THIN-FILM CAPACITANCE MODELS

R.E. SHOWALTER and HEE CHUL PAK
Department of Mathematics
University of Texas at Austin
Austin, TX 78712 USA

AMS: 35A05, 35B27, 35K50, 35K90, 35M20

Abstract Models for distributed capacitance in a thin film are derived in the form of a system of local RC diffusion equations coupled by a global elliptic equation. Such models contain the local geometry of the distributed capacitance on which charge is stored and the exchange of current flux on its interface with the medium. Certain singular limits are characterized, and the resulting degenerate initial-boundary-value problems are shown to be well posed.

KEY WORDS: Distributed RC circuits, microstructure, parabolic, pseudoparabolic.

1. Introduction

The theory of distributed electrical networks is the natural framework in which to study the behavior of a system in which wavelengths of interest may become comparable to the physical dimensions of the system. Such systems arise in the modeling of integrated circuits operated at high frequencies. Additional applications of such systems arise in acoustics, microwave, and optical device theory. The micro-miniaturization of electronics has directed attention toward distributed RC structures, because significant inductance for circuit applications at frequencies below a few MHZ is not readily available in very small volumes. General classes of RC structures possess extremely valuable properties. Important applications include sharper cutoff filters and larger phase shift with less attenuation than is obtainable with conventional lumped RC circuits. Micro-circuits, molecular electronics, and thin-film circuitry are all important fields in which it has become increasingly difficult to construct accurate models with purely lumped circuit components. Due to the emphasis on small size, the very proximity of a conductive region to a ground plane or to another conductive region introduces stray shunt capacitance that cannot be ignored. Such situations occur in the construction of thin-layer interconnecting conductors over a substrate and in the use of crossover connections in non-planar circuits.

Here we shall model the general effects of a single capacitive layer (possibly imperfectly) bonded to a highly conductive region within a device whose total vertical height is very thin compared to its horizontal dimensions. This is a refinement of the classical RC ladder network which is commonly modeled by a single diffusion equation. Thus, we consider a simple generic circuit composed of a highly conductive and very thin region shunted to ground by way of a dielectric layer which may be either uniformly or periodically distributed. These two cases are considered and contrasted. The models described here also arise as the upper half of a vertically symmetric very thin highly conductive region which contains a horizontal middle layer of dielectric. Such situations arise in the construction of capacitors in thin-film structures and also from imperfections in the construction of interconnections, especially when the use of crossovers is required. It is assumed that the height of the total device is very small and that the conductivity of this region is large *by a comparable factor*. A small parameter is introduced in order to to quantify simultaneously the thinness of the device and the resistivity of the conducting region. The horizontal dielectric layer is described by a plane in which the height determines an appropriate scaling of the corresponding resistive and capacitive parameters. The singular character of the problem is determined by this small parameter, and we obtain an approximation of the device by finding the limiting form of the singular initial-boundary-value problem as this parameter converges to zero. In particular, the singularity introduced by the thinness in the geometry will be precisely compensated by the chosen magnitude of the conductivity in the surrounding material. Without this compensation, the resulting imbalance of scales would effectively decouple these regions of the device, and the model would fail to capture the current flow between the conducting and dielectric components.

We define some notation. Let Ω be a domain in $\mathbf{R}^n$ and let B be a Banach space. We denote by $L^p(\Omega; B)$ the space of measurable functions $f : \Omega \to B$ such that for $1 \le p < \infty$, $\int_\Omega \|f(x)\|_B^p \, dx$ is finite or for $p = \infty$, ess $\sup_{x\in\Omega}\|f(x)\|_B$ is finite. Hence as a special case when $B = \mathbf{R}$, $L^2(\Omega) \equiv L^2(\Omega; \mathbf{R})$ is the Lebesgue space of real-valued functions on the domain Ω that are measurable and square summable in Ω with respect to Lebesque measure. We denote by $C^\infty(\bar\Omega)$ the set of restrictions to $\bar\Omega$ of infinitely differentiable functions in $\mathbf{R}^n$ with compact support. Then $W^{m,2}(\Omega)$, $m \ge 1$, is the Sobolev space obtained by the completion of $C^\infty(\bar\Omega)$ in the norm

$$\|u\|_{W^{m,2}(\Omega)} = \left(\sum_{|\alpha| \le m} \|\nabla^\alpha u\|_{L^2(\Omega)}^2 \right)^{1/2} ,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiple index, $|\alpha| = \sum_{i=1}^n \alpha_i$, $\alpha_i \ge 0$. The space $W_0^{m,2}(\Omega)$ is the closure in $W^{m,2}(\Omega)$ of the set of those infinitely differentiable functions in Ω each of which has compact support in Ω. We denote various spaces of periodic functions by a subscript $\sharp$. For example, if Y denotes the unit cube in $\mathbf{R}^n$, then $C_\sharp^\infty(Y)$ is the space of infinitely differentiable functions on $\mathbf{R}^n$ that are periodic of period Y, and $W_\sharp^{1,2}(Y)$ is the completion for the norm of $W^{1,2}(Y)$ of $C_\sharp^\infty(Y)$.

We recall some classical topics in unbounded operators and the Cauchy problem; see [7] or the first Chapter of [8] for a more extensive treatment. Let V be a Hilbert space. A bilinear form $a(\cdot,\cdot) : V \times V \to \mathbf{R}$ is V*-coercive* if there is $c_0 > 0$ for which

$$|a(u,u)| \ge c_0 \|u\|_V^2 , \qquad u \in V .$$

If $a(\cdot,\cdot)$ is bilinear, continuous and *V-coercive*, then for each $f \in V'$ there is a unique

$$u \in V : a(u,v) = f(v) \ , \qquad v \in V \ .$$

Definition 1 *An unbounded linear operator $A : D \to H$ with domain D in the Hilbert space H is* accretive *if*

$$\langle Ax, x \rangle_H \geq 0 \ , \qquad x \in D \ ,$$

*and it is m-*accretive *if, in addition, $A + I$ maps D onto H.*

Let V be a Hilbert space that is dense in another Hilbert space H. We identify $H = H'$ by its Riesz map, and we assume the identity $V \to H$ is continuous. Let $a(\cdot,\cdot)$ be a continuous bilinear form on V. Then we define D to be the set of all $u \in V$ such that the function $v \mapsto a(u,v)$ is continuous on V *with the H-norm*. For each such $u \in D$ there is then a unique $Au \in H$ such that

$$a(u,v) = \langle Au, v \rangle_H \ , \qquad u \in D \ , v \in V \ ,$$

and this defines a linear operator $A : D \to H$.

It is easy to see how the unbounded operator A with domain D in H constructed as above from the continuous bilinear form $a(\cdot,\cdot)$ on V is related to the continuous $\mathcal{A} \in \mathcal{L}(V, V')$ which is equivalent to $a(\cdot,\cdot)$. In fact, the graph of A is the restriction of the graph of $\mathcal{A}$ to $V \times H$. That is, note that $H' \hookrightarrow V'$ by restriction to V of functionals on H, so $D = \{u \in V : \mathcal{A}u \in H'\}$ and then $Au \in H$ is just that $\mathcal{A}u \in H'$ which corresponds through the identification of H with H'. Thus, when $a(\cdot,\cdot) + \lambda(\cdot,\cdot)_H$ is coercive, it is clear that $\mathcal{A} + \lambda I$ is an isomorphism of V onto V' and $A + \lambda I$ is just its (necessarily onto) restriction to $H \subset V'$. Finally, note that A is accretive on H exactly when the linear operator $\mathcal{A}$ satisfies

$$\mathcal{A}v(v) \geq 0 \ , \qquad v \in V \ .$$

Theorem 2 *Let the operator A be m-accretive on the Hilbert space H. Then for every $u_0 \in D(A)$ and $f \in C^1([0,\infty), H)$ there is a unique solution $u \in C^1([0,\infty), H)$ of the initial-value problem*

$$u'(t) + Au(t) = f(t) \ , \qquad t > 0 \ , \quad u(0) = u_0 \ . \tag{1}$$

If additionally A is self-adjoint, then for each $u_0 \in H$ and Hölder continuous $f \in C^\alpha([0,\infty), H)$, $0 < \alpha < 1$, there is a unique solution $u \in C([0,\infty), H) \cap C^1((0,\infty), H)$ of (1).

2. Homogeneous Thin-Layer Model

We describe a single capacitive layer (possibly imperfectly) bonded to the bottom of a highly conductive region. The resulting device is assumed to be very thin compared to its horizontal dimensions. Let $S \subset \mathbf{R}^2$ be a bounded domain, and denote its points by $\tilde{x} \equiv (x_1, x_2) \in S$. Let $\Omega^\varepsilon \equiv S \times (0, \varepsilon)$ be the corresponding and denote its points by $x \equiv (x_1, x_2, x_3) \in \Omega^\varepsilon$. The region Ω^ε is bonded by a resistive layer to the dielectric layer, S, and the bottom of this is grounded.

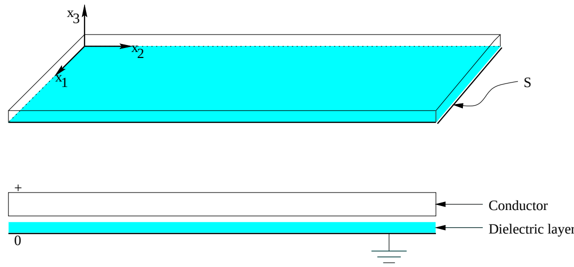

Figure 1: A homogeneous thin-layer model

The distributed voltage in the conductive region is denoted by $u^\varepsilon(x,t)$, and $v^\varepsilon(\tilde{x},t)$ represents the voltage difference across the bottom capacitive layer. The bottom layer is a distributed capacitor for which the properties are scaled in proportion to the height ε of the total device. Thus, it has a capacitance $\frac{1}{\varepsilon}C(\tilde{x})$ inversely proportional to the width and a horizontal conductivity $\varepsilon G(\tilde{x})$ proportional to that width. The vertical resistance of the bonding layer is given similarly by εr_1. The conservation of charge requires the balance of the rate of charge accumulation with horizontal conductance, exchange with the interface, and any loss due to leakage to ground. That is,

$$\frac{\partial}{\partial t}\left\{\frac{1}{\varepsilon}C(\tilde{x})v^\varepsilon(\tilde{x},t)\right\} - \tilde{\nabla}\cdot\varepsilon G(\tilde{x})\tilde{\nabla}v^\varepsilon + \frac{1}{\varepsilon r_1}(v^\varepsilon - u^\varepsilon) + \frac{1}{\varepsilon R}v^\varepsilon = 0 \ ,$$

where εR is the appropriately scaled leakage resistance of the capacitance layer.

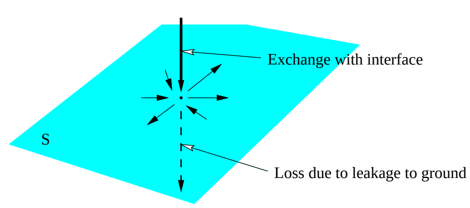

Figure 2: The currents on the bottom layer

The region Ω^ε is assumed to have a correspondingly very high conductivity $\frac{g(\tilde{x})}{\varepsilon^2}$. This is a *material property* which we *assume*. A consequence of the work below is that it is exactly this order of magnitude of conductivity that maintains the coupling of Ω^ε with the capacitive layer S in the limit as $\varepsilon \to 0$.

In this conductive region, we have the conservation of charge

$$\nabla \cdot \mathbf{J} = 0 \ ,$$

where $\mathbf{J}$ is the current field given by Ohm's law as $\mathbf{J} = -\frac{g(\tilde{x})}{\varepsilon^2}\nabla u^{\varepsilon}(x,t)$.

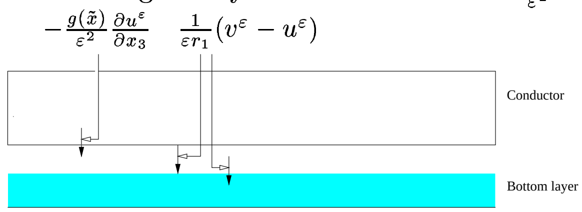

Figure 3: Interface between the conducting region and the bottom layer

On the top, $x_3 = \varepsilon$, we impose a distributed source of current,

$$\frac{g(\tilde{x})}{\varepsilon^2}\nabla u^{\varepsilon}(x,t)\cdot\boldsymbol{\nu} = \frac{g(\tilde{x})}{\varepsilon^2}\frac{\partial u^{\varepsilon}}{\partial x_3} = \frac{1}{\varepsilon}f(\tilde{x},t)\ ,$$

where $\boldsymbol{\nu}$ is the unit outward normal vector. At the interface $x_3 = 0$, the vertical current exchange through the resistive layer is

$$\frac{g(\tilde{x})}{\varepsilon^2}\nabla u^{\varepsilon}(x,t)\cdot\boldsymbol{\nu} = -\frac{g(\tilde{x})}{\varepsilon^2}\frac{\partial u^{\varepsilon}}{\partial x_3} = \frac{1}{\varepsilon r_1}(v^{\varepsilon} - u^{\varepsilon})\ ,$$

where εr_1 is the resistance of interface between the conducting region and the bottom layer.

