial differential equation

$$M\left(\frac{\partial}{\partial t} u(x, t) \right) + L(u(x,t)) = 0,$$

vanishes on the boundary of the domain G for all t in R, and at $t = 0$ is equal to a given function $u_0(x)$ of the space variable x.

The operators M and L are meaningful for functions in $C^2(G)$, but we shall extend the domain of these operators in a meaningful way. This will be accomplished by using the Lax–Milgram theorem on bounded positive-definite bilinear forms in Hilbert space to obtain the corresponding Friedrichs extensions of these operators. The domains of the extended operators are dense subsets of $H_0^1(G)$, and it is in this space that the generalized boundary value problem will be formulated. We shall seek a solution $u(x, t)$ belonging to $H_0^1(G)$ for each fixed t in R, and this will provide the generalization of the vanishing on the boundary of G in view of the remarks in the previous section on the boundary behavior of functions in $H_0^k(G)$.

The following properties of the operators M and L will be assumed.

PROPERTY 1. (P_1). The coefficients occurring in (3.1) and (3.2) are bounded and measurable, and $m(x) \geqq 0$ for x in G.

PROPERTY 2. (P_2). M is uniformly strongly elliptic on G. Hence there is a constant $m_0 > 0$ for which

$$\sum_{i,j=1}^{n} m_{ij}(x)\xi_i\xi_j \geqq m_0 \sum_{i=1}^{n} (\xi_i)^2$$

whenever $\xi = (\xi_1, \cdots, \xi_n)$ is in R^n and x is in G.

PROPERTY 3. (P_3). For $1 \leqq i, j \leqq n$, l_{ij} and m_{ij} belong to $H^2(G)$.

This last assumption is used to relate the operators M and L to the respective bilinear forms

$$B_M(\varphi, \psi) \equiv \sum_{i,j=1}^{n} \left(m_{ij}\frac{\partial}{\partial x_j}\varphi, \frac{\partial}{\partial x_i}\psi\right)_0 \quad (m\varphi, \psi)_0$$

and

$$B_L(\varphi, \psi) \equiv \sum_{i,j=1}^{n} \left(l_{ij}\frac{\partial}{\partial x_j}\varphi, \frac{\partial}{\partial x_i}\psi\right)_0 + \sum_{i=1}^{n} \left(l_i\frac{\partial}{\partial x_i}\varphi, \psi\right) + (l\varphi, \psi)_0$$

for φ, ψ in $C_0^\infty(G)$. It follows from an integration by parts and (P_3) that

$$B_M(\varphi, \psi) = (M\varphi, \psi)_0$$

and

$$B_L(\varphi, \psi) = (L\varphi, \psi)_0.$$

The generalized problem which we shall eventually formulate will be stated in terms of the bilinear forms B_M and B_L. For this reason there is no necessity for the assumption (P_3), and it will be needed only when we wish to consider the linear operators M and L for which it is necessary to be able to differentiate the higher order coefficients.

The inequalities we derive now essentially characterize the bilinear forms B_M and B_L. Letting φ and ψ denote arbitrary elements of $C_0^\infty(G)$, we have from the Cauchy–Schwarz inequalities

$$\begin{aligned} |B_M(\varphi, \psi)| &= \left| \sum_{i,j=1}^{n} (m_{ij}\varphi_{xj}, \psi_{xi})_0 + (m\varphi, \psi)_0 \right| \\ &\leqq \overline{m}\left(\sum_{i=1}^{n} \|\psi_{xi}\|_0^2\right)^{1/2}\left(\sum_{j=1}^{n} \|\varphi_{xj}\|_0^2\right)^{1/2} + \overline{m}\|\varphi\|_0\|\psi\|_0, \end{aligned}$$

where $\overline{m} = \max_{1 \leqq i,j \leqq n}\{\|m_{ij}\|_\infty, \|m\|_\infty\}$. Hence there is a constant $K_m > 0$ such that

$$|B_M(\varphi, \psi)| \leqq K_m\|\varphi\|_1\|\psi\|_1 \tag{3.3}$$

for all φ, ψ in $C_0^\infty(G)$. A similar argument will verify that for some $K_l > 0$ we have

$$|B_L(\varphi, \psi)| \leqq K_l\|\varphi\|_1\|\psi\|_1. \tag{3.4}$$

Hence B_M and B_L are defined by continuity for all φ, ψ in $H_0^1(G)$.

From the ellipticity condition (P_2) we have for φ in $C_0^\infty(G)$

$$B_M(\varphi, \varphi) \geqq m_0 \sum_{i=1}^{n} \|\varphi_{xi}\|_0^2.$$

Poincare's inequality then yields

$$B_M(\varphi, \varphi) \geqq \frac{m_0}{K} \|\varphi\|_0^2,$$

so we have

$$B_M(\varphi, \varphi) \geqq \frac{m_0}{2} \sum_{i=1}^{n} \|\varphi_{xi}\|_0^2 + \frac{m_0}{2K} \|\varphi\|_0^2.$$

Hence there is a constant $k_m > 0$ such that

$$B_M(\varphi, \varphi) \geqq k_m \|\varphi\|_1^2 \tag{3.5}$$

for all φ in $C_0^\infty(G)$.

We shall demonstrate that we may assume without loss of generality that L is elliptic and that

$$B_L(\varphi, \varphi) \geqq k_l \|\varphi\|_1^2 \tag{3.6}$$

for some $k_l > 0$ and all φ in $C_0^\infty(G)$. In particular, $u(x, t)$ is a solution of the problem if and only if $v(x, t) = e^{-\alpha t} u(x, t)$ satisfies the equation

$$M\left(\frac{\partial v}{\partial t}\right) + (\alpha M + L)v = 0.$$

From (3.4) and (3.5) it follows that (3.6) is true for $\alpha M + L$ instead of L if we choose $\alpha \geqq (K_l + k_l)/k_m$. That is, $(L, \varphi, \varphi)_0 \geqq -K_l \|\varphi\|_1^2$, so

$$((\alpha M + L)\varphi, \varphi)_0 \geqq (\alpha k_m - K_l) \|\varphi\|_1^2 \geqq k_l \|\varphi\|_1^2.$$

The ellipticity is verified as follows: letting $\bar{l} = \sup \{|l_{ij}(x)| : x \in G,\ 1 \leqq i, j \leqq n\}$, we have

$$\begin{aligned} \left| \sum_{j,i=1}^{n} l_{ij}(x)\xi_i\xi_j \right| &\leqq \bar{l} \sum_{i=1}^{n} |\xi_i| \sum_{j=1}^{n} |\xi_j| \\ &\leqq \bar{l} n^2 \left(\sum_{i=1}^{n} (\xi_i)^2 \right)^{1/2} \left(\sum_{j=1}^{n} (\xi_j)^2 \right)^{1/2} \\ &= \bar{l} n^2 \sum_{i=1}^{n} (\xi_i)^2. \end{aligned}$$

Hence

$$\sum_{i=1}^{n} \sum_{j=1}^{n} l_{ij}(x)\xi_i\xi_j \geqq -\bar{l} n^2 \sum_{i=1}^{n} (\xi_i)^2,$$

so we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha m_{ij}(x)+l_{ij}(x))\xi_i\xi_j \geqq (\alpha m_0 - \bar{l}n^2)\sum_{i=1}^{n}(\xi_i)^2$$

for all x in G, ξ in R^n, so $\alpha M + L$ is uniformly strongly elliptic for α sufficiently large. As stated above, we shall hereafter assume L is elliptic and that (3.6) is satisfied.

