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A Nonlinear Parabolic–Sobolev Equation

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Let M and L be (nonlinear) operators in a reflexive Banach space B for which $Rg(M + L) = B$ and $|(Mx - My) + \alpha(Lx - Ly)| \geqslant |Mx - My|$ for all $\alpha > 0$ and pairs x, y in $D(M) \cap D(L)$. Then there is a unique solution of the Cauchy problem $(Mu(t))' + Lu(t) = 0$, $Mu(0) = v_0$. When M and L are realizations of elliptic partial differential operators in space variables, this gives existence and uniqueness of generalized solutions of boundary value problems for nonlinear partial differential equations of mixed parabolic–Sobolev type.

1. INTRODUCTION

We shall show that certain initial and boundary value problems on a cylinder are well-posed for partial differential equations of the form

$$D_t(u(x, t) + b(x) L[u]) + L[u] = 0,$$

where $L[u]$ is a nonlinear elliptic operator in divergence form. The coefficient $b(\cdot)$ is assumed bounded, measurable, and nonnegative, so the equation will be parabolic where $b(x) = 0$ and of Sobolev type where $b(x) > 0$. Such equations arise in various applications where the coefficient $b(\cdot)$ denotes a quantity with the dimensions of viscosity. References to these applications and related results were given in [9].

In Section 2 we use the theory of generation of semigroups of nonlinear contractions by (the negatives of) hyper-accretive relations in a Banach space to obtain sufficient conditions for an abstract model of the above equation to be well-posed. A reduction of our equation to the abstract model is obtained in Section 3 as a direct application of the treatment of nonlinear elliptic equations by monotone operator methods in Banach space. Since the applications lead to equations of the above type for which the exposition is particularly easy, we shall restrict our attention to this special class. Gener-

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alizations to other types of equations are immediate consequences of our model in Section 2 so we omit them here. Some of these were given for the linear case in [10].

2. Two Cauchy Problems

Let B be a real reflexive Banach space and suppose we have a pair of (not necessarily linear) operators M: $D(M) \to B$ and L: $D(L) \to B$ with domains in B. We define a subset A of $B \times B$ by $A = \{(x, y)$: for some $z \in D(M) \cap D(L)$, $M(z) = x$ and $L(z) = y\}$. Thus A is a "multivalued operator" or relation on B, given by the composite relation $L \circ M^{-1}$, and its domain is $D(A) = M[D(M) \cap D(L)]$. Functions are identified with their graphs and, hence, viewed as relations; the standard notation for relations is given in [4].

Our interest in the pair (M, L) and the relation $A = L \circ M^{-1}$ arises from the correspondence between the following Cauchy problems. (Note that since B is reflexive every Lipschitz function from the reals to B is strongly-differentiable a.e. [7].)

Definition. Let $u_0 \in D(L) \cap D(M)$. A solution of the Cauchy problem

$$(d/dt)\,(Mu(t)) + Lu(t) = 0, \tag{2.1.a}$$

$$Mu(0) = Mu_0\,, \tag{2.1.b}$$

is a function $u(\cdot)$: $[0, \infty) \to D(L) \cap D(M)$ for which $Mu(\cdot)$ is Lipschitz, (2.1.a) holds a.e. on $[0, \infty)$, and (2.1.b) is satisfied.

Definition. Let $v_0 \in D(A)$. A solution of the Cauchy problem

$$(d/dt)\,(v(t)) + A(v(t)) \ni 0, \tag{2.2.a}$$

$$v(0) = v_0\,, \tag{2.2.b}$$

is a function $v(\cdot) : [0, \infty) \to D(A)$ which is Lipschitz, satisfies (2.2.a) a.e. on $[0, \infty)$, and takes the initial condition (2.2.b).

The correspondence between (2.1) and (2.2) is immediate. If $u(\cdot)$ is a solution of (2.1), we define $v(t) = Mu(t)$ for $t \geqslant 0$ to obtain a solution of (2.2) with $v_0 = Mu_0$. Conversely, if $v(\cdot)$ is a solution of (2.2), we can choose for each $t \geqslant 0$ a $u(t) \in D(M) \cap D(L)$ with $Mu(t) = v(t)$ to obtain a solution of (2.1), where u_0 is any point in $D(M) \cap D(L)$ with $Mu_0 = v_0$. This observation gives the following.

PROPOSITION 1. *Let M and L be operators on a real reflexive Banach space B and define the relation $A = L \circ M^{-1}$ on B. Then, there is a natural correspondence, $v(t) \leftrightarrow Mu(t)$, between solutions of the Cauchy problems* (2.1) *and* (2.2).