The initial charge distribution in the bottom layer is prescribed by

$$\frac{1}{\varepsilon}C(\tilde{x})v^{\varepsilon}(\tilde{x},0) = \frac{1}{\varepsilon}C(\tilde{x})v_0(\tilde{x})\ , \qquad \tilde{x}\in S\ .$$

We assume that the boundary is grounded, so the voltages satisfy

$$u^{\varepsilon} = 0 \quad \text{on} \quad \partial S\times(0,\varepsilon)\ , \qquad v^{\varepsilon} = 0 \quad \text{on} \quad \partial S\ .$$

In summary, we have the following system of equations :

$$-\nabla\cdot\frac{g(\tilde{x})}{\varepsilon^2}\nabla u^{\varepsilon}(x,t) = 0 \qquad x\in\Omega^{\varepsilon}\ ,$$

$$\frac{g(\tilde{x})}{\varepsilon^2}\frac{\partial u^{\varepsilon}}{\partial x_3} = \begin{cases} \frac{1}{\varepsilon}f(\tilde{x},t) & x_3 = \varepsilon, \quad \tilde{x}\in S, \\ \frac{1}{\varepsilon r_1}(u^{\varepsilon} - v^{\varepsilon}) & x_3 = 0\ , \quad \tilde{x}\in S\ , \end{cases}$$

$$u^{\varepsilon} = 0 \qquad \text{on} \quad \partial S\times(0,\varepsilon)\ ,$$

$$\frac{1}{\varepsilon}C\frac{\partial v^{\varepsilon}(\tilde{x},t)}{\partial t} - \tilde{\nabla}\cdot\varepsilon G\tilde{\nabla}v^{\varepsilon} + \frac{1}{\varepsilon r_1}(v^{\varepsilon} - u^{\varepsilon}) + \frac{1}{\varepsilon R}v^{\varepsilon} = 0\ , \qquad \tilde{x}\in S\ , x_3 = 0\ ,$$

$$v^{\varepsilon} = 0 \qquad \text{on} \quad \partial S\ ,$$

and

$$v^{\varepsilon}(\tilde{x},0) = v_0(\tilde{x})\ .$$

We assume that the coefficients satisfy

$$0 < \text{constant} \le g(\tilde{x}), C(\tilde{x}), G(\tilde{x}) \le \text{constant} < \infty$$

and $r_1, R > 0$. We also assume that the initial condition $v_0 \in L^2(S)$ and the initial source $f : [0,T] \to L^2(S)$ is Hölder continuous, that is, for some $0 < \alpha < 1$,

$$\|f(t_1) - f(t_2)\|_{L^2(S)} \le C|t_1 - t_2|^\alpha \quad \text{for } \ t_1, t_2 \in [0,T] \ .$$

Finally, we rescale the vertical axis in order to remove the singular geometry due to the thinness of the structure. With a change of variables, $x_3 \equiv \varepsilon z$, $\frac{\partial}{\partial z} = \varepsilon \frac{\partial}{\partial x_3}$ we get

$$(2)\quad \begin{cases} -\tilde{\nabla} \cdot g \tilde{\nabla} u^\varepsilon(\tilde{x}, z, t) - \dfrac{\partial}{\partial z} \dfrac{g}{\varepsilon^2} \dfrac{\partial}{\partial z} u^\varepsilon(\tilde{x}, z, t) = 0 & (\tilde{x}, z) \text{ in } \Omega^1 \\ \dfrac{g}{\varepsilon^2} \dfrac{\partial u^\varepsilon}{\partial z} = \begin{cases} f(\tilde{x}, t) \\ \dfrac{1}{r_1}(u^\varepsilon - v^\varepsilon) \end{cases} & \begin{array}{l} z = 1 \ , \ \tilde{x} \in S \ , \\ z = 0 \ , \ \tilde{x} \in S \ , \end{array} \\ u^\varepsilon = 0 & \text{on } \partial S \ , \\ C \dfrac{\partial v^\varepsilon}{\partial t} - \tilde{\nabla} \cdot \varepsilon^2 G \tilde{\nabla} v^\varepsilon + \dfrac{1}{r_1}(v^\varepsilon - u^\varepsilon) + \dfrac{1}{R} v^\varepsilon = 0 & \tilde{x} \text{ in } S \ , \ z = 0 \ , \\ v^\varepsilon(\tilde{x}, t) = 0 & \text{on } \partial S \times (0,1) \ , \end{cases}$$

and the initial condition

$$v^\varepsilon(\tilde{x}, 0) = v_0(\tilde{x}) \ .$$

Now all of the singular or degenerate characteristics appear in the coefficients of the system (2) which is posed on a fixed domain Ω^1. Hereafter we set $\Omega \equiv \Omega^1$.

2.1. Elliptic-Parabolic ε-Model

In order to obtain an appropriate weak formulation, we introduce two function spaces,

$$\begin{aligned} \mathcal{V}_0 &\equiv \{u \in W^{1,2}(\Omega) : u = 0 \quad \text{on} \quad \partial S \times (0,1)\} \ , \\ \mathcal{V}_1 &\equiv W_0^{1,2}(S) \qquad (z = 0) \ . \end{aligned}$$

Then for $\phi \in \mathcal{V}_0$, we multiply both sides of the first equation by ϕ and integrate over Ω with Green's Theorem and boundary conditions to obtain

$$(3)\quad \begin{aligned} &\int_\Omega g \tilde{\nabla} u^\varepsilon(\tilde{x}, z, t) \cdot \tilde{\nabla} \phi(\tilde{x}, z) \, d\tilde{x} \, dz + \int_\Omega \frac{g}{\varepsilon^2} \frac{\partial}{\partial z} u^\varepsilon(\tilde{x}, z, t) \frac{\partial}{\partial z} \phi(\tilde{x}, z) \, d\tilde{x} \, dz \\ &\quad + \int_S \frac{1}{r_1}(u^\varepsilon(\tilde{x}, 0, t) - v^\varepsilon(\tilde{x}, t)) \phi(\tilde{x}, 0) \, d\tilde{x} = \int_S f(\tilde{x}, t) \phi(\tilde{x}, 1) \, d\tilde{x} \ , \end{aligned}$$

where ν is the unit outward normal vector. Similarly for $\varphi \in \mathcal{V}_1$, multiplying the two sides of the second equation by φ and integrating over S show that

$$
\begin{aligned}
(4) \qquad & \int_S C \frac{\partial v^\varepsilon}{\partial t} \varphi \, d\tilde{x} + \int_S \varepsilon^2 G \tilde{\nabla} v^\varepsilon \cdot \tilde{\nabla} \varphi \, d\tilde{x} + \int_S \frac{1}{R} v^\varepsilon \varphi \, d\tilde{x} \\
& + \int_S \frac{1}{r_1} (v^\varepsilon - u^\varepsilon(\tilde{x}, 0, t)) \varphi \, d\tilde{x} = 0 \ .
\end{aligned}
$$

Thus a solution of system (2) satisfies $u(t) \in \mathcal{V}_0$, $v(t) \in \mathcal{V}_1$ for $0 < t < T$, and (3), (4) hold for each $\phi \in \mathcal{V}_0$, $\varphi \in \mathcal{V}_1$. Conversely, it follows directly that an appropriately smooth solution of (3) and (4) will satisfy the system (2) above.

In order to more clearly display the structure of the equations (3) and (4), we specify some notation. Define bilinear forms a_1^ε, a_2^ε as follows. For u_1, $u_2 \in \mathcal{V}_0$,

$$
a_1^\varepsilon(u_1, u_2) \equiv \int_\Omega \left(g \tilde{\nabla} u_1 \cdot \tilde{\nabla} u_2 + \frac{g}{\varepsilon^2} \frac{\partial u_1}{\partial z} \frac{\partial u_2}{\partial z} \right) d\tilde{x} \, dz \ .
$$

This determines a family of operators $\mathcal{A}_1^\varepsilon \in \mathcal{L}(\mathcal{V}_0, \mathcal{V}_0')$ by

$$
\mathcal{A}_1^\varepsilon u_1(u_2) \equiv a_1^\varepsilon(u_1, u_2), \qquad u_1, u_2 \in \mathcal{V}_0 \ .
$$

It is easy to see that the family of linear operators $\mathcal{A}_1^\varepsilon : \mathcal{V}_0 \to \mathcal{V}_0'$ is uniformly $\mathcal{V}_0$-coercive for $0 < \varepsilon \leq 1$. For v_1, $v_2 \in \mathcal{V}_1$,

$$
a_2^\varepsilon(v_1, v_2) \equiv \int_S \left(\varepsilon^2 G \tilde{\nabla} v_1 \cdot \tilde{\nabla} v_2 + \frac{1}{R} v_1 v_2 \right) d\tilde{x} \ ,
$$

and this determines the family of operators $\mathcal{A}_2^\varepsilon \in \mathcal{L}(\mathcal{V}_1, \mathcal{V}_1')$ as before. We specify the *trace* at $z = 0$ and $z = 1$ as follows. Define $\gamma_0 : \mathcal{V}_0 \to L^2(S)$, its dual $\gamma_0^* : L^2(S) \to \mathcal{V}_0'$, $\gamma_1 : \mathcal{V}_0 \to L^2(S)$, and $\gamma_1^* : L^2(S) \to \mathcal{V}_0'$ by

$$
\begin{aligned}
\gamma_0 u(\varphi) &\equiv \int_S u(\tilde{x}, 0) \varphi(\tilde{x}) \, d\tilde{x} \ , \qquad \gamma_0^* v(\phi) \equiv \int_S v(\tilde{x}) \phi(\tilde{x}, 0) \, d\tilde{x} = \langle v, \gamma_0 \phi \rangle_{L^2(S)} \ , \\
\gamma_1 u(\varphi) &\equiv \int_S u(\tilde{x}, 1) \varphi(\tilde{x}) \, d\tilde{x} \ , \qquad \gamma_1^* f(\phi) \equiv \int_S f(\tilde{x}) \phi(\tilde{x}, 1) \, d\tilde{x} = \langle f, \gamma_1 \phi \rangle_{L^2(S)} \ .
\end{aligned}
$$

Using these definitions we can rewrite the equations (3) and (4) in the following form.
Find $u^\varepsilon \in C([0,T]; \mathcal{V}_0)$ and $v^\varepsilon \in C([0,T]; L^2(S)) \cap C^1((0,T); L^2(S))$:

$$
(5) \quad u^\varepsilon(t) \in \mathcal{V}_0 : \quad \mathcal{A}_1^\varepsilon u^\varepsilon(t) + \frac{1}{r_1} \gamma_0^* (\gamma_0 u^\varepsilon(t) - v^\varepsilon(t)) = \gamma_1^*(f(t)) \qquad \text{in } \mathcal{V}_0'
$$

$$
(6) \quad v^\varepsilon(t) \in \mathcal{V}_1 : \quad C(\tilde{x}) \frac{\partial v^\varepsilon(t)}{\partial t} + \mathcal{A}_2^\varepsilon v^\varepsilon(t) + \frac{1}{r_1} (v^\varepsilon(t) - \gamma_0 u^\varepsilon(t)) = 0 \qquad \text{in } \mathcal{V}_1'
$$

with initial condition $v^\varepsilon(\tilde{x}, 0) = v_0(\tilde{x})$.

Suppose that there exists a solution $(u^\varepsilon, v^\varepsilon)$ of the system (5), (6). From equation (5), we have

$$
(7) \qquad u^\varepsilon = \left(\mathcal{A}_1^\varepsilon + \frac{1}{r_1} \gamma_0^* \gamma_0 \right)^{-1} \left(\gamma_1^*(f) + \frac{1}{r_1} \gamma_0^* v^\varepsilon \right) \ .
$$

Substituting (7) into equation (6), we get

$$
\begin{aligned}
&\frac{\partial v^\varepsilon}{\partial t} + \frac{1}{C}\left\{\frac{1}{r_1}\left(I - \gamma_0\left(\mathcal{A}_1^\varepsilon + \frac{1}{r_1}\gamma_0^*\gamma_0\right)^{-1}\frac{1}{r_1}\gamma_0^*\right) + \mathcal{A}_2^\varepsilon\right\} v^\varepsilon \\
&\quad = \frac{1}{C}\left\{\frac{1}{r_1}\gamma_0\left(\mathcal{A}_1^\varepsilon + \frac{1}{r_1}\gamma_0^*\gamma_0\right)^{-1}\gamma_1^*\right\} f \ ,
\end{aligned}
\tag{8}
$$

in $\mathcal{V}_1'$. Of course, we have $v^\varepsilon(t) \in \mathcal{V}_1$ at each $t \in [0,T]$. Define a corresponding unbounded operator $\mathbf{A}^\varepsilon : D(\mathbf{A}^\varepsilon) \to \mathcal{H}$ on the Hilbert space $\mathcal{H} \equiv L^2(S)$ with the scalar product

$$
\langle v, w\rangle_{\mathcal{H}} \equiv \int_S C(\tilde{x})v(\tilde{x})w(\tilde{x})\, d\tilde{x} = \left\langle C^{1/2}v, C^{1/2}w\right\rangle_{\mathcal{H}} \ .
$$

The domain of $\mathbf{A}^\varepsilon$ is $D(\mathbf{A}^\varepsilon) \equiv \{v \in \mathcal{V}_1 : \mathbf{A}^\varepsilon v \in \mathcal{H}\}$, and it is defined on this domain by

$$
\mathbf{A}^\varepsilon \equiv \frac{1}{C}\left\{\frac{1}{r_1}\left(I - \gamma_0\left(\mathcal{A}_1^\varepsilon + \frac{1}{r_1}\gamma_0^*\gamma_0\right)^{-1}\frac{1}{r_1}\gamma_0^*\right) + \mathcal{A}_2^\varepsilon\right\} \ .
$$

Proposition 3 *Each $\mathbf{A}^\varepsilon : D(\mathbf{A}^\varepsilon) \to \mathcal{H}$ is self-adjoint and m-accretive.*

Proof. From the calculation

$$
\begin{aligned}
(\mathbf{A}^\varepsilon)^* &= \left\{\frac{1}{C}\left(\frac{1}{r_1}\left(I - \gamma_0\left(\mathcal{A}_1^\varepsilon + \frac{1}{r_1}\gamma_0^*\gamma_0\right)^{-1}\frac{1}{r_1}\gamma_0^*\right) + \mathcal{A}_2^\varepsilon\right)\right\}^* \\
&= \frac{1}{C}\left(\frac{1}{r_1}\left(I^* - \gamma_0^{**}\left\{\left(\mathcal{A}_1^\varepsilon + \frac{1}{r_1}\gamma_0^*\gamma_0\right)^{-1}\right\}^*\frac{1}{r_1}\gamma_0^*\right) + (\mathcal{A}_2^\varepsilon)^*\right) \\
&= \frac{1}{C}\left(\frac{1}{r_1}\left(I - \gamma_0\left((\mathcal{A}_1^\varepsilon)^* + \frac{1}{r_1}\gamma_0^*\gamma_0^{**}\right)^{-1}\frac{1}{r_1}\gamma_0^*\right) + \mathcal{A}_2^\varepsilon\right) = \mathbf{A}^\varepsilon \ ,
\end{aligned}
$$

we see that $\mathbf{A}^\varepsilon$ is self-adjoint. Since $\gamma_0(\mathcal{A}_1^\varepsilon + \frac{1}{r_1}\gamma_0^*\gamma_0)^{-1}\frac{1}{r_1}\gamma_0^*$ is a contraction on $\mathcal{H}$, it follows that $I - \gamma_0(\mathcal{A}_1^\varepsilon + \frac{1}{r_1}\gamma_0^*\gamma_0)^{-1}\frac{1}{r_1}\gamma_0^*$ is continuous and monotone. Adding a positive multiple of this to $\mathcal{A}_2^\varepsilon$ then yields a coercive operator whose restriction to $\mathcal{H}$ is m-accretive. □

Hence, by Theorem 2, there exists a unique solution of the initial-value problem for (8). Then we define $u^\varepsilon(t)$ by (7) to obtain the unique solution $u^\varepsilon \in C([0,T];\mathcal{V}_0)$, $v^\varepsilon \in C([0,T];L^2(S))$ of the system (5) and (6).