We are ready to obtain the extensions of M and L by means of the Lax–Milgram theorem [25, p. 171]. This asserts that there exists a linear transformation M_0 with domain $D(M_0)$ dense in $H_0^1(G)$ for which $B_M(\varphi, \psi) = (M_0\varphi, \psi)_0$ whenever φ is in $D(M_0)$ and ψ in $H_0^1(G)$. The range of M_0 is all of $H^0(G)$, and M_0 has an inverse which is a bounded mapping of $H^0(G)$ into $H_0^1(G)$. From (P_3) it follows that $(M\varphi, \psi)_0 = (M_0\varphi, \psi)_0$ for all φ, ψ in $C_0^\infty(G)$, so M_0 is a (weak) extension of M, also known as the *minimal operator* associated with M, or the *Friedrichs extension*. See [25, p. 173], [31, pp. 329–335] and [21]. The discussion above can be duplicated to obtain the Friedrichs extension L_0 of L with domain $D(L_0)$.

The generalized initial boundary value problem may now be formulated in $H_0^1(G)$ as follows: Find a strongly differentiable [18, p. 59] mapping $t \to u(t)$ of R into $H_0^1(G)$ such that

$$B_M(u'(t), \varphi) + B_L(u(t), \varphi) = 0 \tag{3.7}$$

for each t in R and φ in $C_0^\infty(G)$ with $u(0) = u_0$, where u_0 is a given "initial" function in $H_0^1(G)$.

The proof of the following existence-uniqueness theorem is the context of the next section.

THEOREM 3.1. *Assume (P_1) and (P_2). There is a unique bounded linear operator B on $H_0^1(G)$ which extends $-M_0^{-1}L_0$. If u_0 is an element of $H_0^1(G)$, then there is a unique strongly differentiable mapping $t \to u(t)$ of R into $H_0^1(G)$ such that*

$$u'(t) = Bu(t) \tag{3.8}$$

for all t in R and $u(0) = u_0$.

COROLLARY 3.1. *The vector-valued function $u(t)$ satisfies* (3.7).

COROLLARY 3.2. *If $u(t)$ belongs to $D(L_0)$ then $u'(t)$ is in $D(M_0)$ and*

$$M_0u'(t) + L_0u(t) = 0 \tag{3.9}$$

for all t in R.

4. Existence and uniqueness. The operators M_0 and L_0 are bijections onto $H^0(G)$ from $D(M_0)$ and $D(L_0)$ respectively. We shall show that the bijection $M_0^{-1}L_0$ from $D(L_0)$ onto $D(M_0)$ can be uniquely extended as a bounded linear operator from $H_0^1(G)$ onto itself and that the appropriate exponential of this bounded operator provides the unique solution of the problem in $H_0^1(G)$ as stated in § 3.

We shall verify that the bijection $M_0^{-1}L_0$ is bounded with respect to the norm $\|\cdot\|_1$. If φ is in $C_0^\infty(G)$ it follows from (3.4) and (3.5) that

$$k_m\|M_0^{-1}L_0\varphi\|_1^2 \leqq (L_0\varphi, M_0^{-1}L_0\varphi)_0 \leqq K_l\|\varphi\|_1\|M_0^{-1}L_0\varphi\|_1,$$

so we have

$$\|M_0^{-1}L_0\varphi\|_1 \leqq (K_l/k_m)\|\varphi\|_1. \tag{4.1}$$

The constant K_l/k_m depends only on L, M and the domain G, so (4.1) is true for all φ in $C_0^\infty(G)$. Since this set is dense in $H_0^1(G)$ it follows that $M_0^{-1}L_0$ is bounded and has a unique extension to a bounded linear operator on $H_0^1(G)$. We shall let B denote the extension of $-M_0^{-1}L_0$ and remark that L_0 is defined only on $D(L_0)$ while $B = -M_0^{-1}L_0$ has been defined on all $H_0^1(G)$ by continuity.

By an elementary argument we can verify that the range of B is all of $H_0^1(G)$ and that its inverse is bounded. Letting φ belong to $C_0^\infty(G)$ we have from (3.6) and (3.3)

$$\begin{aligned} k_l\|L_0^{-1}M_0\varphi\|_1^2 &\leqq (M_0\varphi, L_0^{-1}M_0\varphi)_0 \\ &\leqq K_m\|\varphi\|_1\|L_0^{-1}M_0\varphi\|_1, \end{aligned}$$

so we have

$$\|L_0^{-1}M_0\varphi\|_1 \leqq (K_m/k_l)\|\varphi\|_1$$

for all φ in $C_0^\infty(G)$; hence $B^{-1} = -L_0^{-1}M_0$ is bounded from $D(M_0)$ to $D(L_0)$. Since $D(M_0)$ is dense in $H_0^1(G)$, B is onto $H_0^1(G)$. In particular if g is in $H_0^1(G)$ there is a sequence $\{g_n\}$ from $D(M_0)$ which converges to g in the topology of $H_0^1(G)$. The boundedness of B^{-1} on $D(M_0)$ implies that the sequence $f_n = B^{-1}g_n$ is Cauchy in $D(L)$, hence converges to some element f in $H_0^1(G)$. From the continuity of B we conclude

$$B(f) = \lim\{B(f_n): n \to \infty\} = g.$$

The construction of B is indicated by Fig. 1.

$$\begin{tikzcd}
H_0^1(G) \arrow[rr, "B", "\text{isomorphism}"'] & & H_0^1(G) \\
D(L_0) \arrow[u] \arrow[rr, "-M_0^{-1}L_0", "\text{bijection}"'] \arrow[dr, "L_0"'] & & D(M_0) \arrow[u] \arrow[dl, "-M_0"] \\
C_0^\infty(G) \arrow[u] \arrow[r, "L", "\text{injection}"'] & H^0(G) & C_0^\infty(G) \arrow[u] \arrow[l, "-M"', "\text{injection}"]
\end{tikzcd}$$

FIG. 1

From the boundedness of B we are able to construct the exponential of the operator tB for each real number t. This will yield a one-parameter group $\{E(t): t \text{ in } R\}$ of bounded operators on $H_0^1(G)$, and these will be used to construct the solution of the generalized problem. For each real number t, define $E(t)$ by means of the power series

$$\exp(tB) = \sum_{k=0}^{\infty} (tB)^k/k!.$$

Then $E(t)$ is the limit in the uniform operator topology of $\mathscr{L}(H_0^1(G))$ of the sequence

$$\sum_{k=0}^{n} (tB)^k/k!.$$

The convergence of this sequence follows from the completeness of the space $\mathscr{L}(H_0^1(G))$ of bounded linear operators on $H_0^1(G)$, and this is a consequence of the completeness of $H_0^1(G)$. By means of the classical arguments on the convergence of power series with absolute values replaced by the norm $\|\cdot\|_1$, we can show that the indicated power series in tB is convergent for all t in R and that the convergence is uniform on compact subsets of R. In this manner we obtain for each real t the bounded linear operator $E(t)$ on $H_0^1(G)$ whose norm satisfies

$$\|E(t)\|_1 \leqq \exp(|t|\,\|B\|_1).$$

For the purpose of reference we collect the properties of this group of operators on $H_0^1(G)$:

(4.2)
(a) $\{E(t): t \text{ in } R\}$ is an Abelian group, and $E(t_1 + t_2) = E(t_1)E(t_2)$, $E(0) = I$.
(b) Each $E(t)$ is a bounded linear operator on $H_0^1(G)$ and the dependence on t is continuous in the uniform operator topology.
(c) $E(t)$ is differentiable in the uniform operator topology, and

$$E'(t) = B \cdot E(t).$$

The group of bounded operators $E(t)$ can now be used to construct our weak solution. Let u_0 be the given "initial" function in $H_0^1(G)$ and define

$$u(t) = E(t)u_0 \tag{4.3}$$

for each t in R. From (4.2 c) it follows that

$$u'(t) = B \cdot u(t) \tag{4.4}$$

in the strong topology of $H_0^1(G)$. Furthermore we see from (4.2 a) that $u(0) = u_0$ and from (4.2 b) that $u(t)$ is a continuous function of t in the strong topology of $H_0^1(G)$.