COROLLARY (Uniqueness). *Let $v_0 = Mu_0$. If there is at most one solution of* (2.1), *then there is at most one solution of* (2.2). *If there is at most one solution of* (2.2) *and if $u_1(\cdot)$ and $u_2(\cdot)$ are solutions of* (2.1), *then $Mu_1(t) = Mu_2(t)$ for all $t \geqslant 0$.*

COROLLARY (Existence). *Let $v_0 = Mu_0$. There exists a solution of* (2.1) *if and only if there exists a solution of* (2.2).

A sufficient condition for uniqueness in (2.2) is that the operator A be accretive.

DEFINITION. The relation A on the space B is accretive if for every $\alpha > 0$ and $(x_1, y_1), (x_2, y_2) \in A$

$$\|(x_1 + \alpha y_1) - (x_2 + \alpha y_2)\| \geqslant \| x_1 - x_2 \|.$$

If A is accretive and if $v_1(\cdot)$ and $v_2(\cdot)$ are solutions of (2.2), then

$$\| v_1(t) - v_2(t)\| \leqslant \| v_1(0) - v_2(0)\|, \qquad t \geqslant 0,$$

so the solution depends continuously on the initial data. A sufficient condition for (2.2) to have a solution for every $v_0 \in D(A)$ is that A be hyper-accretive.

DEFINITION. The relation A on the space B is hyper-accretive if it is accretive and the range $Rg(I + A)$ is all of B.

The above results are given in [4]. Also, see [1, 6].

The properties on M and L that reflect hyper-accretiveness of A are immediate.

PROPOSITION 2. *Assume the hypotheses of Proposition* 1. *Then $Rg(I + A) = Rg(M + L)$, and A is accretive if and only if*

$$\|(Mz_1 - Mz_2) + \alpha(Lz_1 - Lz_2)\| \geqslant \| Mz_1 - Mz_2 \|, \qquad \alpha > 0,\ z_1, z_2 \in D(M) \cap D(L). \tag{2.3}$$

The preceding discussion gives the following sufficient conditions for the problem (2.1) to be well-posed.

THEOREM 1. *Let M: $D(M) \to B$ and L: $D(L) \to B$ be operators in the real* *reflexive Banach space* B; *assume* $Rg(M+L) = B$ *and* (2.3). *Then for every* $u_0 \in D(M) \cap D(L)$ *there is a solution of* (2.1). *If* $u_1(\cdot)$ *and* $u_2(\cdot)$ *are solutions of* (2.1.a), *then*

$$\| Mu_1(t) - Mu_2(t) \| \leqslant \| Mu_1(0) - Mu_2(0) \|, \qquad t \geqslant 0.$$

Hence, if $u_1(\cdot)$ *and* $u_2(\cdot)$ *are solutions of the Cauchy problem* (2.1), *then* $Mu_1(t) = Mu_2(t)$, $t \geqslant 0$.

Remarks.

(1) Accretiveness of A can be expressed in terms of the duality map, J, of B into the dual, B^*: $(x, f) \in J$ if $f(x) = \| x \|^2 = \| f \|^2$. The condition (2.3) is equivalent to requiring that for every $z_1, z_2 \in D(L) \cap D(M)$ we have $f(Lz_1 - Lz_2) \geqslant 0$ for some $f \in J(Mz_1 - Mz_2)$ [5]. In a Hilbert space H with inner product $(\cdot, \cdot)_H$ this becomes

$$(Mz_1 - Mz_2, Lz_1 - Lz_2)_H \geqslant 0, \qquad z_1, z_2 \in D(M) \cap D(L). \tag{2.4}$$

For linear operators this is the right-angle condition [8, 10].

(2) We can allow L to be multivalued with minor modifications in the above discussion.

(3) If M is injective on $D(M) \cap D(L)$, there is at most one solution of (2.1) in the situation of Theorem 1.

3. Degenerate Parabolic-Sobolev Equations

Let G be a bounded open subset of Euclidean space $\mathbb{R}^n$ whose points are given by $x = (x_1, \ldots, x_n)$. The space of (equivalence classes of) functions on G which are measurable (with Lebesgue measure dx on G) and have summable pth powers is denoted by $L^p(G)$, $1 < p < \infty$. Let D_j be the partial derivative $\partial / \partial x_j$, $1 \leqslant j \leqslant n$, and D_0 be the identity. $W^{1,p}$ is the Sobolev space of those $\phi \in L^p(G)$ with the (distribution) derivatives $D_j \phi \in L^p(G)$, $1 \leqslant j \leqslant n$, and with the norm

$$\| \phi \|_{1,p} = \left(\sum_{j=0}^{n} \| D_j \phi \|_{L^p}^p \right)^{1/p}.$$

Then $W^{1,p}$ is a reflexive and separable Banach space containing the family $C_0^\infty(G)$ of infinitely differentiable functions with compact support in G. For $\phi \in W^{1,p}$, we shall denote by $D\phi = \{D_j \phi : 0 \leqslant j \leqslant n\}$ the indicated point in the product $L^p(G)^{n+1}$. Thus, with appropriate hypotheses on a function $F: G \times \mathbb{R}^{n+1} \to \mathbb{R}$, we can define a first-order nonlinear partial differential operator on G by $\phi \mapsto F(\cdot, D\phi(\cdot))$. See [3] for information and references on Sobolev spaces.