Theorem 4 (Regularity of the solution u^ε, v^ε) *We have*

$$
u^\varepsilon \in C([0,T];\mathcal{V}_0) \text{ and } v^\varepsilon \in C([0,T];L^2(S)) \cap C^1((0,T);W_0^{1,2}(S) \cap W^{2,2}(S)) \ .
$$

Proof. From the equation (5), we have that $u^\varepsilon \in C([0,T];\mathcal{V}_0)$. So from the equation (6), we get $v^\varepsilon \in C([0,T];L^2(S)) \cap C^1((0,T);W_0^{1,2}(S) \cap W^{2,2}(S))$. □

2.2. Strong Convergence

We consider convergence of (5) and (6) as $\varepsilon \to 0$. In order to describe the limiting problem, we define the operators $\mathcal{A}_1$, $\mathcal{A}_2$ as

$$\mathcal{A}_1 u_1(u_2) \equiv \int_S g\tilde{\nabla} u_1 \cdot \tilde{\nabla} u_2 \, d\tilde{x} \ , \qquad \mathcal{A}_2 v_1(v_2) \equiv \int_S \frac{1}{R} v_1 v_2 \, d\tilde{x} \ ,$$

for $u_1, u_2 \in W_0^{1,2}(S)$ and $v_1, v_2 \in L^2(S)$. Then the operator $\mathbf{A} : L^2(S) \to L^2(S)$ defined by

$$\mathbf{A} \equiv \frac{1}{C} \left\{ \frac{1}{r_1} \left(I - \frac{1}{r_1} \left(\mathcal{A}_1 + \frac{1}{r_1} \right)^{-1} \right) + \mathcal{A}_2 \right\}$$

is self-adjoint and m-accretive as well as *bounded*. So there exists the unique solution $v \in C^1([0,T]; L^2(S))$ of the limit problem :

$$\frac{\partial}{\partial t} v + \mathbf{A} v = \frac{1}{C} \left\{ \frac{1}{r_1} \left(\mathcal{A}_1 + \frac{1}{r_1} \right)^{-1} \right\} f$$

with initial condition $v(\tilde{x}, 0) = v_0(\tilde{x})$. That is, there exists the unique solution $u \in C([0,T]; W_0^{1,2}(S))$ and $v \in C^1([0,T]; L^2(S))$ of the system of differential equations

$$\mathcal{A}_1 u + \frac{1}{r_1}(u - v) = f \tag{9}$$

$$C \frac{\partial v}{\partial t} + \mathcal{A}_2 v + \frac{1}{r_1}(v - u) = 0 \ , \tag{10}$$

with initial condition

$$v(\tilde{x}, 0) = v_0(\tilde{x}) \ .$$

The following theorem is our goal.

Theorem 5 *For each $0 < \varepsilon \le 1$, let $(u^\varepsilon, v^\varepsilon)$ be the solution of the system of differential equations (5) and (6) with initial condition $v^\varepsilon(\tilde{x}, 0) = v_0(\tilde{x})$ and $(u, v) \in C([0,T]; W_0^{1,2}(S)) \times C^1([0,T]; L^2(S))$ be the solution of the system of differential equations (9) and (10) with initial condition $v(\tilde{x}, 0) = v_0(\tilde{x})$. Then u^ε converges strongly to u in $C([0,T]; \mathcal{V}_0)$ and v^ε converges strongly to v in $C([0,T]; L^2(S))$.*

In order to show that v^ε converges strongly to v in $C([0,T]; L^2(S))$, we notice that v^ε is the solution of the ε-problem :

$$\frac{\partial}{\partial t} v^\varepsilon + \mathbf{A}^\varepsilon v^\varepsilon = \frac{1}{C} \left\{ \frac{1}{r_1} \gamma_0 \left(\mathcal{A}_1^\varepsilon + \frac{1}{r_1} \gamma_0^* \gamma_0 \right)^{-1} \gamma_1^* \right\} f \equiv h^\varepsilon \ .$$

and v is the solution of the limit problem :

$$\frac{\partial}{\partial t} v + \mathbf{A} v = \frac{1}{C} \left\{ \frac{1}{r_1} \left(\mathcal{A}_1 + \frac{1}{r_1} \right)^{-1} \right\} f \equiv h \ .$$

We shall apply the following consequence of the classical Trotter Convergence Theorem.

Theorem 6 *Let* $\mathbf{A}$ *and* $\mathbf{A}^\varepsilon$*,* $0 < \varepsilon \le 1$*, be m-accretive. Suppose that*

$$(I + \mathbf{A}^\varepsilon)^{-1} F \to (I + \mathbf{A})^{-1} F \qquad \text{in } H \ ,$$

for all $F \in H$*. Assume that* $h^\varepsilon \to h$ *in* $L^1([0,T];H)$ *and* $v_0^\varepsilon \to v_0$ *in* H*. Then*

$$v^\varepsilon \to v \qquad \text{in } C([0,T];H) \ ,$$

where v^ε *and* v *are the solutions of the differential equations*

$$\frac{\partial}{\partial t} v^\varepsilon + \mathbf{A}^\varepsilon v^\varepsilon = h^\varepsilon \quad v^\varepsilon(0) = v_0^\varepsilon \ ; \qquad\qquad \frac{\partial}{\partial t} v + \mathbf{A} v = h \quad v(0) = v_0 \ .$$

2.2.1. Approximation of the stationary problems

In this section, we first show that for a given $F \in L^2(S)$, $\bar{v}^\varepsilon$ *converges weakly* to $\bar{v}$ in $L^2(S)$, where $\bar{v}^\varepsilon \in \mathcal{V}_1$ and $\bar{v} \in L^2(S)$ are the solutions of the ε problem $\bar{v}^\varepsilon + \mathbf{A}^\varepsilon \bar{v}^\varepsilon = \frac{1}{C} F$ and the limit problem $\bar{v} + \mathbf{A}\bar{v} = \frac{1}{C} F$, respectively.

Define $\bar{u}^\varepsilon$: $\bar{u}^\varepsilon \equiv (\mathcal{A}_1^\varepsilon + \frac{1}{r_1}\gamma_0^* \gamma_0)^{-1} (\frac{1}{r_1}\gamma_0^* \bar{v}^\varepsilon)$. Then we have

$$\mathcal{A}_1^\varepsilon \bar{u}^\varepsilon(\phi) + \frac{1}{r_1}(\gamma_0 \bar{u}^\varepsilon - \bar{v}^\varepsilon)(\gamma_0 \phi) \ = \ 0 \tag{11}$$

$$C\bar{v}^\varepsilon(\varphi) + \mathcal{A}_2^\varepsilon \bar{v}^\varepsilon(\varphi) + \frac{1}{r_1}(\bar{v}^\varepsilon - \gamma_0 \bar{u}^\varepsilon)(\varphi) \ = \ F(\varphi), \tag{12}$$

$\phi \in \mathcal{V}_0$, $\varphi \in \mathcal{V}_1$. We substitute $\phi \equiv \bar{u}^\varepsilon$ and $\varphi \equiv \bar{v}^\varepsilon$, and add the two equations to get

$$\|\sqrt{g}\tilde{\nabla}\bar{u}^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon^2}\left\|\frac{\partial}{\partial z}\bar{u}^\varepsilon\right\|_{L^2(\Omega)}^2 + \|\sqrt{C}\bar{v}^\varepsilon\|_{L^2(S)}^2 + \|\varepsilon\sqrt{G}\tilde{\nabla}\bar{v}^\varepsilon\|_{L^2(S)}^2$$
$$+ \frac{1}{R}\|\bar{v}^\varepsilon\|_{L^2(S)}^2 + \frac{1}{r_1}\|\bar{u}^\varepsilon(\tilde{x},0) - \bar{v}^\varepsilon\|_{L^2(S)}^2 = \int_S F(\tilde{x})\bar{v}^\varepsilon(\tilde{x})\, d\tilde{x} \ .$$

This shows that each of

$$\|\tilde{\nabla}\bar{u}^\varepsilon\|_{L^2(\Omega)} \ , \quad \left\|\frac{1}{\varepsilon}\frac{\partial}{\partial z}\bar{u}^\varepsilon\right\|_{L^2(\Omega)} \ , \quad \|\bar{v}^\varepsilon\|_{L^2(S)} \ , \quad \|\varepsilon\tilde{\nabla}\bar{v}^\varepsilon\|_{L^2(S)}$$

is bounded. It follows that a subsequence $\bar{u}^{\varepsilon_j}$ converges weakly to some $\bar{u}$ in $\mathcal{V}_0$ and $\bar{v}^{\varepsilon_j} \rightharpoonup \bar{v}$ in $L^2(S)$.

Lemma 7 *The sequence* $\bar{u}^{\varepsilon_j}$ *converges strongly to some* $\bar{u}$ *in* $L^2(\Omega)$ *for which* $\bar{u} \equiv \bar{u}(\tilde{x})$ *is independent of* z*,* $\nabla \bar{u}^{\varepsilon_j} \rightharpoonup \nabla \bar{u}$ *weakly in* $L^2(\Omega)$*, and so* $\bar{u} \in W_0^{1,2}(S)$*.*

Proof. Since $\bar{u}^{\varepsilon_j}$ converges weakly in $W^{1,2}(\Omega)$ to a $\bar{u} \in W^{1,2}(\Omega)$, $\bar{u}^{\varepsilon_j}$ converges strongly to $\bar{u}$ in $L^2(\Omega)$ and $\nabla \bar{u}^{\varepsilon_j} \rightharpoonup \nabla \bar{u}$ weakly in $L^2(\Omega)$. From the fact that

$\|\frac{1}{\varepsilon_j}\frac{\partial \bar{u}^{\varepsilon_j}}{\partial z}\|_{L^2(\Omega)} \leq$ constant, we have $\frac{\partial \bar{u}^{\varepsilon_j}}{\partial z} \to 0$ strongly as $\varepsilon_j \to 0$. With the conditions $\nabla \bar{u}^{\varepsilon_j} \rightharpoonup \nabla \bar{u}$ and $\frac{\partial \bar{u}^{\varepsilon_j}}{\partial z} \to 0$, we get $\frac{\partial \bar{u}}{\partial z} = 0$, that is, $\bar{u} \equiv \bar{u}(\tilde{x})$ is independent of z. □

So we have for $\phi(\tilde{x}, z) \equiv \varphi(\tilde{x}) \in \mathcal{V}_1$,

$$\begin{aligned} \mathcal{A}_1^{\varepsilon_j} \bar{u}^{\varepsilon_j}(\phi) &= \int_\Omega g \tilde{\nabla} \bar{u}^{\varepsilon_j} \cdot \tilde{\nabla}\phi \, d\tilde{x}\, dz \\ &\to \int_\Omega g \tilde{\nabla} \bar{u} \cdot \tilde{\nabla}\phi \, d\tilde{x}\, dz \left(= \int_S g \tilde{\nabla} \bar{u} \cdot \tilde{\nabla}\phi \, d\tilde{x} \right) \equiv \mathcal{A}_1 \bar{u}(\phi) \ . \end{aligned}$$

Using the weak continuity of the trace map, since $\bar{v}^{\varepsilon_j} \rightharpoonup \bar{v}$ in $L^2(S)$ and $\|\varepsilon \tilde{\nabla} \bar{v}^\varepsilon\|_{L^2(S)}$ is bounded, we get

$$\begin{aligned} \mathcal{A}_2^{\varepsilon_j} \bar{v}^{\varepsilon_j}(\varphi) &\equiv \int_S \left(\varepsilon_j^2 G \tilde{\nabla} \bar{v}^{\varepsilon_j} \cdot \tilde{\nabla}\varphi + \frac{1}{R} \bar{v}^{\varepsilon_j} \varphi \right) d\tilde{x} \\ &= \varepsilon_j \int_S \varepsilon_j G \tilde{\nabla} \bar{v}^{\varepsilon_j} \cdot \tilde{\nabla}\varphi \, d\tilde{x} + \int_S \frac{1}{R} \bar{v}^{\varepsilon_j} \varphi \, d\tilde{x} \\ &\to \int_S \frac{1}{R} \bar{v} \varphi \, d\tilde{x} \equiv \mathcal{A}_2 \bar{v}(\varphi) \ . \end{aligned}$$

In summary, taking weak limits in (11) and (12) yield the limit system

$$\begin{aligned} \mathcal{A}_1 \bar{u} + \frac{1}{r_1}(\bar{u} - \bar{v}) &= 0 \\ C\bar{v} + \mathcal{A}_2 \bar{v} + \frac{1}{r_1}(\bar{v} - \bar{u}) &= F \ , \end{aligned}$$

with operators defined at the beginning of Section 2.2. We have shown that for a given $F \in L^2(S)$, by uniqueness of the solution, we have the original sequence $\bar{u}^\varepsilon$ converges weakly to $\bar{u}$ in $\mathcal{V}_0$ and $\bar{v}^\varepsilon$ converges weakly to $\bar{v}$ in $L^2(S)$. Also note that the system above is equivalent to the single equation

$$\bar{v} + \mathbf{A}\bar{v} = \frac{1}{C} F \ .$$

Next we show that $\bar{v}^\varepsilon$ converges *strongly* to $\bar{v}$ in $L^2(S)$, and $\bar{u}^\varepsilon$ converges *strongly* to $\bar{u}$ in $\mathcal{V}_0$. We write the operators $\mathcal{A}_1^\varepsilon$ and $\mathcal{A}_2^\varepsilon$ in two parts :

$$\begin{aligned} \mathcal{A}_1^\varepsilon u(\phi) &\equiv \int_\Omega g \tilde{\nabla} u \cdot \tilde{\nabla}\phi \, d\tilde{x}\, dz + \frac{1}{\varepsilon^2} \int_\Omega g \frac{\partial u}{\partial z} \frac{\partial \phi}{\partial z} \, d\tilde{x}\, dz \\ &\equiv \mathcal{A}_1 u(\phi) + \frac{1}{\varepsilon^2} \mathcal{B}_1 u(\phi) \ , \qquad u, \phi \in \mathcal{V}_0 \ , \\ \mathcal{A}_2^\varepsilon v(\varphi) &\equiv \int_S \frac{1}{R} v \varphi \, d\tilde{x} + \varepsilon^2 \int_S G \tilde{\nabla} v \cdot \tilde{\nabla}\varphi \, d\tilde{x} \\ &\equiv \mathcal{A}_2 v(\varphi) + \varepsilon^2 \mathcal{B}_2 v(\varphi) \ , \qquad v, \varphi \in \mathcal{V}_1 \ . \end{aligned}$$

Add the equations (11) and (12) to get

$$C\bar{v}^\varepsilon(\varphi) + \mathcal{A}_1\bar{u}^\varepsilon(\phi) + \frac{1}{\varepsilon^2}\mathcal{B}_1\bar{u}^\varepsilon(\phi) + \mathcal{A}_2\bar{v}^\varepsilon(\varphi) + \varepsilon^2\mathcal{B}_2\bar{v}^\varepsilon(\varphi)$$

$$+\frac{1}{r_1}(\gamma_0\bar{u}^\varepsilon - \bar{v}^\varepsilon)(\gamma_0\phi - \varphi) = F\varphi\ .$$