We shall verify that the solution given by (4.3) is the *only* such solution to the generalized problem. Letting $u(t)$ denote any such solution, we consider the real-valued function

$$\alpha(t) = (u(t), u(t))_1.$$

By the Cauchy–Schwarz inequality and (4.4) we have

$$|\alpha'(t)| = 2|(Bu(t), u(t))_1| \leqq 2\|B\|_1\alpha(t)$$

for all real t. This yields the estimate $\alpha(t) \leqq \exp(2\|B\|_1|t|)\alpha(0)$ from which we have

$$\|u(t)\|_1 \leqq \|u(0)\|_1 \exp(\|B\|_1|t|). \tag{4.5}$$

An immediate consequence of (4.5) is the uniqueness of the solution, for the difference of any two solutions is a solution which is initially zero, hence zero for all t in R.

Finally we must verify (3.7). Since $u(t)$ belongs to $H_0^1(G)$, there is a sequence $\{\varphi_n\}$ in $C_0^\infty(G)$ converging to $u(t)$. The boundedness of B on $H_0^1(G)$ implies that $\{B\varphi_n\}$ converges to $u'(t)$. But $M_0(B\varphi_n) + L_0(\varphi_n) = 0$ for all n, so we see

$$\begin{aligned} B_M(u'(t), \varphi) + B_L(u(t), \varphi) &= \lim_{n\to\infty} B_M(B\varphi_n, \varphi) + \lim_{n\to\infty} B_L(\varphi_n, \varphi) \\ &= \lim_{n\to\infty} [(M_0(B\varphi_n), \varphi)_0 + (L_0\varphi_n, \varphi)_0] \equiv 0. \end{aligned}$$

Having obtained the weak solution to the generalized problem under consideration, we shall relate the extended operators L_0 and M_0 on their respective domains to the operators L_1 and M_1 which are just the extensions of L and M respectively to the domain $H^2(G)$ in the sense of generalized derivatives. *Hereafter we shall always assume* (P_3). An integration by parts shows that for all f in $H_0^1(G) \cap H^2(G)$ and g in $H_0^1(G)$ we have

$$(M_1 f, g)_0 = B_M(f, g),$$

and from the characterization of $D(M_0)$ in the Lax–Milgram theorem it follows that

$$H_0^1(G) \cap H^2(G) \subset D(M_0)$$

and that $M_0(f) = M_1(f)$ when f belongs to $H_0^1(G) \cap H^2(G)$. Likewise we have

$$H_0^1(G) \cap H^2(G) \subset D(L_0)$$

and $L_0 = L_1$ on $H_0^1(G) \cap H^2(G)$.

5. Regularity of the weak solution. The group of operators $\{E(t): t \text{ in } R\}$ has enabled us to construct a solution by (4.3) of the generalized problem in the weak sense of (3.7). We shall in this section show that each of the subspaces $H_0^1(G) \cap H^p(G)$ remains invariant under the family $\{E(t)\}$, where the integer p depends on the differentiability of the coefficients in L and M as well as the boundary of G. These results are based on the regularity problem for the Dirichlet problem. The invariance of these subspaces implies that the solution $u(t)$ given by (4.3) is just as smooth in the L^2 sense as is the initial function u_0. In fact the special case $L = M$ possesses the solution $u(x, t) = e^{-t}u_0(x)$, and this example shows that we may not in general expect the solution to be more smooth in the space variable than is the initial function. Thus the invariance of the subspaces is the strongest possible result. Finally we shall show that under certain smoothness conditions on the coefficients, boundary and initial function u_0, the solution is an analytic function of the time variable and is uniformly continuous (or differentiable) in the space variable.

In order to show that B leaves invariant the spaces $H_0^1(G) \cap H^p(G)$ we shall make use of the results on the Dirichlet problem as presented in [12, pp. 270–307]. The following criterion will be used to specify the assumptions of smoothness on the generalized problem.

DEFINITION 5.1. The generalized initial boundary value problem (3.7) is *p-smooth* for the integer $p \geqq 2$, if

(i) the coefficients in (3.1) and (3.2) satisfy for $1 \leqq i; j \leqq n$; l_{ij}, $m_{ij} \in C^{p-1}(\mathrm{cl}(\mathbf{G}))$; $m, l, l_i \in C^{p-2}(\mathrm{cl}(\mathbf{G}))$, with $m(x) \geqq 0$ for x in $\mathrm{cl}(\mathbf{G})$;
(ii) M and L are uniformly strongly elliptic in G; and
(iii) the boundary ∂G is of class C^p.

From [12] there is then for any f in $H^{p-2}(G)$ a unique pair u, v in $H_0^1(G) \cap H^p(G)$ for which $L_0 u = f$ and $M_0 v = f$.

Assume that the generalized problem is p-smooth and let v belong to $H_0^1(G) \cap H^p(G)$. $L_0(v)$ is in $H^{p-2}(G)$, so there is a unique u in $H_0^1(G) \cap H^p(G)$ for which $M_0 u = -L_0(v)$. Thus $u = -M_0^{-1} L_0 v$ is in $H_0^1(G) \cap H^p(G)$, so we see that B maps $H_0^1(G) \cap H^p(G)$ into itself. Furthermore B is onto $H_0^1(G) \cap H^p(G)$ from itself, since we need only solve the Dirichlet problem

$$L_0 v = -M_0 u, \qquad v \text{ in } H_0^1(G)$$

for a given u in $H_0^1(G) \cap H^p(G)$ to obtain the v in $H_0^1(G) \cap H^p(G)$ for which $u = -M_0^{-1} L_0 v$. We conclude that B maps each of these subspaces $H_0^1(G) \cap H^q(G)$ onto itself for $p \geqq q \geqq 2$.

Remark. We shall hereafter assume that the problem is at least 2-smooth. It follows that if f is in $H^0(G)$ there is a unique v in $H_0^1(G) \cap H^2(G)$ with $M_0 v = f$; hence the domain $D(M_0)$ is contained in $H_0^1(G) \cap H^2(G)$, and by a previous remark thus equal to $H_0^1(G) \cap H^2(G)$. Similarly, $D(L_0) = H_0^1(G) \cap H^2(G)$. We collect these results in the following statement.

PROPOSITION 5.1. *Let the generalized problem be p-smooth for some integer $p \geqq 2$. Then the domains $D(L_0)$ and $D(M_0)$ of the respective Friedrich's extensions coincide with $H_0^1(G) \cap H^2(G)$ and the bounded extension B of $-M_0^{-1} L_0$ on $H_0^1(G)$ leaves invariant each of the subspaces $H_0^1(G) \cap H^q(G)$, where $2 \leqq q \leqq p$.*

We shall make use of the closed graph theorem [18, p. 47] to show that

$$B : H_0^1(G) \cap H^p(G) \to H_0^1(G) \cap H^p(G)$$

is bounded with respect to the norm $\|\cdot\|_p$. The linear operator B is said to be *closed* if whenever $x_n \to x_0$ and $Bx_n \to x_1$ it is necessarily true that $x_1 = Bx_0$. The closed graph theorem asserts that any such closed linear operator is necessarily bounded; its proof depends on the completeness of the space. We remark that since $H_0^1(G) \cap H^p(G)$ is a linear subset of the Hilbert space $H^p(G)$ and since $\|\cdot\|_1 \leqq \|\cdot\|_p$ on this space, $H_0^1(G) \cap H^p(G)$ is a (complete) Hilbert space with the norm $\|\cdot\|_p$.

We shall have need of similar results as this on the boundedness of a linear operator with respect to stronger topologies on subspaces, so we prove a fundamental lemma which with the above discussion implies that B is bounded on $H_0^1(G) \cap H^p(G)$.