We shall define a nonlinear elliptic problem from a given family of functions $A_i : G \times \mathbb{R}^{n+1} \to \mathbb{R}$ for which we assume the following:

Each $A_i(x, \xi)$ is measurable in x for fixed ξ and continuous in ξ for a.e. x. There is a real p, $1 < p < \infty$, a $g \in L^q(G)$ where $q = p/(p-1)$, and $c > 0$ such that

$$| A_i(x, \xi)| \leqslant c \sum_{j=0}^{n} | \xi_j |^{p-1} + g(x), \qquad x \in G, \quad \xi \in \mathbb{R}^{n+1}, \quad 0 \leqslant i \leqslant n. \tag{3.1}$$

For x in G and each pair $\xi, \eta \in \mathbb{R}^{n+1}$,

$$\sum_{i=0}^{n} (A_i(x, \xi) - A_i(x, \eta)) (\xi_i - \eta_i) \geqslant 0. \tag{3.2}$$

There is a $c_0 > 0$ such that

$$\sum_{i=0}^{n} A_i(x, \xi) \xi_i \geqslant c_0 \sum_{i=0}^{n} | \xi_i |^p, \qquad x \in G, \quad \xi \in \mathbb{R}^{n+1}. \tag{3.3}$$

From (3.1) it follows [2; 3, pp. 73–75] that for $\phi \in W^{1,p}$, each $A_i(\cdot, D\phi(\cdot)) \in L^q(G)$, and we can define a Dirichlet form

$$a(\phi, \psi) = \sum_{i=0}^{n} \int_G A_i(x, D\phi(x)) D_i\psi(x) \, dx, \qquad \phi, \psi \in W^{1,p}. \tag{3.4}$$

Let V be a closed subspace of $W^{1,p}$ which contains $C_0^\infty(G)$; denote the dual by V^* and the duality by $\langle \cdot, \cdot \rangle$. Then a (nonlinear) operator T: $V \to V^*$ is determined by

$$\langle T\phi, \psi \rangle = a(\phi, \psi), \qquad \phi, \psi \in V. \tag{3.5}$$

For each $\phi \in V$, the restriction of $T\phi$ to $C_0^\infty(G)$ is the distribution on G given by

$$T\phi = -\sum_{i=1}^{n} D_i A_i(\cdot, D\phi(\cdot)) + A_0(\cdot, D\phi(\cdot)).$$

Assume $p \geqslant 2$. Then by identifying $L^2(G)$ with its dual $L^2(G)^*$ we have $V \hookrightarrow L^2(G) = L^2(G)^* \hookrightarrow V^*$, where the indicated injections are continuous and have dense ranges, and $\langle \phi, \psi \rangle = (\phi, \psi)_{L^2(G)}$ for $\phi \in L^2(G)$, $\psi \in V$. Let

$b(\cdot) \in L^\infty(G)$ be given with $b(x) \geqslant 0$ a.e., and define a pair of operators on the Hilbert space $H = L^2(G)$ by

$$D(L) = \{\phi \in V : T\phi \in H\}, \qquad L\phi = T\phi,$$
$$D(M) = \{\phi \in V : b(\cdot)\, T\phi \in H\}, \quad M\phi = \phi + b(\cdot)\, T\phi.$$

Since $b(\cdot)$ is bounded we have $D(L) \subset D(M)$. For a pair $\phi, \psi \in D(L)$ we have

$$(L\phi - L\psi, M\phi - M\psi)_H = \langle T\phi - T\psi, \phi - \psi\rangle + \int_G b(x)\,(T\phi - T\psi)^2\,dx.$$

From (3.2) it follows that the first term is non-negative; the second term is also nonnegative, so (2.4) is satisfied.

We wish to show $Rg(M + L) = H$. The linear and continuous map $\phi \to (1 + b(\cdot))^{-1}\phi : V \to V^*$ has range in H and satisfies

$$\langle (1 + b(\cdot))^{-1}\phi, \phi\rangle = ((1 + b(\cdot))^{-1}\phi, \phi)_H \geqslant 0, \qquad \phi \in V.$$