We subtract from this the expression

$$C\bar{v}(\varphi) + \mathcal{A}_1\bar{u}(\phi) + \mathcal{A}_2\bar{v}(\varphi) + \frac{1}{r_1}(\bar{u} - \bar{v})(\gamma_0\phi - \varphi)$$

to obtain the identity

$$C(\bar{v}^\varepsilon - \bar{v})(\varphi) + \mathcal{A}_1(\bar{u}^\varepsilon - \bar{u})(\phi) + \frac{1}{\varepsilon^2}\mathcal{B}_1\bar{u}^\varepsilon(\phi) + \varepsilon^2\mathcal{B}_2\bar{v}^\varepsilon(\varphi) + \mathcal{A}_2(\bar{v}^\varepsilon - \bar{v})(\varphi)$$

$$+\frac{1}{r_1}(\gamma_0\bar{u}^\varepsilon - \bar{u} - (\bar{v}^\varepsilon - \bar{v}))(\gamma_0\phi - \varphi)$$

$$= F\varphi - \left(C\bar{v}(\varphi) + \mathcal{A}_1\bar{u}(\phi) + \mathcal{A}_2\bar{v}(\varphi) + \frac{1}{r_1}(\bar{u} - \bar{v})(\gamma_0\phi - \varphi)\right)\ .$$

Now substitute $\phi \equiv \bar{u}^\varepsilon - \bar{u}$ and $\varphi \equiv \bar{v}^\varepsilon - \bar{v}$ to get

$$C\|\bar{v}^\varepsilon - \bar{v}\|^2_{L^2(S)} + \mathcal{A}_1(\bar{u}^\varepsilon - \bar{u})(\bar{u}^\varepsilon - \bar{u}) + \frac{1}{\varepsilon^2}\mathcal{B}_1(\bar{u}^\varepsilon - \bar{u})(\bar{u}^\varepsilon - \bar{u}) + \varepsilon^2\mathcal{B}_2\bar{v}^\varepsilon(\bar{v}^\varepsilon)$$

$$+\mathcal{A}_2(\bar{v}^\varepsilon - \bar{v})(\bar{v}^\varepsilon - \bar{v}) + \frac{1}{r_1}\|\gamma_0\bar{u}^\varepsilon - \bar{u} - (\gamma_0\bar{v}^\varepsilon - \bar{v})\|^2_{L^2(S)}$$

$$= \varepsilon^2\mathcal{B}_2\bar{v}^\varepsilon(\bar{v}) + \left(F - C\bar{v} - \mathcal{A}_2\bar{v} + \frac{1}{r_1}(\bar{u} - \bar{v})\right)(\bar{v}^\varepsilon - \bar{v})$$

$$-\left(\mathcal{A}_1\bar{u} + \frac{1}{r_1}\gamma_0^*(\bar{u} - \bar{v})\right)(\bar{u}^\varepsilon - \bar{u})\ .$$

Since the right side converges to zero, it follows that so also does each term on the left. In particular, we see $\bar{v}^\varepsilon$ converges strongly to $\bar{v}$ in $L^2(S)$ and $\bar{u}^\varepsilon$ converges strongly to $\bar{u}$ in $\mathcal{V}_0$.

2.2.2. Approximation of the evolution problem

In this Section, we finish the proof of Theorem 5.

Proposition 8 *Suppose $w^\varepsilon \rightharpoonup w$ in $L^2(S)$. Then for $i = 0, 1$, $(\mathcal{A}_1^\varepsilon + \frac{1}{r_1}\gamma_0^*\gamma_0)^{-1}\gamma_i^* w^\varepsilon$ converges weakly to $(\mathcal{A}_1 + \frac{1}{r_1})^{-1}w$ in $\mathcal{V}_0$, which is independent of z.*

Proof. Let $k^\varepsilon \equiv (\mathcal{A}_1^\varepsilon + \frac{1}{r_1}\gamma_0^*\gamma_0)^{-1}\gamma_i^* w^\varepsilon$, $i = 0, 1$. Then we have $\mathcal{A}_1^\varepsilon k^\varepsilon(\phi) + \frac{1}{r_1}\gamma_0^*\gamma_0 k^\varepsilon(\phi) = \gamma_i^* w^\varepsilon(\phi)$. As we have shown in Section 2.2.1, the energy estimates show that $\|k^\varepsilon\|_{\mathcal{V}_0}$ and $\|\frac{1}{\varepsilon}\frac{\partial k^\varepsilon}{\partial z}\|_{L^2(\Omega)}$ are bounded so k^{ε_j} converges weakly to some k in $\mathcal{V}_0$ for which k is independent of z. Thus for $\phi \in \mathcal{V}_1$,

$$\mathcal{A}_1^{\varepsilon_j} k^{\varepsilon_j}(\phi) + \frac{1}{r_1}\gamma_0 k^{\varepsilon_j}(\gamma_0\phi) = w^\varepsilon(\gamma_i\phi) \quad \rightarrow \quad \mathcal{A}_1 k(\phi) + \frac{1}{r_1}k(\phi) = w(\phi) \ .$$

This says that $k = (\mathcal{A}_1 + \frac{1}{r_1})^{-1}w$, and k^ε converges weakly to k in $\mathcal{V}_0$. $\square$

Corollary 9 *Let* $h^\varepsilon \equiv \frac{1}{C}\frac{1}{r_1}\gamma_0(\mathcal{A}_1^\varepsilon + \frac{1}{r_1}\gamma_0^*\gamma_0)^{-1}\gamma_1^* f$, $h \equiv \frac{1}{C}\frac{1}{r_1}(\mathcal{A}_1 + \frac{1}{r_1})^{-1} f$. *Then* h^ε *converges strongly to* h *in* $L^1([0,T]; L^2(S))$.

Proof. Let $k^\varepsilon(t) \equiv (\mathcal{A}_1^\varepsilon + \frac{1}{r_1}\gamma_0^*\gamma_0)^{-1}\gamma_1^* f(t)$ and $k(t) \equiv (\mathcal{A}_1 + \frac{1}{r_1})^{-1} f(t)$. Then for each fixed t, $k^\varepsilon(t)$ converges weakly to $k(t)$ in $\mathcal{V}_0$ for which $k(t)$ is independent of z. Thus $k^\varepsilon(t)$ converges strongly to $k(t)$ in $L^2(\Omega)$. Since we have $\gamma_0 k^\varepsilon(t) \to k(t)$ for all $t \in [0,T]$ and

$$\|h^\varepsilon(t)\|_{L^2(S)} \le \max_{\tilde{x}\in S}\left\{\frac{1}{C(\tilde{x})}\right\}\frac{1}{r_1}\|f(t)\|_{L^2(S)} \ ,$$

by Lebesgue Dominated Convergence Theorem, we obtain that $h^\varepsilon \to h$ in $L^1([0,T]; L^2(S))$. $\square$

By the result of Section 2.2.1 and Theorem 6, v^ε converges strongly to v in $C([0,T]; L^2(S))$. Define

$$\begin{aligned} u^\varepsilon(t) &\equiv \left(\mathcal{A}_1^\varepsilon + \frac{1}{r_1}\gamma_0^*\gamma_0\right)^{-1}\left(\gamma_1^* f(t) + \frac{1}{r_1}\gamma_0^* v^\varepsilon(t)\right) \\ u(t) &\equiv \left(\mathcal{A}_1 + \frac{1}{r_1}\right)^{-1}\left(f(t) + \frac{1}{r_1}v(t)\right) \ . \end{aligned}$$

Then by Proposition 8, for fixed $t \in [0,T]$, $u^\varepsilon(t)$ converges weakly to $u(t)$ in $\mathcal{V}_0$ for which $u(t)$ is independent of z. Also, we subtract from

$$\mathcal{A}_1^\varepsilon u^\varepsilon(t)(\phi) + \frac{1}{r_1}\gamma_0^*\gamma_0 u^\varepsilon(t)(\phi) = \gamma_1^* f(t)(\phi) + \frac{1}{r_1}\gamma_0^* v^\varepsilon(t)(\phi)$$

the expression $\mathcal{A}_1 u(t)(\phi) + \frac{1}{r_1}u(t)(\gamma_0\phi)$ to get the identity

$$\begin{aligned} &\mathcal{A}_1(u^\varepsilon(t) - u(t))(\phi) + \frac{1}{\varepsilon^2}\mathcal{B}_1 u^\varepsilon(t)(\phi) + \frac{1}{r_1}\left(\gamma_0 u^\varepsilon(t) - u(t)\right)(\gamma_0\phi) \\ &= f(t)(\gamma_1\phi) + \frac{1}{r_1}v^\varepsilon(t)(\gamma_0\phi) - \mathcal{A}_1 u(t)(\phi) - \frac{1}{r_1}u(t)(\gamma_0\phi) \end{aligned}$$

(with operators defined in Section 2.2.1). We substitute $\phi \equiv u^\varepsilon(t) - u(t)$, and then as $\varepsilon \to 0$, the right side converges to zero. It follows that u^ε converges strongly to u in $C([0,T]; \mathcal{V}_0)$. This completes the proof of Theorem 5.

3. Two-periodic Thin-Layer Model

Let $S \subset \mathbf{R}^2$ be a bounded domain, and denote its points by $\tilde{x} \equiv (x_1, x_2) \in S$. Let the thickness of the medium be $\varepsilon > 0$, denote by $\Omega^\varepsilon \equiv S \times (0, \varepsilon)$ the corresponding region in $\mathbf{R}^3$, and denote its points by $x \equiv (x_1, x_2, x_3) \in \Omega^\varepsilon$. The region Ω^ε is a good conductor of conductivity $\frac{g(\tilde{x})}{\varepsilon^2}$, and it is bonded by a resistive layer on the bottom, $x_3 = 0$ to a *periodic array* of capacitors.

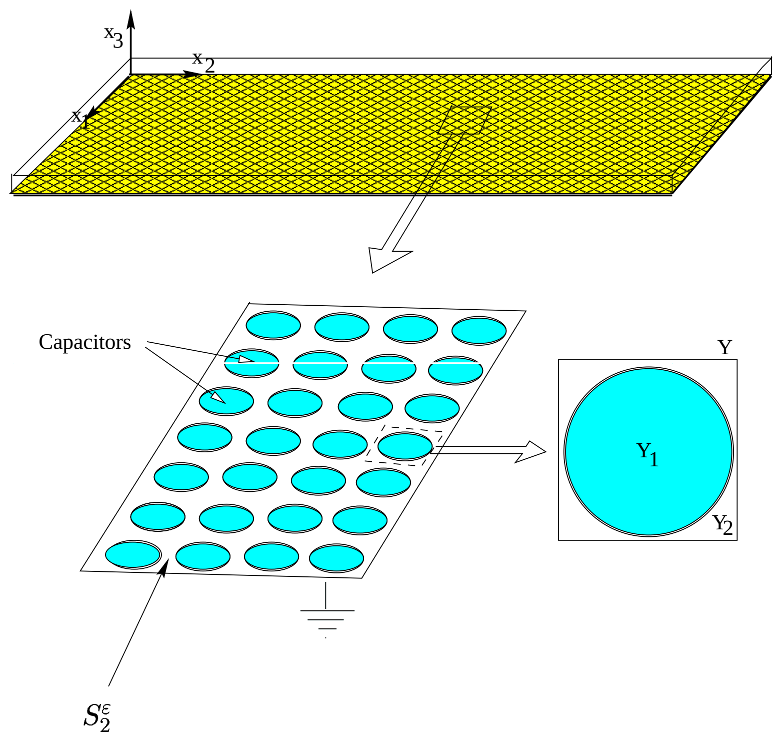

Figure 4: A two-periodic thin-layer model

These have period εY, where $Y \equiv [0,1]^2$ is the unit square, and their thickness is also scaled by ε. Let Y be given in the complementary parts, Y_1, Y_2, where Y_1 represents a scaled copy of one of the capacitors in the layer. Denote by $\chi_j(y)$ the characteristic function of Y_j for $j = 1, 2$, extended Y-periodically to all of $\mathbf{R}^2$. Thus, $\chi_1(y) + \chi_2(y) = 1$. The global domain S is divided into two subdomains, S_1^ε, S_2^ε, given by

$$S_j^\varepsilon \equiv \left\{ \tilde{x} \in S : \chi_j\left(\frac{\tilde{x}}{\varepsilon}\right) = 1 \right\} , \qquad j = 1, 2,$$

where S_1^ε represents the periodic distribution of capacitance along the bottom. We shall assume that the domain $\{y \in \mathbf{R}^2 : \chi_2(y) = 1\}$ is smooth and connected, so the capacitors are necessarily *isolated*. Hereafter we set $\chi_j^\varepsilon(\tilde{x}) \equiv \chi_j(\frac{\tilde{x}}{\varepsilon})$. Due to the ε-scaling of the thickness, the medium Ω^ε is bonded along the bottom by a layer of resistance εr_1 to a layer in S_1^ε with capacitance $\frac{1}{\varepsilon}C(\tilde{x})$ and horizontal conductivity $\varepsilon G(\tilde{x})$, and it is connected directly to ground in S_2^ε by the resistance εr_1. Let $u^\varepsilon(x, t)$ denote the voltage distribution in Ω^ε and $v^\varepsilon(\tilde{x}, t)$ the voltage difference across S_1^ε.

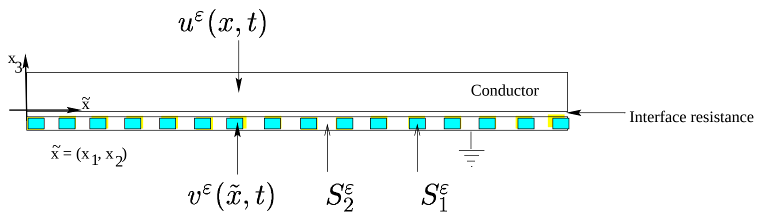

Figure 5: A periodic bottom layer

In the conductive region Ω^ε, we have

$$\nabla \cdot \mathbf{J} = 0 \ ,$$

where $\mathbf{J}$ is the conductive current field given by Ohm's law as $\mathbf{J} = -\frac{g(\tilde{x})}{\varepsilon^2}\nabla u^\varepsilon(x,t)$.

On the top, $x_3 = \varepsilon$, we impose a distributed source of current,

$$\frac{g(\tilde{x})}{\varepsilon^2}\nabla u^\varepsilon(x,t)\cdot\boldsymbol{\nu} = \frac{g(\tilde{x})}{\varepsilon^2}\frac{\partial u^\varepsilon}{\partial x_3} = \frac{1}{\varepsilon}f(\tilde{x},t) \ ,$$

where $\boldsymbol{\nu}$ is the unit outward normal vector.