FUNDAMENTAL LEMMA. *Let X_i $(i = 1, 2)$ be Banach spaces with respective norms $|\cdot|_i$. Let Y_i be a subset of X_i which is a Banach space with norm $\|\cdot\|_i$ and assume $|y|_i \leqq \|y\|_i$ when y belongs to Y_i. Let T be a bounded linear transformation from X_1 to X_2 such that T maps Y_1 into Y_2. Then T is bounded from Y_1 to Y_2.*

Proof. We need only show that T is closed as a transformation of Y_1 into Y_2. Hence let $\{y_n : n \geqq 2\}$ be a sequence in Y_1 for which $\|y_n - y_0\|_1 \to 0$ and

$\|Ty_n - y_1\|_2 \to 0$ as $n \to \infty$, where $y_0 \in Y_1$ and $y_1 \in Y_2$. Since

$$\begin{aligned} |y_1 - Ty_0|_2 &\leqq |y_1 - Ty_n|_2 + |T(y_n - y_0)|_2 \\ &\leqq |y_1 - Ty_n|_2 + |T||y_n - y_0|_1 \\ &\leqq \|y_1 - Ty_n\|_2 + |T|\,\|y_n - y_0\|_1, \end{aligned}$$

we have $y_1 = Ty_0$, so T is closed, hence bounded.

The significance of the boundedness of B on $H_0^1(G) \cap H^p(G)$ is that the group of operators $\{E(t) : t \text{ in } R\}$ is bounded on $H_0^1(G) \cap H^p(G)$. We state this as the main result of this section.

THEOREM 5.1. *If the generalized problem is p-smooth, then the group of operators $\{E(t) : t \text{ in } R\}$ leaves invariant the subspace $H_0^1(G) \cap H^p(G)$. For each t in R, $E(t)$ is a bijection of $H_0^1(G) \cap H^p(G)$ onto itself and is bounded with respect to the norm $\|\cdot\|_p$.*

In fact we could duplicate the discussion on the construction of the $E(t)$ but replace the norm $\|\cdot\|_1$ by $\|\cdot\|_p$ since B is bounded with respect to $\|\cdot\|_p$ and thus obtain the corresponding results with $H_0^1(G)$ replaced by $H_0^1(G) \cap H^p(G)$.

Since we always assume $p \geqq 2$ it follows that $H_0^1(G) \cap H^2(G)$ is invariant under $\{E(t) : t \text{ in } R\}$. Hence if u_0 is in $H_0^1(G) \cap H^2(G)$ the solution $u(t)$ of the equation (4.4) as given by (4.3) belongs to $H_0^1(G) \cap H^2(G)$ for each t in R. Furthermore it follows from (4.4) and the invariance of $H_0^1(G) \cap H^2(G)$ under B that $u'(t)$ belongs to $H_0^1(G) \cap H^2(G)$. But this is the domain of the extended operators M_0, so we may apply M_0 to both sides of (4.4) to obtain the equation

$$M_0u'(t) + L_0u(t) = 0. \tag{5.1}$$

That is, $M_0u'(t)$ and $L_0u(t)$ are both in $H^0(G)$, so (5.1) is equivalent to (3.7).

Since the group of operators constructed above leaves invariant the subspaces $H_0^1(G) \cap H^q(G)$ for $p \geqq q \geqq 2$ under the assumption of p-smoothness, it follows that this group also leaves invariant each of their (point-set) complements. That is, if u_0 is in $H_0^1(G) \cap H^{p-1}(G)$ but not in $H^p(G)$ then the same is true of $u(t)$ for each t in R. Thus our transformation group preserves smoothness but does not improve it.

We can use the Sobolev lemma to obtain a sufficient condition for the solution $u(t)$ to be a continuous function of the space variable and infinitely differentiable in the time variable.

PROPOSITION 5.2. *Let the generalized problem be p-smooth and u_0 belong to $H_0^1(G) \cap H^p(G)$, with $p \geqq [n/2] + 1$. Then for each t in R, $u(t)$ can be identified (a.e.) with a uniformly continuous function of x, denoted by $u(x, t)$, and the mapping $t \to u(x, t)$ is infinitely differentiable. The function $u(x, t)$ vanishes identically on the boundary ∂G.*

Proof. From Theorem 5.1 it follows that $u(t)$ belongs to $H_0^1(G) \cap H^p(G)$ for every t in R, hence by Sobolev's lemma it can be identified with a uniformly continuous function $u(x, t)$ on G. Also from Sobolev's inequality it follows that if $\delta \neq 0$

$$\begin{aligned} &|\delta^{-1}(u(x, t + \delta) - u(x, t)) - B \cdot u(x, t)| \\ &\qquad = |(\delta^{-1}(E(\delta) - I) - B)u(x, t)| \\ &\qquad \leqq C_s\|(\delta^{-1}(E(\delta) - I) - B)u(t)\|_p, \end{aligned}$$

where the constant C_s depends only on n and ∂G. Since the group $\{E(t):t \text{ in } R\}$ is infinitely differentiable in the uniform operator topology induced by $\|\cdot\|_p$ and its kth derivative is $B^k \cdot E(t)$, the last term in the above inequality converges to zero as $\delta \to 0$. This establishes the differentiability of $u(x, t)$ and the equality

$$\frac{\partial}{\partial t}u(x, t) = B \cdot u(x, t)$$

for each x in G. A repetition of this argument will show that $u(x, t)$ is infinitely differentiable with respect to t and that its derivatives agree with the corresponding derivatives of $u(t)$ in $H_0^1(G) \cap H^p(G)$.

In fact we see that $u(x, t)$ is *analytic* in t, for the remainder term

$$R_n(x, t) = \frac{\partial^{n+1}}{\partial t^{n+1}}u(x, T)t^{n+1}/(n + 1)!$$

(where $|T| < |t|$) of the Taylor formula converges to zero as n increases. That is,

$$\begin{aligned}|R_n(x, t)| &= |((tB)^{n+1}/(n + 1)!)u(x, T)| \\ &\leqq C_s\|(tB)^{n+1}/(n + 1)!\|_p\|u_0\|_p \exp(\|tB\|_p)\end{aligned}$$

by Sobolev's lemma, and the convergence of the power series for $\exp(tB)$ in $\mathscr{L}(H_0^1(G) \cap H^p(G))$ implies that its $(n + 1)$st term converges to zero in $\mathscr{L}(H_0^1(G) \cap H^p(G))$.

Finally we note that the uniform continuity of $u(x, t)$ in the space variable and its belonging to $H_0^1(G)$ imply that it vanishes on the boundary.

COROLLARY. *The solution $u(t)$ of the generalized problem can be identified with a function $u(x, t)$ in $C^m(\mathrm{cl}(\mathbf{G}))$ for each t in R, where $m = p - [n/2] - 1$. Hence a classical solution of the problem exists if $p \geqq [n/2] + 3$.*

6. Asymptotic behavior. We shall investigate the asymptotic behavior of the solution of the problem under consideration. The additional assumptions of symmetry of the operators or of constant coefficients are reasonable from the standpoint of physical motivation. We shall show in this section that under the appropriate conditions the solution $u(t)$ of our problem decays exponentially along with its derivatives up through a specified order. Furthermore we shall obtain more regularity type results which will imply that if the initial function has a given number of derivatives vanishing on the boundary then the solution has this same property.

Assume throughout the remainder of this section that M is symmetric and that the statements (P_1) and (P_2) of § 3 are valid. By letting u_0 in $H_0^1(G)$ be arbitrary, it follows from the strong differentiability of $u(t)$ and the symmetry of the bilinear form B_M on $H_0^1(G)$ that the real-valued function

$$\gamma(t) = B_M(u(t), u(t))$$

is continuously differentiable and that

$$\tfrac{1}{2}\gamma'(t) = B_M(u'(t), u(t)).$$

From (3.7), (3.6) and (3.3), respectively, we see that

$$\tfrac{1}{2}\gamma'(t) = -B_L(u(t), u(t))$$
$$\leqq -k_l\|u(t)\|_1^2 \leqq -k_l/K_m\gamma(t).$$

Hence for all $t \geqq 0$ we have

$$\gamma(t) \leqq \gamma(0)\exp(-2k_l/K_m t).$$

Using (3.5) and (3.3) we then obtain the estimate

$$\|u(t)\|_1 \leqq (K_m/k_m)^{1/2}\|u_0\|_1 \exp(-k_l/K_m t) \tag{6.1}$$

for $t \geqq 0$. This estimate (6.1) implying the exponential decay of the solution and its first derivatives in the sense of their L^2-norms is true in particular whenever M has constant coefficients, for then it can be written in a symmetric form.