Letting $T_1\phi = (1 + b(\cdot))^{-1}\phi + T\phi$, we have an operator $T_1 : V \to V^*$ which is monotone, by (3.2),

$$\langle T_1\phi - T_1\psi, \phi - \psi\rangle \geqslant 0, \qquad \phi, \psi \in V,$$

demicontinuous [3, pp. 76–77], and coercive by (3.3),

$$\langle T_1\phi, \phi\rangle \geqslant \langle T\phi, \phi\rangle \geqslant h(\|\phi\|_V)\,\|\phi\|_V,$$

where $h(r) = r^{p-1}$ satisfies $\lim_{r\to\infty} h(r) = \infty$. Hence T_1 maps V onto V^* [2, 3]. Let $w \in L^2(G)$. Then $(1 + b(\cdot))^{-1}\, w \in L^2(G) \subset V^*$ so there is a $\phi \in V$ with $T_1\phi = (1 + b(\cdot))^{-1}w$. Thus $T\phi = (1 + b(\cdot))^{-1}\,(w - \phi) \in L^2(G)$ and we have $\phi \in D(L) \cap D(M)$ with $M\phi + L\phi = w$.

THEOREM 2. *Let V be a closed subspace of $W^{1,p}$, $2 \leqslant p < \infty$, and $b(\cdot) \in L^\infty(G)$ with $b(x) \geqslant 0$,* a.e. *$x \in G$. Assume* (3.1), (3.2), *and* (3.3); *let T: $V \to V^*$ be given by* (3.4) *and* (3.5). *Finally, assume $u_0 \in V$ is given with $Tu_0 \in L^2(G)$. Then there is a unique function u: $[0, \infty) \to V$ for which $Tu(t) \in L^2(G)$ for every $t \geqslant 0$, u and $b(\cdot)\, Tu : [0, \infty) \to L^2(G)$ are Lipschitz, hence strongly differentiable at a.e. $t \in [0, \infty)$,*

$$(d/dt)(u(t) + b(x)\, Tu(t)) + Tu(t) = 0, \tag{3.6}$$

in $L^2(G)$ for a.e. $t \in [0, \infty)$, and $u(0) = u_0$.

Proof. From the preceding discussion and Theorem 1 we obtain all the statements except the Lipschitz continuity and the initial value. These follow from the fact that T is monotone, and hence

$$\|\phi - \psi\|_H^2 \leqslant (M\phi - M\psi, \phi - \psi)_H \qquad \phi, \psi \in D(M),$$

so the Cauchy–Schwartz inequality gives

$$\|\phi - \psi\|_H \leqslant \|M\phi - M\psi\|_H, \qquad \phi, \psi \in D(M).$$

This last inequality shows that M is injective, giving the initial condition, and that $u(\cdot)$, and hence $bTu(\cdot)$, are Lipschitz.

The equation (3.6) represents a weak form of a partial differential equation on the cylinder $G \times (0, \infty)$. The solution u: $[0, \infty) \to V$ leads in the usual way [2, 3, 8, 9, 11] to an (equivalence class of) functions v: $G \times (0, \infty) \to \mathbb{R}$ related by $v(\cdot, t) = u(t)$ in $L^2(G)$, and this function v is a weak solution of the equation

$$D_t \left\{ v(x, t) - b(x) \sum_{i=1}^{n} D_i A_i(x, Dv(x, t)) + b(x) A_0(x, Dv(x, t)) \right\}$$

$$- \sum_{i=1}^{n} D_i A_i(x, Dv(x, t)) + A_0(x, Dv(x, t)) = 0.$$

Let Γ be a measurable subset of the boundary, ∂G, and let V be the closure in $W^{1,p}$ of those infinitely differentiable functions on G with support disjoint from Γ. Then $u(t) \in V$, $t \geqslant 0$, gives a boundary condition

$$v(x, t) = 0, \qquad x \in \Gamma, \qquad t \geqslant 0.$$

The condition that $u(t) \in D(L)$, $t \geqslant 0$, gives us from (3.5) the identity

$$(Tu(t), \psi)_H = a(u(t), \psi), \qquad \psi \in V, \qquad t \geqslant 0,$$

and this gives (formally) a "natural" or "variational" boundary condition from the divergence theorem,

$$\sum_{i=1}^{n} A_i(x, Dv(x, t))\, \nu_i(x) = 0, \qquad x \in \partial G \sim \Gamma, \quad t \geqslant 0,$$

where $\nu(x) = (\nu_1(x),..., \nu_n(x))$ is the unit outward normal on ∂G.

Remark. Suppose the viscosity coefficient is constant, say $b(x) = b > 0$ for $x \in G$, and let L be a hyper-accretive operator. Then $A = L(I + bL)^{-1}$ is the Yosida approximation for L and, letting $v_b(t)$ denote the solution of

(2.2) (with M = identity) and $u(t)$ the solution of (2.1), we have [1, Theorem 2.2]

$$u(t) = \lim_{b \to 0} v_b(t), \qquad \text{for } t \geqslant 0$$

and the limit is uniform on bounded intervals. In our application above, this shows that a particular Sobolev equation can be used to approximate parabolic equations [8]. This approximation has been a means of constructing the solution of (2.2) [6, 7, 11].

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