At the interface $x_3 = 0$, when $\tilde{x} \in S_1^\varepsilon$, the vertical current exchange through the resistive interface is given by

$$\frac{g(\tilde{x})}{\varepsilon^2}\nabla u^\varepsilon(x,t)\cdot\boldsymbol{\nu} = -\frac{g(\tilde{x})}{\varepsilon^2}\frac{\partial u^\varepsilon}{\partial x_3} = \frac{1}{\varepsilon r_1}(v^\varepsilon - u^\varepsilon) \ ,$$

where εr_1 is the resistance of the interface between the conducting region and the bottom layer. But when $\tilde{x} \in S_2^\varepsilon$,

$$\frac{g(\tilde{x})}{\varepsilon^2}\nabla u^\varepsilon(x,t)\cdot\boldsymbol{\nu} = -\frac{g(\tilde{x})}{\varepsilon^2}\frac{\partial u^\varepsilon}{\partial x_3} = -\frac{1}{\varepsilon r_1}u^\varepsilon \ .$$

On the bottom layer, S_1^ε is a distributed capacitance in which the conservation of charge requires that for $\tilde{x} \in S_1^\varepsilon$,

$$\frac{\partial}{\partial t}\left\{\frac{1}{\varepsilon}Cv^\varepsilon(\tilde{x},t)\right\} - \tilde{\nabla}\cdot\varepsilon G\tilde{\nabla}v^\varepsilon(\tilde{x},t) + \frac{1}{\varepsilon r_1}(v^\varepsilon(\tilde{x},t) - u^\varepsilon(\tilde{x},0,t)) + \frac{1}{\varepsilon R}v^\varepsilon(\tilde{x},t) = 0 \ ,$$

where εR is the leakage resistance across S_1^ε to the ground.

The initial charge distribution in the bottom layer is prescribed by

$$\frac{1}{\varepsilon}C(\tilde{x})v^\varepsilon(\tilde{x},0) = \frac{1}{\varepsilon}C(\tilde{x})v_0(\tilde{x}), \quad \tilde{x} \in S_1^\varepsilon \ .$$

We assume that the lateral boundary of the region Ω^ε is grounded, so the voltages are all zero there,

$$u^\varepsilon = 0 \quad \text{on} \quad \partial S \times (0,\varepsilon) \ ,$$

and also each of the capacitor cells is grounded on its boundary,

$$v^\varepsilon = 0 \quad \text{on} \quad \partial S_1^\varepsilon \; .$$

We assume that the coefficients satisfy $0 < \text{constant} \leq g(\tilde{x}), C(\tilde{x}), G(\tilde{x}) \leq \text{constant} < \infty$, and $r_1,\ R > 0$. We also assume that the initial condition $v_0 \in L^2(S)$ and the initial source $f : [0,\infty) \to L^2(S)$ is Hölder continuous as in Section 2.

In order to remove the singular geometry due to the thinness of the structure, we rescale the vertical axis with a change of variables, $x_3 \equiv \varepsilon z$, $\frac{\partial}{\partial z} = \varepsilon \frac{\partial}{\partial x_3}$ to get the scaled system

$$(13)\begin{cases} -\tilde{\nabla}\cdot g\tilde{\nabla}u^\varepsilon(\tilde{x},z,t) - \dfrac{\partial}{\partial z}\dfrac{g}{\varepsilon^2}\dfrac{\partial}{\partial z}u^\varepsilon(\tilde{x},z,t) = 0 & (\tilde{x},z) \text{ in } \Omega\ , \\ \dfrac{g}{\varepsilon^2}\dfrac{\partial u^\varepsilon}{\partial z} = \begin{cases} f(\tilde{x},t) \\ \dfrac{1}{r_1}u^\varepsilon \\ \dfrac{1}{r_1}(u^\varepsilon - v^\varepsilon) \end{cases} & \begin{array}{l} z=1, \tilde{x}\in S\ , \\ z=0, \tilde{x}\in S_2^\varepsilon\ , \\ z=0, \tilde{x}\in S_1^\varepsilon\ , \end{array} \\ u^\varepsilon = 0 & \text{on } \partial S\times(0,1)\ , \\ C\dfrac{\partial v^\varepsilon}{\partial t} - \tilde{\nabla}\cdot\varepsilon^2 G\tilde{\nabla}v^\varepsilon + \dfrac{1}{r_1}(v^\varepsilon - u^\varepsilon) + \dfrac{1}{R}v^\varepsilon = 0\ , & \tilde{x}\in S_1^\varepsilon\ ,\ z=0\ , \\ v^\varepsilon(\tilde{x},t) = 0 & \text{on } \partial S_1^\varepsilon\ , \end{cases}$$

and the initial condition $v^\varepsilon(\tilde{x},0) = v_0(\tilde{x})$ in $L^2(S_1^\varepsilon)$, wherein and hereafter we set $\Omega^1 = \Omega$. Now the parameter $\varepsilon > 0$ affects the geometry only through the periodic array S_1^ε at $z = 0$. All additional effects are contained in the size of various coefficients in (13).

3.1. Elliptic-Parabolic ε-Model

First, we describe a weak formulation of the scaled problem. We define function spaces $\mathcal{V}_0$, and $\mathcal{V}_1^\varepsilon$ as follows :

$$\mathcal{V}_0 \equiv \{u \in W^{1,2}(\Omega) : u = 0 \quad \text{on} \quad \partial S\times(0,1)\}\ , \qquad \mathcal{V}_1^\varepsilon \equiv W_0^{1,2}(S_1^\varepsilon)\ .$$

By the zero-extension to S_2^ε, we regard $v \in W_0^{1,2}(S_1^\varepsilon)$ as an element in $L^2(S)$. Then for $\phi \in \mathcal{V}_0$, we multiply by ϕ on the both sides of the first equation and integrate it over Ω to obtain

$$(14)\quad \begin{aligned} &\int_\Omega g\tilde{\nabla}u^\varepsilon(\tilde{x},z,t)\cdot\tilde{\nabla}\phi(\tilde{x},z)\,d\tilde{x}\,dz + \int_\Omega \frac{g}{\varepsilon^2}\frac{\partial}{\partial z}u^\varepsilon(\tilde{x},z,t)\frac{\partial}{\partial z}\phi(\tilde{x},z)\,d\tilde{x}\,dz \\ &\quad + \int_S \frac{1}{r_1}(u^\varepsilon(\tilde{x},0,t) - \chi_1^\varepsilon(\tilde{x})v^\varepsilon(\tilde{x},t))\phi(\tilde{x},0)\,d\tilde{x} = \int_S f(\tilde{x},t)\phi(\tilde{x},1)\,d\tilde{x}\ . \end{aligned}$$

For $\varphi \in \mathcal{V}_1^\varepsilon$, multiplying φ on the both sides of the second equation and integrating it over S_1^ε, we have

$$\begin{aligned} &\int_{S_1^\varepsilon} C\frac{\partial v^\varepsilon}{\partial t}\varphi\, d\tilde{x} + \int_{S_1^\varepsilon} \varepsilon^2 G\tilde{\nabla} v^\varepsilon \cdot \tilde{\nabla}\varphi\, d\tilde{x} + \int_{S_1^\varepsilon} \frac{1}{R} v^\varepsilon \varphi\, d\tilde{x} \\ &\quad + \int_{S_1^\varepsilon} \frac{1}{r_1}(v^\varepsilon - u^\varepsilon(\tilde{x},0))\varphi\, d\tilde{x} = 0\ . \end{aligned} \tag{15}$$

In order to more clearly display the structure of the equations (14) and (15), we define the bilinear forms $a_1^\varepsilon(\cdot,\cdot)$, $a_2^\varepsilon(\cdot,\cdot)$ as follows.
For $u_1,\ u_2 \in \mathcal{V}_0$,

$$a_1^\varepsilon(u_1,u_2) \equiv \int_\Omega \left(g\tilde{\nabla} u_1 \cdot \tilde{\nabla} u_2 + \frac{g}{\varepsilon^2}\frac{\partial u_1}{\partial z}\frac{\partial u_2}{\partial z} \right) d\tilde{x}\, dz\ .$$

This determines a corresponding family of operators $\mathcal{A}_1^\varepsilon \in \mathcal{L}(\mathcal{V}_0, \mathcal{V}_0')$ by

$$\mathcal{A}_1^\varepsilon u_1(u_2) \equiv a_1^\varepsilon(u_1,u_2)\ , \qquad u_1,u_2 \in \mathcal{V}_0\ .$$

For $v_1,\ v_2 \in \mathcal{V}_1^\varepsilon$,

$$a_2^\varepsilon(v_1,v_2) \equiv \int_{S_1^\varepsilon} \left(\varepsilon^2 G\tilde{\nabla} v_1 \cdot \tilde{\nabla} v_2 + \frac{1}{R} v_1 v_2 \right) d\tilde{x}$$

determines $\mathcal{A}_2^\varepsilon \in \mathcal{L}(\mathcal{V}_1^\varepsilon, (\mathcal{V}_1^\varepsilon)')$ similarly by

$$\mathcal{A}_2^\varepsilon v_1(v_2) \equiv a_2^\varepsilon(v_1,v_2), \qquad v_1,v_2 \in \mathcal{V}_1^\varepsilon\ .$$

Using this notation we can rewrite the system (13) in the following form.

Find $(u^\varepsilon, v^\varepsilon) \in C([0,\infty);\mathcal{V}_0) \times C([0,\infty);L^2(S_1^\varepsilon)) \cap C^1((0,\infty);\mathcal{V}_1^\varepsilon)$:

$$\mathcal{A}_1^\varepsilon u^\varepsilon(t) + \frac{1}{r_1}\gamma_0^*(\gamma_0 u^\varepsilon(t) - \chi_1^\varepsilon v^\varepsilon(t)) \quad = \quad \gamma_1^*(f(t)) \quad \text{in} \quad \mathcal{V}_0' \tag{16}$$

$$C\frac{\partial v^\varepsilon(t)}{\partial t} + \mathcal{A}_2^\varepsilon v^\varepsilon(t) + \frac{1}{r_1}(v^\varepsilon(t) - \gamma_0 u^\varepsilon(t)) \quad = \quad 0 \qquad \text{in} \quad (\mathcal{V}_1^\varepsilon)' \tag{17}$$

with initial condition

$$v^\varepsilon(\tilde{x},0) = v_0(\tilde{x}) \quad \tilde{x} \in S_1^\varepsilon\ .$$

The trace operators $\gamma_0, \gamma_1 : \mathcal{V}_0 \to L^2(S)$ are given as before in Section 2.1. Define a unbounded operator $\mathbf{A}^\varepsilon : D(\mathbf{A}^\varepsilon) \to L^2(S_1^\varepsilon)$ on the Hilbert space $\mathcal{H}^\varepsilon \equiv L^2(S_1^\varepsilon)$ with the scalar product

$$\langle v,w\rangle_{\mathcal{H}^\varepsilon} \equiv \int_{S_1^\varepsilon} C(\tilde{x})v(\tilde{x})w(\tilde{x})\, d\tilde{x}\ .$$

Note that $\mathcal{H}^\varepsilon \subset (\mathcal{V}_1^\varepsilon)'$. The domain of $\mathbf{A}^\varepsilon$ is $D(\mathbf{A}^\varepsilon) \equiv \{v \in \mathcal{V}_1^\varepsilon : \mathbf{A}^\varepsilon v \in \mathcal{H}^\varepsilon\}$, where the value of this operator is given on this domain by

$$\mathbf{A}^\varepsilon \equiv \frac{1}{C}\left\{ \frac{1}{r_1}\left(I - \gamma_0 \left(\mathcal{A}_1^\varepsilon + \frac{1}{r_1}\gamma_0^*\gamma_0 \right)^{-1} \gamma_0^*\chi_1^\varepsilon \right) + \mathcal{A}_2^\varepsilon \right\}\ .$$

Then the system (16), (17) is equivalent to

$$\frac{\partial}{\partial t}v^\varepsilon(\tilde{x}) + \mathbf{A}^\varepsilon v^\varepsilon(\tilde{x}) = \frac{1}{C}\frac{1}{r_1}\gamma_0\left(\mathcal{A}_1^\varepsilon + \frac{1}{r_1}\gamma_0^*\gamma_0\right)^{-1}\gamma_1^* f$$

Since $\mathbf{A}^\varepsilon$ is m-accretive and self-adjoint, from Theorem 2, it follows that there is a unique solution v^ε in $C([0,\infty); L^2(S_1^\varepsilon)) \cap C^1((0,\infty); L^2(S_1^\varepsilon))$ of the Cauchy problem for (1). Define $u^\varepsilon(t)$ by

$$u^\varepsilon(t) \equiv \left(\mathcal{A}_1^\varepsilon + \frac{1}{r_1}\gamma_0^*\gamma_0\right)^{-1}\left(\gamma_1^*(f(t)) + \frac{1}{r_1}\gamma_0^*\chi_1^\varepsilon v^\varepsilon(t)\right) ,$$

where $\chi_1^\varepsilon v^\varepsilon(\tilde{x}, t) \equiv v^\varepsilon(\tilde{x}, t)$ for $\tilde{x} \in S_1^\varepsilon$ and 0 otherwise. Then we obtain the unique solution $u^\varepsilon \in C([0,\infty); \mathcal{V}_0)$, $v^\varepsilon \in C([0,\infty); \mathcal{V}_1^\varepsilon) \cap C^1((0,\infty); L^2(S_1^\varepsilon))$ to the system (16) and (17).

Theorem 10 (Regularity of the solution u^ε,v^ε) *We have that*

$$u^\varepsilon \in C([0,\infty); \mathcal{V}_0) \text{ and } v^\varepsilon \in C([0,\infty); L^2(S_1^\varepsilon)) \cap C^1((0,\infty); \mathcal{V}_1^\varepsilon \cap W^{2,2}(S_1^\varepsilon)) .$$

3.2. Two-scale convergence

In this section we introduce the notion of two-scale convergence that will be used in the following sections. Let Ω be an open set in $\mathbf{R}^n$ and $Y \equiv [0,1]^n$.

Definition 11 *A sequence of functions u^ε in $L^2(\Omega)$ is said to* two-scale converge *to a limit $U_0(x,y) \in L^2(\Omega \times Y)$ (denoted by $u^\varepsilon \xrightarrow{2} U_0$) if for any test function $\Psi(x,y) \in C_0^\infty\Omega; C_\sharp^\infty(Y))$, we have*

$$\lim_{\varepsilon\to 0}\int_\Omega u^\varepsilon(x)\Psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega\int_Y U_0(x,y)\Psi(x,y)\, dx\, dy .$$

In [1], we can find the following remarks.