Because of the boundedness of the operator B on $H_0^1(G)$ it has made no difference whether we consider (5.1) or the equation

$$-M_0u'(t) + L_0u(t) = 0.$$

However it is apparent in the previous paragraph that the sign of M is fundamental in obtaining the estimate (6.1) describing the asymptotic behavior in the norm $\|\cdot\|_1$ for the solution. Without this sign consideration we would only obtain an estimate of the form (4.5) which allows the solution to grow exponentially with the time variable. The estimate (6.1) is valid only for $t \geqq 0$, but this is the case of physical interest. The previously used estimate also implies that for $t \leqq 0$

$$\gamma(t) \geqq \gamma(0)\exp(-k_l/K_m t)$$

and by (3.3) and (3.5) would follow

$$\|u(t)\|_1 \geqq (k_m/K_m)^{1/2}\|u_0\|_1 \exp(k_l/K_m|t|) \tag{6.2}$$

whenever $t \leqq 0$. The inequalities (4.5), (6.1) and (6.2) describe the behavior of $u(t)$ in the large: the solution grows exponentially as $t \to -\infty$ and decays exponentially as $t \to \infty$ whenever M is symmetric.

We should note that in order for the above results to be significant we must assume that (3.6) is true for the "original" operator L. That is, by replacing L by $\alpha M + L$ we actually obtain the solution $e^{\alpha t}u(t)$ which is bounded by $(K_m/k_m)^{1/2}\|u_0\|_1$ $\exp((\alpha - kl/Km)t)$. But our sufficient choice for α given in § 3 implies that $\alpha - k_l/K_m = K_l/k_m + k_l/k_m - k_l/K_m$, and this quantity will in general be positive. In this event we would not be able to show that the solution decayed exponentially for $t \to \infty$. An example of this is the case $M = -d^2/dx^2$, $L = I$ and $u_0(x) = \{\sinh(x),$ $0 \leqq x \leqq \tfrac{1}{2};\ \sinh(1 - x),\ \tfrac{1}{2} \leqq x \leqq 1\}$. The solution $u(x, t) = u_0(x)e^t$ in $H_0^1(G)$ grows exponentially.

We will obtain some bounds on the higher order derivatives of the solution. To do so let us assume that the generalized problem is $(k + 1)$-smooth, k being an integer $\geqq 1$, and that M and L have constant coefficients.

Our first task is to show that the space $H_0^{1+k}(G)$ is invariant under the group of operators $\{E(t)\}$. Since B has already been shown to be bounded with respect to the $(k + 1)$-norm, it will suffice to show that B maps $H_0^{1+k}(G)$ into itself. Hence let ψ be an element of $C_0^\infty(G)$. The regularity results previously obtained imply that $B\psi$ belongs to $H^{1+k}(G)$. If $|\alpha| \leqq k$ then since $D^\alpha\psi$ belongs to $C_0^\infty(G)$ we have $BD^\alpha\psi$ belongs to $H_0^1(G) \cap H^2(G)$ and hence

$$M_0(BD^\alpha\psi) + L_0(D^\alpha\psi) = 0.$$

But M_0 and L_0 have constant coefficients, so we see

$$\begin{aligned} M_0(BD^\alpha\psi) &= -L_0(D^\alpha\psi) = -D^\alpha(L_0\psi) \\ &= D^\alpha(M_0B\psi) = M_0(D^\alpha B\psi). \end{aligned}$$

That is, we have

$$D^\alpha(B\psi) = B(D^\alpha\psi) \tag{6.3}$$

belongs to $H_0^1(G)$ whenever $|\alpha| \leqq k$, so in particular $B\psi$ must be in $H_0^{1+k}(G)$. Since B maps $C_0^\infty(G)$ into $H_0^{1+k}(G)$ and is bounded with respect to the $(k+1)$-norm, it follows that B maps all of $H_0^{1+k}(G)$ into itself. Also it is easy to show that (6.3) is true for all ψ in H_0^{1+k}; the argument is similar to that used below to verify (6.4).

We have shown that each $E(t)$ maps $H_0^{1+k}(G)$ onto itself and we shall verify that when $|\alpha| \leqq k$

$$D^\alpha E(t)\psi = E(t)D^\alpha\psi \tag{6.4}$$

for each ψ in $H_0^{1+k}(G)$. Let $E_n(t)$ denote the nth partial sum of the series which defined $E(t)$. Since D^α commutes with B it also commutes with each $E_n(t)$. Thus for any φ in $C_0^\infty(G)$ we have

$$\begin{aligned} (E(t)D^\alpha\psi, \varphi)_0 &= \lim_{n\to\infty} (E_n(t)D^\alpha\psi, \varphi)_0 = \lim_{n\to\infty} (D^\alpha E_n(t)\psi, \varphi)_0 \\ &= \lim_{n\to\infty} (E_n(t)\psi, (-1)^{|\alpha|}D^\alpha\varphi)_0 = (E(t)\psi, (-1)^{|\alpha|}D^\alpha\varphi)_0 \\ &= (D^\alpha E(t)\psi, \varphi)_0. \end{aligned}$$

The desired estimates on the derivatives of a solution to the generalized problem are now easily obtained. Let u_0 be given in $H_0^{1+k}(G)$. Then $u(t) = E(t)u_0$ belongs to $H_0^{1+k}(G)$ and from (6.4) it follows that $D^\alpha u(t)$ is the unique solution in $H_0^1(G)$ of the generalized problem with initial condition $D^\alpha u(0) = D^\alpha u_0$. Hence we have the estimate

$$\|D^\alpha u(t)\|_1 \leqq (K_m/k_m)^{1/2}\|D^\alpha u_0\|_1 \exp\left(-\frac{kl}{Km}t\right) \tag{6.5}$$

for all α with $|\alpha| \leqq k$.

From the inequality (6.5) one can proceed by means of the Sobolev lemma to obtain pointwise bounds on the solution and various derivatives. The smoothness of the problem now depends only on the differentiability of the boundary ∂G, so the largest number k for which the solution belongs to $H_0^{1+k}(G)$ and (6.5) is true when $|\alpha| \leqq k$ depends on the boundary ∂G and the initial function u_0.

7. The nonhomogeneous problem. The objective in this section is to extend the previous results to the nonhomogeneous equation

$$M_1 u'(t) + L_1 u(t) = f(t) \tag{7.1}$$

with a solution in $H^2(G)$ satisfying a nonhomogeneous time-varying boundary condition. Note that for any v in $H^2(G)$ the expression $M_1 v$ denotes the element of $H^0(G)$ defined as a linear combination of v and its first and second order strong derivatives as specified by (3.1). It follows that the linear mapping $v \mapsto M_1 v$ is bounded from $H^2(G)$ to $H^0(G)$, and we have shown that M_0 is the restriction of M_1 to the subspace $H_0^1(G) \cap H^2(G)$. The corresponding statements hold for the operator L_1.

We shall first prove the following result.

LEMMA 7.1. *Assume that the (associated homogeneous) problem is 2-smooth and $f(t)$ is strongly continuous in $H^0(G)$. There is a unique mapping $t \mapsto w(t)$ of R into $H_0^1(G) \cap H^2(G)$ with a strongly continuous derivative which satisfies (7.1) and the initial condition $w(0) = 0$.*

Proof. The operator M_0^{-1} is continuous from $H^0(G)$ into $H_0^1(G)$, so it follows from the Fundamental Lemma of § 5 that it not only maps $H^0(G)$ onto $H_0^1(G) \cap H^2(G)$ but is continuous with respect to the stronger norm $\|\cdot\|_2$ on $H_0^1(G) \cap H^2(G)$. The strong continuity of $f(t)$ implies that $M_0^{-1}f(t)$ is strongly continuous with respect to $\|\cdot\|_2$. Also the continuity of the mapping $\xi \mapsto E(\xi)$ in the uniform operator topology of $\mathscr{L}(H_0^1(G) \cap H^2(G))$ implies that for each t in R the function

$$T \mapsto E(t - T)M_0^{-1}f(T)$$

from R into $H_0^1(G) \cap H^2(G)$ is strongly continuous.

By means of the calculus of vector-valued functions [18, pp. 56–58] we have given for each real number t an element of $H_0^1(G) \cap H^2(G)$ denoted by

$$w(t) = \int_0^t E(t - T)M_0^{-1}f(T)\,dT.$$

The integral is taken as a limit of Riemann sums with respect to the norm $\|\cdot\|_2$. From the differentiability of $E(t)$ it follows that $w(t)$ is differentiable with respect to $\|\cdot\|_2$ and that

$$\begin{aligned} w'(t) &= \int_0^t E'(t - T)M_0^{-1}f(T)\,dT + E(0)M_0^{-1}f(t) \\ &= \int_0^t B \cdot E(t - T)M_0^{-1}f(T)\,dT + M_0^{-1}f(t). \end{aligned}$$

The continuity and linearity of B then implies that

$$w'(t) = Bw(t) + M_0^{-1}f(t).$$

Each term of this last equation belongs to $H_0^1(G) \cap H^2(G)$ so we have

$$M_0 w'(t) + L_0 w(t) = f(t),$$

where $w(t)$ has a strongly continuous derivative in $H_0^1(G) \cap H^2(G)$ and $w(0) = 0$.

The uniqueness of $w(t)$ follows from the corresponding result for the homogeneous equation by linearity.

We shall proceed by means of this lemma to the case of time-varying boundary conditions. The boundary condition is given by a function $t \mapsto \beta(t)$ from R to $H^2(G)$ with a strongly continuous derivative in the $\|\cdot\|_2$-norm. The initial function u_0 belongs to $H^2(G)$, and these functions satisfy a compatibility condition

$$u_0 - \beta(0) \in H_0^1(G). \tag{7.2}$$

Define a function in $H^0(G)$ by

$$F(t) = f(t) - M_1\beta'(t) - L_1\beta(t)$$

for each t in R. The continuity of β and β' in $H^2(G)$ implies that $F(t)$ is continuous in $H^0(G)$. From the preceding lemma we know that the function

$$v(t) = \int_0^t E(t - T)M_0^{-1}F(T)\,dT$$

in $H_0^1(G) \cap H^2(G)$ satisfies the equation

$$M_0v'(t) + L_0v(t) = F(t)$$

and the initial condition $v(0) = 0$. Now we define the function

$$u(t) = \beta(t) + E(t)(u_0 - \beta(0)) + v(t) \tag{7.3}$$

which has a strongly continuous derivative in $H^2(G)$. Furthermore we may verify directly that $u(t)$ satisfies the requirements in the following theorem which is the main result of this section.

THEOREM 7.1. *Let the (associated homogeneous) problem be 2-smooth, $f(t)$ be strongly continuous in $H^0(G)$, $\beta(t)$ have a strongly continuous derivative in $H^2(G)$, and u_0 be a function in $H^2(G)$ for which* (7.2) *is satisfied. There is a unique strongly differentiable function $u(t)$ in $H^2(G)$ given by* (7.3) *which satisfies* (7.1) *and for which $u(t) - \beta(t)$ is in $H_0^1(G)$ for all t in R, and $u(0) = u_0$.*

Remark. In verifying (7.1) it is essential to note that $M_1M_0^{-1} = I$ on $H^0(G)$ and hence $M_1B = -L_0$ on $H_0^1(G) \cap H^2(G)$.

In the same manner we can verify the following result.

COROLLARY. *Let the problem be p-smooth ($p \geqq 2$), $f(t)$ be strongly continuous in $H^{p-2}(G)$, $\beta(t)$ have a strongly continuous derivative in $H^p(G)$, u_0 belong to $H^p(G)$ and satisfy* (7.2). *Then there is a strongly differentiable mapping $u(t)$ of R into $H^p(G)$ satisfying* (7.1) *with $u(t) - \beta(t)$ belonging to $H_0^1(G)$ for all real t and $u(0) = u_0$.*

8. Remarks on parabolic equations. In this section we shall briefly discuss an interesting relationship between the solution $u_\lambda(t)$ of the pseudoparabolic equation

$$(\lambda L_0 + I)u_\lambda'(t) + L_0u_\lambda(t) = 0 \tag{8.1}$$

and the solution $u(t)$ of the parabolic equation

$$u'(t) + L_0u(t) = 0, \tag{8.2}$$

both of which satisfy the same initial condition and a homogeneous boundary condition. From the very form of these equations one might expect that for λ

sufficiently small the solution $u_\lambda(t)$ is "close" to $u(t)$ in some generalized sense. We shall show that this is exactly the situation. This result is normally assumed in the formulation of these boundary value problems from a physical model, since one often takes $u(t)$ as an approximation for $u_\lambda(t)$ by assuming that the viscosity coefficient λ is zero.

The generalized solution of the parabolic equation (8.2) can be constructed by means of the semigroup theory of Hille and Yosida. This method is used in [25]. The extended operator $-L_0$ is such that its resolvent set contains all of the positive real axis and furthermore

$$\|(\lambda L_0 + I)^{-1}\|_0 \leqq (\lambda l_0 + 1)^{-1}$$

for all positive numbers λ and a constant l_0 depending only on L_0 and the domain G. These are precisely the conditions for which the Hille–Yosida theorem can be used to construct a strongly continuous semigroup of bounded linear operators $\{S(t): t \geqq 0\}$ with the property that if u_0 belongs to $D(L_0)$ then the function

$$u(t) = S(t)u_0 \tag{8.3}$$

is strongly continuous in $L^2(G)$, belongs to $D(L_0)$ and satisfies $u(0) = u_0$, $u'(t) = -L_0u(t)$ for $t \geqq 0$.

The semigroup $\{S(t): t \geqq 0\}$ is constructed as follows. Define for each number $\lambda > 0$ an operator

$$L_\lambda = (I + \lambda L_0)^{-1}L_0$$

and show that it is a bounded operator on $L^2(G)$. Also for any v in $D(L_0)$ we have

$$\lim_{\lambda \to 0} \|L_\lambda v - L_0 v\|_0 = 0.$$

Since L_λ is bounded we can define for each number t the bounded operator

$$E_\lambda(t) = \exp(-tL_\lambda).$$

It can then be shown that, for those $t \geqq 0$, $E_\lambda(t)$ converges to an operator $S(t)$ in the strong sense as λ converges to zero, and that $\{S(t): t \geqq 0\}$ is the desired semigroup.

The relation between the solution of the parabolic problem given by (8.3) and the solution to the equation (8.1) is now clear. The operator L_λ above can be expressed as $L_\lambda = M_0^{-1}L_0$ for the special case $M_0 = \lambda L_0 + I$ which we are considering, hence $E_\lambda(t)$ is for each $\lambda > 0$ the group of bounded operators constructed in § 4 for the equation (8.1). The solution to (8.1) is then given by

$$u_\lambda(t) = E_\lambda(t)u_0.$$

In order for the parabolic problem to be meaningful we require that u_0 belong to $D(L_0)$. The statement above that $E_\lambda(t)$ converges in the strong sense to $S(t)$ is exactly the result we seek. That is, for $t \geqq 0$ and u_0 in $D(L_0)$ we have

$$\lim_{\lambda \to 0} \|u_\lambda(t) - u(t)\|_0 = 0, \tag{8.4}$$

and this is the precise form of the statement that $u(t)$ is "close" to $u_\lambda(t)$ when λ is small.