Remark 12 *1. For any $\Psi(x,y) \in L^2(\Omega; C_\sharp(Y))$, we have*

$$\lim_{\varepsilon\to 0}\int_\Omega \Psi\left(x, \frac{x}{\varepsilon}\right)^2 dx = \int_\Omega\int_Y \Psi(x,y)^2\, dx\, dy .$$

2. **(Nguetseng)** *Any bounded sequence u^ε in $L^2(\Omega)$ has a two-scale convergent subsequence.*

3. Let $\chi_1(\frac{x}{\varepsilon})v^\varepsilon$ and $\varepsilon\chi_1(\frac{x}{\varepsilon})\nabla v^\varepsilon$ be bounded in $L^2(\Omega)$. Then, there exists a function $V(x,y) \in L^2(\Omega; W_\sharp^{1,2}(Y))$ and a subsequence for which $\chi_1(\frac{x}{\varepsilon_j})v_j^\varepsilon$ and $\varepsilon_j\chi_1(\frac{x}{\varepsilon_j})\nabla v_j^\varepsilon$ two-scale converge to $\chi_1(y)V(x,y)$ and $\chi_1(y)\nabla_y V(x,y)$, respectively.

These extend to the following situation which is better suited to our needs here.

Proposition 13 (two-scale convergence with parameter) *Suppose that u^ε is a bounded sequence in $L^\infty([0,\infty);L^2(\Omega))$. Then there is a subsequence u^{ε_j} of u^ε and a function $U_0(t,x,y) \in L^\infty([0,\infty);L^2(\Omega\times Y))$ such that for any $\Psi(t,x,y)$ in $L^1([0,\infty);C_0^\infty(\Omega;C_\sharp^\infty(Y)))$*

$$\lim_{\varepsilon_j\to 0}\int_0^\infty\int_\Omega u^{\varepsilon_j}(t,x)\Psi\left(t,x,\frac{x}{\varepsilon_j}\right)dx\,dt=\int_0^\infty\int_Y\int_\Omega U_0(t,x,y)\Psi(t,x,y)\,dx\,dy\,dt\ .$$

Proof. Let $\mathcal{F}_\varepsilon(\Psi)\equiv\int_0^\infty\int_\Omega u^\varepsilon(t,x)\Psi(t,x,\frac{x}{\varepsilon})\,dx\,dt$ and $\mathcal{D}\equiv L^2(\Omega;C_\sharp(Y))$. There is a positive constant C such that $\|u^\varepsilon\|_{L^\infty(0,\infty;L^2(\Omega))}\le C$. Since

$$\begin{aligned}|\mathcal{F}_\varepsilon(\Psi)| &\le \int_0^\infty \|u^\varepsilon(t)\|_{L^2(\Omega)}\left\|\Psi\left(t,x,\frac{x}{\varepsilon}\right)\right\|_{L^2(\Omega)}dt\\ &\le C\int_0^\infty\max_{y\in Y}\|\Psi(t,x,y)\|_{L^2(\Omega)}\,dt=C\int_0^\infty\|\Psi(t)\|_{\mathcal{D}}\,dt\end{aligned}$$

and $\Psi\in L^1([0,\infty);\mathcal{D})$. We have $\mathcal{F}_\varepsilon\in(L^1([0,\infty);\mathcal{D}))'$. So a subsequence $\mathcal{F}_{\varepsilon_j}$ is *weak**-convergent to some $\mathcal{U}_0\in(L^1(0,\infty;\mathcal{D}))'$. Hence for any Ψ in $L^1([0,\infty);\mathcal{D})$, we have

$$\left\|\Psi\left(t,x,\frac{x}{\varepsilon_j}\right)\right\|_{L^2(\Omega)}\le\|\Psi(t)\|_{\mathcal{D}}\ ,$$

so by Lebesgue Dominated Convergence Theorem,

$$\begin{aligned}|\mathcal{U}_0(\Psi)|=\lim_{\varepsilon_j\to 0}|\mathcal{F}_{\varepsilon_j}(\Psi)| &\le \limsup_{\varepsilon_j\to 0}\int_0^\infty\|u^{\varepsilon_j}(t)\|_{L^2(\Omega)}\left\|\Psi\left(t,x,\frac{x}{\varepsilon_j}\right)\right\|_{L^2(\Omega)}dt\\ &\le C\int_0^\infty\limsup_{\varepsilon_j\to 0}\left\|\Psi\left(t,x,\frac{x}{\varepsilon_j}\right)\right\|_{L^2(\Omega)}dt\\ &= C\int_0^\infty\|\Psi(t)\|_{L^2(\Omega\times Y)}\,dt\ .\end{aligned}$$

Since $L^1([0,\infty);\mathcal{D})$ is dense in $L^1([0,\infty);L^2(\Omega\times Y))$, it follows that $\mathcal{U}_0$ is in $(L^1([0,\infty);\ L^2(\Omega\times Y)))'$. By Riesz Representation Theorem, we have

$$\mathcal{U}_0(\Psi)=\int_0^\infty\langle U_0(t),\Psi(t)\rangle_{L^2(\Omega\times Y)}\,dt$$

for some $U_0\in L^\infty([0,\infty);L^2(\Omega\times Y))$. Therefore

$$\begin{aligned}\lim_{\varepsilon_j\to 0}\int_0^\infty\int_\Omega u^{\varepsilon_j}(t,x)\Psi\left(t,x,\frac{x}{\varepsilon_j}\right)dx\,dt &\equiv \lim_{\varepsilon_j\to 0}\mathcal{F}_{\varepsilon_j}(\Psi)\\ &= \mathcal{U}_0(\Psi)\\ &= \int_0^\infty\int_Y\int_\Omega U_0(t,x,y)\Psi(t,x,y)\,dx\,dy\,dt\ .\end{aligned}$$

□

3.3. Homogenization

In this Section, we investigate the convergence of the solutions of (16) and (17) as $\varepsilon \to 0$. The two-scale limit is also the solution of a Cauchy problem. To find the form of this limit problem, we consider the two-scale convergence of the corresponding stationary problems for which energy estimates are simpler. This is justified by the end of this Section. First, we display the operator $\mathbf{A}$ which corresponds to the limit problem. For this, we define operators $\mathcal{A}_1 : \mathcal{V}_1 \to (\mathcal{V}_1)'$, $\mathcal{A}_2 : L^2(S; W_0^{1,2}(Y_1)) \to (L^2(S; W_0^{1,2}(Y_1)))'$ by

$$\begin{aligned}
\mathcal{A}_1 u(\phi) &\equiv \int_S g\tilde{\nabla} u(\tilde{x}) \cdot \tilde{\nabla}\phi(\tilde{x})\, d\tilde{x} \\
\mathcal{A}_2 V(\Psi) &\equiv \int_{Y_1}\int_S G\tilde{\nabla}_{\tilde{y}} V(\tilde{x},\tilde{y}) \cdot \tilde{\nabla}_{\tilde{y}}\Psi(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y} + \int_{Y_1}\int_S \frac{1}{R} V(\tilde{x},\tilde{y})\Psi(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y}\ .
\end{aligned}$$

Define an unbounded operator $\mathbf{A} : D(\mathbf{A}) \to L^2(S \times Y_1)$ on the Hilbert space $\mathcal{H} \equiv L^2(S \times Y_1)$ with the scalar product

$$\langle V, W\rangle_{\mathcal{H}} \equiv \int_{S\times Y_1} C(\tilde{x}) V(\tilde{x},\tilde{y}) W(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y}\ .$$

The domain of $\mathbf{A}$ is $D(\mathbf{A}) \equiv \{V \in L^2(S; W_0^{1,2}(Y_1)) : \mathbf{A}V \in \mathcal{H}\}$, and it is defined on this domain by

$$\mathbf{A}(V) \equiv \frac{1}{C}\left\{\frac{1}{r_1}\left\{V - \left(\mathcal{A}_1 + \frac{1}{r_1}\right)^{-1}\int_{Y_1} V(\cdot,\tilde{y})\, d\tilde{y}\right\} + \mathcal{A}_2 V\right\}\ .$$

The following lemma justifies the existence and uniqueness of the solution to the limit problem.

Lemma 14 $\mathbf{A} : D(\mathbf{A}) \to L^2(S \times Y_1)$ *is m-accretive and self-adjoint.*

Proof. The m-accretivity is the same as Proposition 3. We observe that

$$\begin{aligned}
&\left\langle \left(\mathcal{A}_1 + \frac{1}{r_1}\right)^{-1}\int_{Y_1} V(\tilde{x},\tilde{y})\, d\tilde{y}\ ,\ W(\tilde{x},\tilde{y})\right\rangle_{L^2(S\times Y_1)} \\
&= \int_S\int_{Y_1}\left(\mathcal{A}_1 + \frac{1}{r_1}\right)^{-1}\int_{Y_1} V(\tilde{x},\tilde{y})\, d\tilde{y}\, W(\tilde{x},\tilde{y})\, d\tilde{y}\, d\tilde{x} \\
&= \int_S\left(\mathcal{A}_1 + \frac{1}{r_1}\right)^{-1}\int_{Y_1} V(\tilde{x},\tilde{y})\, d\tilde{y}\int_{Y_1} W(\tilde{x},\tilde{y})\, d\tilde{y}\, d\tilde{x} \\
&= \int_S\left(\mathcal{A}_1^* + \frac{1}{r_1}\right)\left(\mathcal{A}_1 + \frac{1}{r_1}\right)^{-1}\int_{Y_1} V(\tilde{x},\tilde{y})\, d\tilde{y}\left(\mathcal{A}_1 + \frac{1}{r_1}\right)^{-1}\int_{Y_1} W(\tilde{x},\tilde{y})\, d\tilde{y}\, d\tilde{x} \\
&= \left\langle V(\tilde{x},\tilde{y}), \left(\mathcal{A}_1 + \frac{1}{r_1}\right)^{-1}\int_{Y_1} W(\tilde{x},\tilde{y})\, d\tilde{y}\right\rangle_{L^2(S\times Y_1)}\ .
\end{aligned}$$

From this, we have that $\mathbf{A}$ is self-adjoint. $\square$

Let $v^\varepsilon \in C([0,\infty); L^2(S_1^\varepsilon))$ be the solution of the ε-problem :

$$\frac{\partial}{\partial t} v^\varepsilon(\tilde{x}) + \mathbf{A}^\varepsilon v^\varepsilon(\tilde{x}) = \frac{1}{C}\frac{1}{r_1}\gamma_0\left(\mathcal{A}_1^\varepsilon + \frac{1}{r_1}\gamma_0^*\gamma_0\right)^{-1}\gamma_1^* f(\tilde{x}) \equiv h^\varepsilon(\tilde{x}) \ ,$$

$v^\varepsilon(\tilde{x},0) = v_0(\tilde{x})$ ($\tilde{x} \in S_1^\varepsilon$), and let $V_0 \in C([0,\infty); L^2(S \times Y_1))$ be the solution of the limit problem:

$$\frac{\partial}{\partial t} V_0 + \mathbf{A} V_0 = \frac{1}{C}\frac{1}{r_1}\left(\mathcal{A}_1 + \frac{1}{r_1}\right)^{-1} f \equiv h, \qquad V_0(\tilde{x},\tilde{y},0) = v_0(\tilde{x}) \ .$$

In order to examine the convergence of the solutions, we extend the solutions $v^\varepsilon(t)$ in $L^2(S_1^\varepsilon)$ to $\chi_1^\varepsilon v^\varepsilon(t)$ in $L^2(S)$ and similarly extend $V_0(t)$ in $L^2(S \times Y_1)$ to $\chi_1(\tilde{y})V_0(t)$ in $L^2(S \times Y)$.

Theorem 15 *For each $T > 0$, we have*

$$\int_0^T \chi_1^\varepsilon v^\varepsilon(t)\, dt \xrightarrow{2} \int_0^T \chi_1(\tilde{y}) V_0(t)\, dt \ .$$

Remark 16 *We may not display all the information that we have. In fact, roughly speaking, the convergence is a kind of weak*-convergence with respect to $L^\infty(0,\infty)$-norm and two-scale with respect to $L^2(S)$-norm.*

In order to prove Theorem 15, we use Proposition 13.

Lemma 17 $\|\chi_1^\varepsilon v^\varepsilon(t)\|_{L^\infty([0,\infty);L^2(S))}$ *is bounded.*

Proof. Let $\{S_\varepsilon(t) : t \geq 0\}$ be the semi-group generated by $-\mathbf{A}^\varepsilon$. Then from the facts that

$$\begin{aligned}
\|v^\varepsilon(t)\|_{L^2(S_1^\varepsilon)} &= \left\| S_\varepsilon(t) v^\varepsilon(0) + \int_0^t S_\varepsilon(t-s) h^\varepsilon(s)\, ds \right\|_{L^2(S_1^\varepsilon)} \\
&\leq \|\chi_1^\varepsilon v^\varepsilon(0)\|_{L^2(S)} + \int_0^t \|h^\varepsilon(s)\|_{L^2(S)}\, ds \\
&\leq \|v_0\|_{L^2(S)} + \int_0^\infty \|h^\varepsilon(s)\|_{L^2(S)}\, ds \leq \text{ constant},
\end{aligned}$$

we have that for all $t \geq 0$, $\|\chi_1^\varepsilon v^\varepsilon(t)\|_{L^2(S)}$ is bounded. $\square$

Hence, there is a subsequence v^{ε_j} of v^ε and W in $L^\infty([0,\infty); L^2(S \times Y))$ such that for $\Psi(t,\tilde{x},\tilde{y}) \in L^1([0,\infty); C_0^\infty(S; C_\sharp^\infty(Y)))$,

$$\begin{aligned}
&\lim_{\varepsilon_j \to 0} \int_0^\infty \int_S \chi_1^{\varepsilon_j}(\tilde{x}) v^{\varepsilon_j}(t,\tilde{x}) \Psi\left(t, \tilde{x}, \frac{\tilde{x}}{\varepsilon_j}\right) d\tilde{x}\, dt \\
&\quad = \int_0^\infty \int_Y \int_S W(t,\tilde{x},\tilde{y}) \Psi(t,\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y}\, dt \ .
\end{aligned}$$

In particular, for $\Psi(\tilde{x},\tilde{y}) \in C_0^\infty(S; C_\sharp^\infty(Y))$, we take $\chi_{[0,T]}(t)\Psi(\tilde{x},\tilde{y})$ as a test function to obtain

$$\int_0^T \chi_1^{\varepsilon_j} v^{\varepsilon_j}(t)\, dt \xrightarrow{2} \int_0^T W(t)\, dt \ ,$$

and also we have the following lemma.

Lemma 18 *For any $\lambda > 0$,*

$$\chi_1^{\varepsilon_j} \widehat{v^{\varepsilon_j}}(\lambda) \xrightarrow{2} \widehat{W}(\lambda) \ ,$$

where $\widehat{W}$ represents Laplace transform : $\widehat{W}(\lambda) \equiv \int_0^\infty e^{-\lambda t} W(t)\, dt$.