This result can be generalized to the equation

$$(\lambda M_0 + I)u_\lambda'(t) + L_0 u_\lambda(t) = 0, \tag{8.5}$$

for which we have the following.

THEOREM. *Assume that the generalized problem* (8.5) *is* 3-*smooth and* u_0 *belongs to* $H_0^1(G) \cap H^2(G)$. *Then for all* $t \geqq 0$ *the solution* $u(x, t)$ *of the parabolic equation* (8.2) *given by* (8.3) *is the* $\|\cdot\|_0$-*limit of the solutions* $u_\lambda(x, t)$ *of the pseudoparabolic equation* (8.5). (See [37].)

The proof of this result is modeled after the proof of the Hille–Yosida Theorem [39], but the details are considerably more involved since there are two different operators to consider.

9. The Schauder estimates. We shall begin an independent but parallel study of the problem considered previously, and this investigation is based on the solution of the Dirichlet problem by the method of Schauder. In this context the operators M and L are studied on the Banach space of functions with uniformly Hölder continuous second order derivatives, and we shall see that the product operator $M^{-1}L$ is bounded on this space. This will enable the construction of the solution by exponentiating this bounded operator. In proving the boundedness of $M^{-1}L$, we shall make use of the Schauder estimates (up to the boundary) and the closed graph theorem, so the completeness of the function spaces used is essential.

The existence, uniqueness and regularity results are essentially the same as those obtained previously. That is, the solution is obtained directly as the exponential of a bounded operator, and this operator leaves certain subspaces invariant. There will be no need of an analogue of Sobolev's lemma since convergence in the function space will imply pointwise convergence, hence this method always yields a pointwise solution.

A function $v(x)$ is said to belong to the class $C^{m+\alpha}(\mathrm{cl}(\mathbf{G}))$, where m is a nonnegative number and $0 < \alpha < 1$, if v belongs to $C^m(\mathrm{cl}(\mathbf{G}))$ and all of its mth order derivatives are uniformly Hölder continuous of exponent α. By this last statement we mean

$$H_\alpha^m(v) = \sup\left\{\frac{D^j v(x) - D^j v(y)}{|x - y|^\alpha} : x, y \in G, |j| = m\right\}$$

is finite. We define on $C^{m+\alpha}(\mathrm{cl}(\mathbf{G}))$ a norm

$$|v|_{m+\alpha} = |v|_m + H_\alpha^m(v),$$

where

$$|v|_m = \sum_{i=0}^{m} \sup\{|D^j v(x)| : x \in G, |j| = i\}.$$

Furthermore one can show that $C^{m+\alpha}(\mathrm{cl}(\mathbf{G}))$ is complete with respect to the norm $|\cdot|_{m+\alpha}$, so it is a Banach space.

The boundary ∂G is in the class $C^{m+\alpha}$ whenever there is at each point of ∂G a neighborhood S in which ∂G has a parametric representation of the form

$$x_i = g(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n),$$

where g belongs to $C^{m+\alpha}(\mathrm{cl}(\mathbf{S}))$.

The operators M and L will be assumed to have the forms

$$M = \sum_{i,j=1}^{n} m_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} m_i(x)\frac{\partial}{\partial x_i} - m(x),$$

$$L = \sum_{i,j=1}^{n} l_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} l_i(x)\frac{\partial}{\partial x_i} - l(x).$$

The following assumptions will always be made:

(A_1): Each of the coefficients which appears above belongs to $C^\alpha(\mathrm{cl}(\mathbf{G}))$ and the coefficients $m(x)$, $l(x)$ are nonnegative.

(A_2): M and L are uniformly elliptic, hence there are positive constants m_0 and l_0 for which

$$\sum_{i,j=1}^{n} m_{ij}(x)\xi_i\xi_j \geqq m_0 \sum_{i=1}^{n} (\xi_i)^2,$$

$$\sum_{i,j=1}^{n} l_{ij}(x)\xi_i\xi_j \geqq l_0 \sum_{i=1}^{n} (\xi_i)^2$$

whenever ξ belongs to R^n and x belongs to G.

The technique which we shall use here is totally dependent on the existing results on the solution of the Dirichlet problem. That is, given a function f in $C^\alpha(\mathrm{cl}(\mathbf{G}))$, find a function u for which

$$Lu = f$$

in G and $u(x) = 0$ when x is on ∂G. In proving the existence of a solution of such a problem by the method of continuity, the following a priori estimate is essential [2], [12], [29].

THEOREM 9.1. *Assume $(A_1), (A_2)$, that f belongs to $C^\alpha(\mathrm{cl}(\mathbf{G}))$ and that ∂G is of class $C^{2+\alpha}$. If u is a function in $C^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ for which $Lu = f$ in G and $u = 0$ on ∂G, then*

$$|u|_{2+\alpha} \leqq K_L|f|_\alpha, \tag{9.1}$$

where K_L depends only on L and G.

This is a very strong result and is used to prove the following existence theorem for the Dirichlet problem.

THEOREM 9.2. *Assume $(A_1), (A_2)$, that f belongs to $C^\alpha(\mathrm{cl}(\mathbf{G}))$ and that ∂G is of class $C^{2+\alpha}$. Then there exists a unique function u in $C^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ for which $L(u) = f$ in G and $u = 0$ on ∂G.*

Concerning the differentiability of solutions of the Dirichlet problem we have the following result.

THEOREM 9.3. *Let p be a nonnegative integer for which f and all the coefficients which appear in L belong to $C^{p+\alpha}(\mathrm{cl}(\mathbf{G}))$ and for which ∂G is of class $C^{p+2+\alpha}$. Then any function u in $C^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ for which $Lu = f$ in G and $u = 0$ on ∂G belongs to $C^{p+2+\alpha}(\mathrm{cl}(\mathbf{G}))$.*

Corresponding results are of course valid for the operator M.

We are now ready to study the behavior of L and M on the appropriate function space. Define $C_0^{m+\alpha}(\mathrm{cl}(\mathbf{G}))$ as the set of functions in $C^{m+\alpha}(\mathrm{cl}(\mathbf{G}))$ that vanish on

∂G. With the norm $|\cdot|_{m+\alpha}$, $C_0^{m+\alpha}(\mathrm{cl}(\mathbf{G}))$ is a Banach subspace of $C^{m+\alpha}(\mathrm{cl}(\mathbf{G}))$, because convergence with respect to $|\cdot|_{m+\alpha}$ implies uniform convergence of the function and hence preserves the zero condition on the boundary. From the results stated above for the Dirichlet problem it is immediate that L maps $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ onto $C^\alpha(\mathrm{cl}(\mathbf{G}))$ in a one-to-one manner. From (9.1) it follows that L^{-1} is bounded, so from the closed graph theorem it is immediate that L is a linear homeomorphism of $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ onto $C^\alpha(\mathrm{cl}(\mathbf{G}))$. The same is true of M, so we may conclude that $M^{-1}L$ is a bounded linear operator on $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$.

For each real number t we construct the exponential of the bounded operator $-tM^{-1}L$ by means of the power series

$$E(t) = \exp(-tM^{-1}L) = \sum_{k=0}^{\infty} (-tM^{-1}L)^k/k!$$

This power series converges with respect to the uniform operator topology induced on $\mathscr{L}(C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G})))$ by the norm $|\cdot|_{2+\alpha}$ on $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$. It is not difficult to verify that the family $\{E(t): t \text{ in } R\}$ is an infinitely differentiable group of bounded linear operators and that

$$E'(t) = -M^{-1}\cdot L\cdot E(t) \tag{9.2}$$

for all t in R. This group of linear operators provides the existence portion of the following result.

THEOREM 9.4. *Assume that* (A_1) *and* (A_2) *are true,* ∂G *is of class* $C^{2+\alpha}$ *and that* u_0 *is a given function in* $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$. *There is a unique strongly differentiable mapping*

$$t \mapsto u(t)$$

of R *into* $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ *for which*

$$Mu'(t) + Lu(t) = 0 \tag{9.3}$$

in $C^\alpha(\mathrm{cl}(\mathbf{G}))$ *for all real* t *and* $u(0) = u_0$. *This mapping is infinitely differentiable.*

Proof. Define $u(t) = E(t)u_0$. It is immediate that $u(0) = u_0$ and that $u(t)$ is infinitely differentiable. Furthermore since M and L are both bijections of $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ onto $C^\alpha(\mathrm{cl}(\mathbf{G}))$ it follows from (9.2) that (9.3) is true.

We shall verify the uniqueness of the solution. The solution must necessarily satisfy the integral equation

$$u(t) = u(0) - M^{-1}L\int_0^t u(T)\,dT$$

because of the boundedness and linearity of $M^{-1}L$ on $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$. The integral is taken as usual as the limit in the $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ topology of Riemann sums. From this equation we have

$$|u(t)|_{2+\alpha} \leqq |u(0)|_{2+\alpha} + |M^{-1}L|_{2+\alpha}\int_0^t |u(T)|_{2+\alpha}\,dT \tag{9.4}$$

for all t in R.

LEMMA 9.1 (Gronewall). *If φ is continuous and nonnegative on $R^+ = \{r \in R : r \geqq 0\}$ and if*

$$\varphi(\xi) \leqq c + m \int_0^\xi \varphi(T)\, dT$$

for all $\xi \geqq 0$ then

$$\varphi(\xi) \leqq c \exp(m\xi).$$

Proof. From the hypotheses we have

$$\frac{1}{m} \frac{d}{dt} \left\{ \ln\left(c + m \int_0^t \varphi(T)\, dT \right) \right\} \leqq 1$$

so

$$\ln \left[\left(c + m \int_0^t \varphi(T)\, dT \right) \Big/ c \right] \leqq mt.$$

Hence

$$c + m \int_0^t \varphi(T)\, dT \leqq c \exp(mt)$$

and the result is immediate from this inequality.

This lemma together with (9.4) shows that *any* solution of the problem satisfies

$$|u(t)|_{2+\alpha} \leqq |u(0)|_{2+\alpha} \exp(|M^{-1}L|\, |t|). \tag{9.5}$$

In particular the difference between any two solutions satisfies (9.5) with $u(0) = 0$, hence the solutions are identical.

The solution thus obtained can easily be seen to be a solution in the pointwise sense. For each real number t, $u(t)$ belongs to $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ and is therefore a real-valued function of the space variable whose value at the point x of G is denoted by $u(x, t)$. Furthermore for any real $\delta \neq 0$ we have

$$\begin{aligned} |\delta^{-1}(u(x, t + \delta) - u(x, t)) &+ M^{-1}L[u(x, t)]| \\ &= |(\delta^{-1}(E(\delta) - I) + M^{-1}L)[u(x, t)]| \\ &\leqq |(\delta^{-1}(E(\delta) - I) + M^{-1}L)u(t)|_{2+\alpha} \\ &\leqq |\delta^{-1}(E(\delta) - I) + M^{-1}L|_{2+\alpha} |u(t)|_{2+\alpha} \end{aligned}$$

so the mapping $t \mapsto u(x, t)$, x in G, is differentiable, in fact infinitely differentiable, since the group $\{E(t) : t \text{ in } R\}$ is infinitely differentiable. Consequently Theorem 9.4 implies that the equation (9.3) possesses a pointwise solution $u(x, t)$ which belongs to $C^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ for each t in R, vanishes on the boundary ∂G and is infinitely differentiable with respect to the time variable t.

The results on the regularity of the solution are completely analogous to those obtained previously, and the same methods may be used as before. In particular we use the results stated above on the regularity of the solution to the Dirichlet problem to prove the following.

PROPOSITION 9.1. *Let p be a nonnegative integer and assume ∂G is of class $C^{p+2+\alpha}$. Let the operator L satisfy (A_1) and (A_2) and assume that its coefficients belong to $C^{p+\alpha}(\mathrm{cl}(\mathbf{G}))$. Then L is a linear homeomorphism of $C_0^{p+2+\alpha}(\mathrm{cl}(\mathbf{G}))$ onto $C^{p+\alpha}(\mathrm{cl}(\mathbf{G}))$.*

Proof. The results above for the Dirichlet problem show that L is a bijection as stated, so the boundedness of L and L^{-1} is the only question. But this is settled by the Fundamental Lemma of § 5.

COROLLARY. *Let p be a nonnegative integer such that ∂G is of class $C^{p+\alpha+2}$ and the operators M and L satisfy (A_1) and (A_2), and their coefficients belong to $C^{p+\alpha}(\mathrm{cl}(\mathbf{G}))$. Then $M^{-1}L$ is a linear homeomorphism of $C_0^{p+\alpha+2}(\mathrm{cl}(\mathbf{G}))$ onto itself.*

From the boundedness of $M^{-1}L$ with respect to the norm $|\cdot|_{p+\alpha+2}$ on $C_0^{p+\alpha+2}(\mathrm{cl}(\mathbf{G}))$ it follows as before that the group of operators $\{E(t):t \text{ in } R\}$ is bounded on and leaves invariant the space $C_0^{p+\alpha+2}(\mathrm{cl}(\mathbf{G}))$. This yields the following result on the regularity of solutions.

THEOREM 9.5. *Under the assumptions of the corollary above, the solution $u(t)$ of the problem* (9.3), (9.4) *belongs to $C_0^{p+\alpha+2}(\mathrm{cl}(\mathbf{G}))$ for each t in R if and only if u_0 belongs to $C_0^{p+\alpha+2}(\mathrm{cl}(\mathbf{G}))$.*

The nonhomogeneous problem can be handled in much the same way as was done previously. The main result in this direction is the following.

THEOREM 9.6. *Assume that (A_1) and (A_2) are true and the ∂G is of class $C^{2+\alpha}$. Let $f(t)$ be a (strongly) continuous function of R into $C^{\alpha}(\mathrm{cl}(\mathbf{G}))$ and $\beta(t)$ a continuously differentiable function of R into $C^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$. Let u_0 belong to $C^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ and satisfy the "compatibility condition" $u_0 = \beta(0)$ on ∂G. (That is, $u_0 - \beta(0)$ is in $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$.) Then there exists a unique continuously differentiable function $u(t)$ of R into $C^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ such that*

(i) $Mu'(t) + Lu(t) = f(t)$,
(ii) $u(0) = u_0$, *and*
(iii) $u(t) = \beta(t)$ *on the boundary* ∂G.

Proof. Define $F(t)$ from R into $C^{\alpha}(\mathrm{cl}(\mathbf{G}))$ by $F(t) = -M\beta'(t) - L\beta(t) + f(t)$. Since M and L are bounded (but not invertible) from $C^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ into $C^{\alpha}(\mathrm{cl}(\mathbf{G}))$, we see that $F(t)$ is continuous. Since M^{-1} is bounded from $C^{\alpha}(\mathrm{cl}(\mathbf{G}))$ onto $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$, we have that $M^{-1}F(t)$ is continuous in $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$, so we can define

$$v(t) = \int_0^t E(t - T)M^{-1}F(T)\,dT$$

in $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$. It follows that the continuously differentiable mapping $t \mapsto v(t)$ satisfies the equation

$$Mv'(t) + Lv(t) = F(t)$$

and initial condition $v(0) = 0$.

Remark. Since M is not invertible (not injective on $C^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$), we do *not* have $M^{-1}M = $ identity. This is of consequence if one wishes to expand $M^{-1}F(t)$ into its three terms.

Now define the continuously differentiable function

$$u(t) = v(t) + \beta(t) + E(t)[u_0 - \beta(0)].$$

This satisfies (i)–(iii) above. The uniqueness follows from Theorem 9.4 by looking at the difference between two such solutions.

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