Proof. For $\Psi(\tilde{x},\tilde{y}) \in C_0^\infty(S; C_\sharp^\infty(Y))$, we take $e^{-\lambda t}\Psi(\tilde{x},\tilde{y})$ as a test function to observe

$$\begin{aligned}
&\lim_{\varepsilon_j \to 0} \int_S \chi_1^{\varepsilon_j} \widehat{v^{\varepsilon_j}}(\lambda, \tilde{x}) \Psi\left(\tilde{x}, \frac{\tilde{x}}{\varepsilon_j}\right) d\tilde{x} \\
&= \lim_{\varepsilon_j \to 0} \int_S \left(\int_0^\infty e^{-\lambda t} \chi_1^{\varepsilon_j} v^{\varepsilon_j}(t,\tilde{x})\, dt \right) \Psi\left(\tilde{x}, \frac{\tilde{x}}{\varepsilon_j}\right) d\tilde{x} \\
&= \lim_{\varepsilon_j \to 0} \int_0^\infty \int_S \chi_1^{\varepsilon_j} v^{\varepsilon_j}(t,\tilde{x}) e^{-\lambda t} \Psi\left(\tilde{x}, \frac{\tilde{x}}{\varepsilon_j}\right) d\tilde{x}\, dt \\
&= \int_0^\infty \int_Y \int_S W(t,\tilde{x},\tilde{y}) e^{-\lambda t} \Psi(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y}\, dt \\
&= \int_Y \int_S \left(\int_0^\infty e^{-\lambda t} W(t,\tilde{x},\tilde{y})\, dt \right) \Psi(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y} \\
&= \int_Y \int_S \widehat{W}(\lambda,\tilde{x},\tilde{y}) \Psi(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y} \ ,
\end{aligned}$$

that is, for fixed $\lambda > 0$, $\chi_1^{\varepsilon_j} \widehat{v^{\varepsilon_j}}(\lambda) \xrightarrow{2} \widehat{W}(\lambda)$. □

Therefore we take Laplace transform on $\frac{\partial}{\partial t} v^\varepsilon(t) + \mathbf{A}^\varepsilon v^\varepsilon(t) = h^\varepsilon(t)$ in $(\mathcal{V}_1^\varepsilon)'$ to get

$$\lambda \widehat{v^\varepsilon} - v^\varepsilon(0) + \mathbf{A}^\varepsilon \widehat{v^\varepsilon} = \widehat{h^\varepsilon} \qquad \text{in } (\mathcal{V}_1^\varepsilon)' \ .$$

Suppose we have the condition that

$$\chi_1^\varepsilon \left(\lambda I + \mathbf{A}^\varepsilon\right)^{-1} h|_{S_1^\varepsilon} \xrightarrow{2} \chi_1(\tilde{y}) \left(\lambda I + \mathbf{A}\right)^{-1} h \tag{18}$$

for all $h \in L^2(S)$. Then we obtain

$$\chi_1^\varepsilon \widehat{v^\varepsilon}(\lambda) \xrightarrow{2} \chi_1(\tilde{y})(\lambda I + \mathbf{A})^{-1}(\widehat{h} + v_0) \ .$$

In fact, from the estimate $\|(\lambda I + \mathbf{A}^\varepsilon)^{-1}\| \leq \frac{1}{\lambda}$, we have

$$\lim_{\varepsilon\to 0} \left\| \chi_1^\varepsilon (\lambda I + \mathbf{A}^\varepsilon)^{-1} \left(\widehat{h^\varepsilon}|_{S_1^\varepsilon} - \widehat{h}|_{S_1^\varepsilon} \right) \right\|_{L^2(S)} \leq \lim_{\varepsilon\to 0} \frac{1}{\lambda} \|\widehat{h^\varepsilon} - \widehat{h}\|_{L^2(S)} = 0 \ ,$$

by Corollary 9, where $\widehat{h}|_{S_1^\varepsilon}$ means the restriction map on S_1^ε. Hence

$$\begin{aligned} \chi_1^\varepsilon \widehat{v^\varepsilon}(\lambda) &= \chi_1^\varepsilon (\lambda I + \mathbf{A}^\varepsilon)^{-1} \widehat{h^\varepsilon}|_{S_1^\varepsilon} + \chi_1^\varepsilon (\lambda I + \mathbf{A}^\varepsilon)^{-1} v_0|_{S_1^\varepsilon} \\ &= \chi_1^\varepsilon (\lambda I + \mathbf{A}^\varepsilon)^{-1} \left(\widehat{h^\varepsilon}|_{S_1^\varepsilon} - \widehat{h}|_{S_1^\varepsilon} \right) + \chi_1^\varepsilon (\lambda I + \mathbf{A}^\varepsilon)^{-1} \left(\widehat{h}|_{S_1^\varepsilon} + v_0|_{S_1^\varepsilon} \right) \\ &\xrightarrow{2} \chi_1(\tilde{y}) (\lambda I + \mathbf{A})^{-1} (\widehat{h} + v_0) \ . \end{aligned}$$

By Lemma 18, this means that $\widehat{W}(\lambda) = \chi_1(\tilde{y})(\lambda I + \mathbf{A})^{-1}(\widehat{h} + v_0)$. By the uniqueness of Laplace transform, we get $W = \chi_1(\tilde{y}) V_0$ and

$$\frac{\partial}{\partial t} V_0 + \mathbf{A} V_0 = h \qquad V_0(0) = v_0$$

in $L^2(S \times Y_1)$. Also, since the solution of above is unique, we have that the original sequence $\int_0^T \chi_1^\varepsilon v^\varepsilon(t)\, dt$ converges weakly to $\int_0^T \chi_1(\tilde{y}) V_0(t)\, dt$.

It remains to verify (18) to finish the proof of Theorem 15. This is done in the next Section.

3.3.1. Two-scale convergence of stationary problems

In this Section, we prove (18), that is, for a given $h \in L^2(S)$, we have

$$\chi_1^\varepsilon \bar{v}^\varepsilon \xrightarrow{2} \chi_1 \bar{V}_0 \ ,$$

where $\bar{v}^\varepsilon \in \mathcal{V}_1^\varepsilon$ and $\bar{V}_0 \in L^2(S; W_0^{1,2}(Y_1))$ are the solutions of the ε-problem $\lambda \bar{v}^\varepsilon(\tilde{x}) + \mathbf{A}^\varepsilon \bar{v}^\varepsilon(\tilde{x}) = \frac{1}{C} h(\tilde{x})$ ($\tilde{x} \in S_1^\varepsilon$) and the limit problem $\lambda \bar{V}_0(\tilde{x}, \tilde{y}) + \mathbf{A} \bar{V}_0(\tilde{x}, \tilde{y}) = \frac{1}{C} h(\tilde{x})$ ($(\tilde{x}, \tilde{y}) \in S \times Y_1$), respectively.

Define $\bar{u}^\varepsilon : \bar{u}^\varepsilon \equiv (\mathcal{A}_1^\varepsilon + \frac{1}{r_1} \gamma_0^* \gamma_0)^{-1} (\frac{1}{r_1} \gamma_0^* \chi_1^\varepsilon \bar{v}^\varepsilon)$. Then we have

$$\mathcal{A}_1^\varepsilon \bar{u}^\varepsilon(\phi) + \frac{1}{r_1} (\gamma_0 \bar{u}^\varepsilon - \chi_1^\varepsilon \bar{v}^\varepsilon)(\gamma_0 \phi) = 0 \qquad \phi \in \mathcal{V}_0 \tag{19}$$

$$\lambda C \bar{v}^\varepsilon(\varphi) + \mathcal{A}_2^\varepsilon \bar{v}^\varepsilon(\varphi) + \frac{1}{r_1} (\chi_1^\varepsilon \bar{v}^\varepsilon - \gamma_0 \bar{u}^\varepsilon)(\varphi) = h(\varphi), \qquad \varphi \in \mathcal{V}_1^\varepsilon \ . \tag{20}$$

We set $\phi \equiv \bar{u}^\varepsilon$, $\varphi \equiv \bar{v}^\varepsilon$ and add the two equations to show that the sequences $\|\tilde{\nabla} \bar{u}^\varepsilon\|_{L^2(\Omega)}$, $\|\frac{1}{\varepsilon} \sqrt{g} \frac{\partial}{\partial z} \bar{u}^\varepsilon\|_{L^2(\Omega)}$, $\|\bar{v}^\varepsilon\|_{L^2(S_1^\varepsilon)}$, $\|\varepsilon \tilde{\nabla} \bar{v}^\varepsilon\|_{L^2(S_1^\varepsilon)}$ are bounded. It follows that a subsequence $\bar{u}^{\varepsilon_j}$ converges weakly to some $\bar{u}$ in $\mathcal{V}_0$ for which $\bar{u}$ is independent of z, and $\chi_1^{\varepsilon_j} \bar{v}^{\varepsilon_j}$ two-scale converges to $\chi_1 \bar{W}$ for some $\bar{W} \in L^2(S; W_\sharp^{1,2}(Y))$.

Let $\bar{V}_0 \equiv \bar{W}|_{S \times Y_1}$, the restriction to $S \times Y_1$. Then, for $\phi(\tilde{x}, z) = \varphi(\tilde{x}) \in \mathcal{V}_1$ on equation (19) and taking limit we have

$$\mathcal{A}_1 \bar{u}(\varphi) + \frac{1}{r_1} \left(\bar{u} - \int_{Y_1} \bar{V}_0(\cdot, y)\, dy \right)(\varphi) = 0 \qquad \text{in } (\mathcal{V}_1)' \ .$$

Since $\chi_1^{\varepsilon_j}\varepsilon_j\tilde{\nabla}\bar{v}^{\varepsilon_j} \xrightarrow{2} \chi_1(\tilde{y})\tilde{\nabla}_{\tilde{y}}\bar{W}$, we get for any $\Psi \in C_0^\infty(S; C_\sharp^\infty(Y))$ with $\Psi = 0$ on $\bar{Y}_2$, $\Psi(\tilde{x}, \frac{\tilde{x}}{\varepsilon_j}) \in \mathcal{V}_1^{\varepsilon_j}$ and so

$$\begin{aligned}
\mathcal{A}_2^{\varepsilon_j}\bar{v}^{\varepsilon_j}\left(\Psi\left(\tilde{x}, \frac{\tilde{x}}{\varepsilon_j}\right)\right) &\equiv \int_{S_1^\varepsilon}\left\{\varepsilon_j G\tilde{\nabla}\bar{v}^{\varepsilon_j}\cdot\left(\varepsilon_j\tilde{\nabla}\Psi\left(\tilde{x}, \frac{\tilde{x}}{\varepsilon_j}\right)\right) + \frac{1}{R}\bar{v}^{\varepsilon_j}\Psi\left(\tilde{x}, \frac{\tilde{x}}{\varepsilon_j}\right)\right\} d\tilde{x} \\
&\to \int_{Y_1}\int_S G\tilde{\nabla}_{\tilde{y}}\bar{V}_0\cdot\tilde{\nabla}_{\tilde{y}}\Psi(\tilde{x},\tilde{y})\,d\tilde{x}\,d\tilde{y} + \int_{Y_1}\int_S \frac{1}{R}\bar{V}_0\Psi\,d\tilde{x}\,d\tilde{y} \\
&\equiv \mathcal{A}_2\bar{V}_0(\Psi)\ .
\end{aligned}$$

Remark 19 *We have that $\bar{W} = 0$ on ∂Y_1.*

Proof. Since $\varepsilon_j\chi_1^{\varepsilon_j}\tilde{\nabla}\bar{v}^{\varepsilon_j} \xrightarrow{2} \chi_1(\tilde{y})\tilde{\nabla}_{\tilde{y}}\bar{W}$, for $\mathbf{\Psi} \in C_0^\infty(S; C_\sharp^\infty(Y))^2$, we get that

$$\lim_{\varepsilon\to 0}\int_S \varepsilon_j\chi_1^{\varepsilon_j}\tilde{\nabla}\bar{v}^{\varepsilon_j}\cdot\mathbf{\Psi}\left(\tilde{x}, \frac{\tilde{x}}{\varepsilon_j}\right)d\tilde{x} = \int_Y\int_S \chi_1(\tilde{y})\tilde{\nabla}_{\tilde{y}}\bar{W}(\tilde{x},\tilde{y})\cdot\mathbf{\Psi}(\tilde{x},\tilde{y})\,d\tilde{x}\,d\tilde{y}\ .$$

By integration by parts of the left-hand side, we also have

$$\begin{aligned}
&-\lim_{\varepsilon\to 0}\int_S \chi_1^{\varepsilon_j}\bar{v}^{\varepsilon_j}\left\{\varepsilon_j\tilde{\nabla}_{\tilde{x}}\cdot\mathbf{\Psi}\left(\tilde{x}, \frac{\tilde{x}}{\varepsilon_j}\right) + \tilde{\nabla}_{\tilde{y}}\cdot\mathbf{\Psi}\left(\tilde{x}, \frac{\tilde{x}}{\varepsilon_j}\right)\right\} d\tilde{x} \\
&\quad = -\int_Y\int_S \chi_1(\tilde{y})\bar{W}(\tilde{x},\tilde{y})\,\tilde{\nabla}_{\tilde{y}}\cdot\mathbf{\Psi}(\tilde{x},\tilde{y})\,d\tilde{x}\,d\tilde{y}\ .
\end{aligned}$$

By the uniqueness of the limit, we have $\bar{W}(\tilde{x},\tilde{y}) = 0$ on ∂Y_1. □

We have shown that for a given function $h \in L^2(S)$, we have the original sequence $\bar{u}^\varepsilon$ converges weakly to $\bar{u}$ in $\mathcal{V}_0$ and $\chi_1^\varepsilon\bar{v}^\varepsilon$ two-scale converges to $\chi_1\bar{V}_0$ for which $(\bar{u}, \bar{V}_0)$ is the solution of a corresponding system (with operators defined at the beginning of Section 3.3)

$$\begin{aligned}
\mathcal{A}_1\bar{u} + \frac{1}{r_1}\left(\bar{u} - \int_{Y_1}\bar{V}_0(\cdot,\tilde{y})\,d\tilde{y}\right) &= 0 \quad \text{in } (\mathcal{V}_1)' \\
\lambda C\bar{V}_0 + \mathcal{A}_2\bar{V}_0 + \frac{1}{r_1}(\bar{V}_0 - \bar{u}) &= h \quad \text{in } \left(L^2(S; W_0^{1,2}(Y_1))\right)'\ .
\end{aligned}$$

We note the limit system is equivalent to

$$\lambda\bar{V}_0(\tilde{x},\tilde{y}) + \mathbf{A}\bar{V}_0(\tilde{x},\tilde{y}) = \frac{1}{C}\chi_1(\tilde{y})h(\tilde{x})\ , \qquad (\tilde{x},\tilde{y}) \in S\times Y_1\ .$$

3.3.2. Approximation of the evolution problem

Theorem 20 *Let $u^\varepsilon \in C([0,\infty);\mathcal{V}_0)$, $v^\varepsilon \in C([0,\infty); L^2(S_1^\varepsilon))\cap C^1((0,\infty); L^2(S_1^\varepsilon))$ be the solutions of the ε-problem of (16), (17) with initial condition $v^\varepsilon(\tilde{x},0) =$*

$v_0(\tilde{x})$, $\tilde{x} \in S_1^\varepsilon$. Suppose $u \in C([0,\infty);\mathcal{V}_1)$ and $V_0 \in C([0,\infty);L^2(S\times Y_1)) \cap C^1((0,\infty);L^2(S;W_0^{1,2}(Y_1)\cap W^{2,2}(Y_1)))$ are the solutions of the system

$$\begin{aligned} \mathcal{A}_1 u(t) + \frac{1}{r_1}\left(u(t) - \int_{Y_1} V_0(\cdot,\tilde{y},t)\,d\tilde{y}\right) &= f(t) \qquad \text{in } (\mathcal{V}_1)' \\ C\frac{\partial}{\partial t}V_0(t) + \mathcal{A}_2 V_0(t) + \frac{1}{r_1}(V_0(t) - u(0)) &= 0 \qquad \text{in } \left(L^2(S;W_0^{1,2}(Y_1))\right)' , \end{aligned}$$

with initial condition $V_0(\tilde{x},\tilde{y},0) = v_0(\tilde{x})$, $(\tilde{x},\tilde{y}) \in S\times Y_1$. Then

$$\int_0^T \chi_1^\varepsilon v^\varepsilon(t)\,dt \xrightarrow{2} \int_0^T \chi_1(\tilde{y})V_0(t)\,dt \quad \text{and} \quad \int_0^T u^\varepsilon(t)\,dt \rightharpoonup \int_0^T u(t)\,dt \quad \text{in } \mathcal{V}_0$$

for every $T > 0$.

Proof. With the consequence of Section 3.3.1 and Theorem 15 in Section 3.3, we have that for each $T > 0$,

$$\int_0^T \chi_1^\varepsilon v^\varepsilon(t)\,dt \xrightarrow{2} \int_0^T \chi_1(\tilde{y})V_0(t)\,dt\ .$$

For each $T > 0$, from the facts that

$$\begin{aligned} \int_0^T u^\varepsilon(t)\,dt &= \left(\mathcal{A}_1^\varepsilon + \frac{1}{r_1}\gamma_0^*\gamma_0\right)^{-1}\left(\int_0^T \gamma_1^* f(t)\,dt + \frac{1}{r_1}\int_0^T \gamma_0^*\chi_1^\varepsilon v^\varepsilon(t)\,dt\right) , \\ \int_0^T u(t)\,dt &= \left(\mathcal{A}_1 + \frac{1}{r_1}\right)^{-1}\left(\int_0^T f(t)\,dt + \frac{1}{r_1}\int_0^T\int_Y \chi_1(\tilde{y})V_0(t,\cdot,\tilde{y})\,d\tilde{y}\,dt\right) , \end{aligned}$$

and since

$$\int_0^T \chi_1^\varepsilon v^\varepsilon(t)\,dt \rightharpoonup \int_0^T\int_Y \chi_1(\tilde{y})V_0(t,\cdot,\tilde{y})\,d\tilde{y}\,dt\ ,$$

by Proposition 8, we obtain that

$$\int_0^T u^\varepsilon(t)\,dt \rightharpoonup \int_0^T u(t)\,dt$$

in $\mathcal{V}_0$ for which $u(\tilde{x},t)$ is independent of z. □

4. Concluding Remarks

4.1. The Homogeneous Thin-layer Model

In Theorem 5, we have obtained the system of differential equations

$$\begin{aligned} \mathcal{A}_1 u + \frac{1}{r_1}(u-v) &= f \\ C\frac{\partial v}{\partial t} + \mathcal{A}_2 v + \frac{1}{r_1}(v-u) &= 0\ , \end{aligned} \tag{21}$$

with initial condition $v(\tilde{x},0) = v_0(\tilde{x})$ as the limit problem describing the thin-layer conductor with a homogeneous capacitive layer. The strong formulation of this system is

$$-\nabla \cdot g(\tilde{x})\nabla u(\tilde{x},t) + \frac{1}{r_1}(u(\tilde{x},t) - v(\tilde{x},t)) = f(\tilde{x},t)$$

$$C\frac{\partial v(\tilde{x},t)}{\partial t} + \frac{1}{r_1}(v(\tilde{x},t) - u(\tilde{x},t)) + \frac{1}{R}v(\tilde{x},t) = 0 \,, \qquad \tilde{x} \in S \,,$$

$$u = 0 \quad \text{on} \quad \partial S$$

$$\text{and} \quad v(\tilde{x},0) = v_0(\tilde{x}) \qquad \tilde{x} \in S \,.$$

Note that the second equation is an ordinary differential equation, that is, the horizontal diffusion along the capacitor, $-\tilde{\nabla} \cdot G(\tilde{x})\tilde{\nabla} v(\tilde{x},t)$, was lost in the limit.

The original solution pair, $u^\varepsilon(t)$ and $v^\varepsilon(t)$, converges *strongly* to $u(t)$ and $v(t)$ in $\mathcal{V}_0$ and $L^2(S)$, respectively, and the convergence is *uniform* in $t \in [0,T]$. Thus, we may regard the solution of the limit problem (21) as a good approximation of the solution of the exact ε-model system (2) for $\varepsilon > 0$ sufficiently small. Note that the components of this limit system satisfy the *pseudo-parabolic* equation

$$\left(C\frac{\partial}{\partial t} + \frac{1}{R}\right)(I + r_1\mathcal{A}_1)\,u + \mathcal{A}_1 u = r_1\left(C\frac{\partial}{\partial t} + \frac{1}{R}\right)f + f \,. \tag{22}$$

This equation is well-posed both forward and backward in time, so there is no regularizing effect.

Theorem 21 (Regularity of solutions) *The solution u, v of the limit problem (21) satisfies $u \in C([0,T];\mathcal{V}_0 \cap W^{2,2}(S))$ and $v \in C^1([0,T];L^2(S))$.*

In fact, the regularity of solutions is *preserved* for all time, since the Sobolev spaces are *invariant* under the generator of the C^0-group of operators which represents the solutions of (22), namely, the *bounded* operator $\frac{1}{r_1}(I - \frac{1}{r_1}(\mathcal{A}_1 + \frac{1}{r_1})^{-1}) = \frac{1}{r_1}(I - (I + r_1\mathcal{A}_1)^{-1})$. Furthermore, this bounded operator is the Yosida approximation of $\mathcal{A}_1$, so as $r_1 \to 0$, it converges strongly to $\mathcal{A}_1$. In particular, it follows from Yosida's proof of the Hille-Yosida Theorem that the solution of (22) depends continuously on r_1 as $r_1 \to 0$.

4.2. The Periodic Thin-layer Model

In Theorem 20, we have found the limit system of differential equations

$$\begin{aligned}
\mathcal{A}_1 u(t) + \frac{1}{r_1}\left(u(t) - \int_{Y_1} V_0(\cdot,\tilde{y},t)\,d\tilde{y}\right) &= f(t) \;\text{ in } (\mathcal{V}_1)' \\
C\frac{\partial}{\partial t}V_0(t) + \mathcal{A}_2 V_0(t) + \frac{1}{r_1}(V_0(t) - u(0)) &= 0 \;\text{ in } \left(L^2(S;W_0^{1,2}(Y_1))\right)' \,,
\end{aligned} \tag{23}$$

with initial condition $V_0(\tilde{x},\tilde{y},0) = v_0(\tilde{x})$, $(\tilde{x},\tilde{y}) \in S \times Y_1$, to be the two-scale limit of the exact ε-model system (16), (17) for the thin-layer conductor with a periodic capacitance layer. The strong formulation of this system is

$$
\begin{gathered}
-\nabla \cdot g(\tilde{x})\nabla u(\tilde{x},t) + \tfrac{1}{r_1}\left(u(\tilde{x},t) - \textstyle\int_{Y_1} V_0(\tilde{x},\tilde{y},t)\,d\tilde{y}\right) = f(\tilde{x},t)\ , \quad \tilde{x} \in S\ , \\
C\tfrac{\partial}{\partial t}V_0(\tilde{x},\tilde{y},t) - G\tilde{\Delta}_{\tilde{y}}V_0(\tilde{x},\tilde{y},t) + \tfrac{1}{r_1}(V_0(\tilde{x},\tilde{y},t) - u(\tilde{x},t)) + \tfrac{1}{R}V_0(\tilde{x},\tilde{y},t) = 0\ , \\
\tilde{y} \in Y_1\ , \qquad \tilde{x} \in S\ , \\
u = 0\ , \quad \text{on} \quad \partial S\ , \\
\text{and} \quad V_0 = 0\ , \quad \text{on} \quad S \times \partial Y_1\ .
\end{gathered}
$$

Note that the second differential equation is a family of parabolic equations, and they retain the local diffusion in the terms $G\tilde{\Delta}_{\tilde{y}}V_0(\tilde{x},\tilde{y},t)$. It survives as a family of independent differential equations with respect to time t and the local variable $\tilde{y}$. That is, for each $\tilde{x} \in S$, we have a second order *diffusion equation* with constant coefficients on the local cell Y_1.

It may seem counter-intuitive that the local diffusion is retained in the limit of the periodic array while it disappears in the homogeneous case which has the larger capacitance layer. The following remarks show this is due to the loss through the Dirichlet boundary conditions on the local cells.

Consider again the case of periodic thin-layer capacitors, but instead of being directly connected along their boundary, we assume they are *insulated* against lateral current flow. This leads to the Neumann boundary condition in (23) instead of the Dirichlet condition, and we lose the local diffusion.

Theorem 22 *Suppose we have*

$$\frac{\partial v^\varepsilon}{\partial \nu}(\tilde{x},t) = 0 \qquad \textit{on } \partial S_1^\varepsilon$$

in place of the boundary condition in the system (13). Then the limit problem is the system

$$
\begin{aligned}
&-\nabla \cdot g(\tilde{x})\nabla u(\tilde{x},t) + \frac{1}{r_1}\left(u(\tilde{x},t) - |Y_1|v(\tilde{x},t)\right) = f(\tilde{x},t) \quad \tilde{x} \in S \\
&C\frac{\partial}{\partial t}v(\tilde{x},t) + \frac{1}{r_1}(v(\tilde{x},t) - u(\tilde{x},t)) + \frac{1}{R}v(\tilde{x},t) = 0\ , \qquad \tilde{x} \in S \\
&\qquad u = 0\ , \quad \textit{on} \quad \partial S\ .
\end{aligned}
\tag{24}
$$

Proof. We have the same result as the case of Dirichlet boundary condition: the limit problem has the strong form

$$-\nabla \cdot g(\tilde{x})\nabla u(\tilde{x},t) + \frac{1}{r_1}\left(u(\tilde{x},t) - \int_{Y_1} V(\tilde{x},\tilde{y},t)\,d\tilde{y}\right) = f(\tilde{x},t) \qquad \tilde{x} \in S$$

$$C\frac{\partial}{\partial t}V(\tilde{x},\tilde{y},t) - G\tilde{\Delta}_{\tilde{y}}V(\tilde{x},\tilde{y},t) + \frac{1}{r_1}(V(\tilde{x},\tilde{y},t) - u(\tilde{x},t)) + \frac{1}{R}V(\tilde{x},\tilde{y},t) = 0\ ,$$

$$\tilde{y} \in Y_1 \quad \tilde{x} \in S$$

$$u = 0, \quad \text{on} \quad \partial S$$

$$\text{and} \quad \frac{\partial V}{\partial \nu} = 0 \quad \text{on} \quad S \times \partial Y_1\ .$$

From the uniqueness of the solution of this system, the observation that if $V(\tilde{x},\tilde{y},t) = v(\tilde{x},t)$ then this system is equivalent to (24), and from the existence of the solution of (24), we conclude that $V(\tilde{x},\tilde{y},t)$ is independent of $\tilde{y}$. □

4.3 Discussion

Models of distributed networks were classically developed by methods of *averaging* in which arguments often depended on intuition. Such conceptual models can also be developed from the *theory of mixtures*. The homogeneous thin-layer model (4.1) is typical of the form of such systems as are derived by these techniques. In theoretical physics and stochastic analysis, there emerged *percolation theory* as the method of choice. In mathematics there has been developed the theory of *homogenization* in which one can say that the *equations* with rapidly oscillating coefficients have been averaged by the asymptotic expansions. Distributed microstructure models such as our periodic thin-layer model (4.3) arise as the limits by this method.

The advantage of all these methods is that macroscopic properties are *deduced* from microscopic properties and the geometry of the structure. The disadvantage is that the exact physics of the microscopic model must be known. For example, in the ε-model of our periodic thin-layer model it was not clear how one should treat the flux at the boundary of the local capacitive cells. The question as to which model is better in a given situation can only be answered by experimental tests. The value of these averaging techniques is that they precisely relate the assumed structure at the micro-scale to the final model derived on the macro-scale.

A related difficulty in the use of asymptotic averaging techniques is that there are many choices possible for the scaling of both geometry and coefficients or other material properties. Often such a choice is *natural*, such as the scaling of the capacitance and conductance coefficients in our models according to the *width* of the layer. In other cases, one chooses a scaling based on the anticipated or possibly necessary outcome. The choice of the ε^{-2} scaling of the high conductance in the two models considered above was based on the requirement that the limiting macro-model should retain the coupling of the conductive region with the capacitive layer. The two methods used here are useful tools which support the modeling process wherever spatial upscaling is necessary.

References

[1] G. Allaire, "Homogenization and two-scale convergence", *SIAM J. Math. Anal.*, **23** (1992), 1482–1518.

[2] T. Arbogast, J. Douglas, and U. Hornung, "Derivation of the double porosity model of single phase flow via homogenization theory", *SIAM J. Math. Anal.*, **21** (1990), 823–836.

[3] G. W. Clark and L. A. Packer, "Two-scale convergence of degenerate evolution equations", *J. Math. Anal.*, **214** (1997), 420–438.

[4] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin-New York, 1966.

[5] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, volume 44 of *Applied Mathematical Sciences*, Springer, New York-Berlin, 1983.

[6] E. Sanchez-Palencia, *Nonhomogeneous media and Vibration theory*, volume 127 of *Lecture Notes in Phys.*, Springer-Verlag, New York, 1980.

[7] R. E. Showalter, *Hilbert space methods for Partial Differential Equations*, volume 1 of *Monographs and Studies in Mathematics*, Pitman, 1977.

[8] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, volume 49 of *Mathematical Surveys and Monographs*, American Mathematical Society, 1997.

[9] R. E. Showalter and C.H. Snyder, "A distributed RC network model with dielectric loss", *IEEE Trans. Circuits Syst.*, CAS-**33** (1986), 707–709.

[10] K. Yosida, *Functional Analysis*, Springer, Berlin-New York, 6th edition, 1980.

[11] V. V. Zhikov, S.M. Kozlov, and O.A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, Berlin-New York, 1991.