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Double-diffusion models from a highly-heterogeneous medium ☆

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Abstract

A distributed microstructure model is obtained by homogenization from an exact micro-model with continuous temperature and flux for heat diffusion through a periodically distributed highly-heterogeneous medium. This composite medium consists of two flow regions separated by a third region which forms the doubly-porous matrix structure. The homogenized system recognizes the multiple scale processes and the microscale geometry of the local structure, and it quantifies the distributed heat exchange across the internal boundaries. The classical double-diffusion models of Rubinstein (1948) and Barenblatt (1960) are obtained in non-isotropic form for the special case of quasi-static coupling in this homogenized system.

Keywords: Mathematical modeling; Homogenization; Double-diffusion model; Two scale convergence; Distributed microstructure model

1. Introduction

Consider a composite structure consisting of two distinct and separated but finely intertwined *flow path* regions embedded in a matrix and periodically distributed in a domain

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Ω with period εY, where $\varepsilon > 0$ and $Y = (0,1)^N$ is the unit cube. In the system of partial differential equations that describes the heat diffusion within this structure, an appropriate scale factor is chosen for the thermal conductivity of the matrix, and the method of two-scale convergence is used to approximate the resulting very singular system by a fully-coupled *distributed microstructure* model. This model captures the interaction between the local scale and global scale processes, it recognizes the geometry of the local cell boundaries, and it quantifies the heat exchange across the internal boundaries of the local structure. The quasi-static case of the distributed coupling in this model yields the classical *double-diffusion* models of Rubinstein [10] and Barenblatt [3]. These classical double-diffusion models are characterized by having two temperatures (or pressures) assigned to each point Ω, one for each of the two distributed components in the composite material. Flow in each component is described by a diffusion equation throughout the domain, and they are coupled by a distributed exchange that is proportional to the difference in the temperatures (or pressures) of the components. See Lee and Hill [9] for additional discussion of such models. Our overall objective is to derive these models by homogenization from a physically meaningful exact model, i.e., a model in which temperature and flux are *continuous*. We achieve this by introducing a third region which separates the other two components and provides the medium for the exchange of flux. The temperature gradient in this exchange region is necessarily very high, i.e., it is inversely proportional to the local width of the exchange region, and the conductivity in this region is of the order of the local width, in order to maintain continuity of the flux. Such ideas are implicit in the classical development of these models, and we have attempted to make them precise in the *exact micro-model* developed below. This model is a direct extension of that presented by Arbogast et al. in [2] for fluid flow in a fissured medium. In fact, their model is recovered by deleting the second flow region in our system below. In [4] and [5], a similar model is presented for the single phase flow in a partially fissured medium; see [6,7,12]. Each of these models is based on a structure composed of two subregions, the *fissure system* and the *matrix system*. In the partially fissured case, both regions are connected and contribute to the global flow. The double-diffusion model was also obtained as a two-scale limit by Hornung [8], but in his micro-model, the flux condition across the internal boundaries was assumed to be given by a difference in pressure, thereby forcing such a relationship in the macro-limit. In our micro-model below, we assume that both temperature and flux are continuous across all internal interfaces.

In order to indicate spaces of *Y-periodic* functions we will use the symbol # as a subscript. For example, $C_\#(Y)$ is the Banach space of continuous, Y-periodic, functions defined on all of $\mathbb{R}^N$. Similarly, $L^2_\#(Y)$ is the Banach space of functions in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ which are Y-periodic. For this space we take the usual norm of $L^2(Y)$ and note that $L^2_\#(Y)$ is equivalent to the space of Y-periodic extensions to $\mathbb{R}^N$ of functions in $L^2(Y)$. The space $H^1_\#(Y)$ is the Banach space of Y-periodic extensions to $\mathbb{R}^N$ of those functions in $H^1(Y)$ for which the trace or boundary values agree on opposite sides of the boundary ∂Y, and its norm is the usual norm of $H^1(Y)$. The linear space $C^\infty_\#(Y) \equiv C_\#(Y) \cap C^\infty(\mathbb{R}^N)$ is dense in both $L^2_\#(Y)$ and $H^1_\#(Y)$. The quotient space $H^1_\#(Y)/\mathbb{R}$ is defined as the space of equivalence classes up to constant functions.

Various spaces of *vector-valued* functions will arise in the developments below. Thus, if $\mathbb{B}$ is a Banach space and X is a topological space, then $C(X;\mathbb{B})$ denotes the space of continuous $\mathbb{B}$-valued functions defined on X with the corresponding supremum norm. For any measure space Ω, we let $L^2(\Omega;\mathbb{B})$ denote the space of square norm-summable functions $f:\Omega\to\mathbb{B}$ such that $\|f(\cdot)\|_{L^2(\Omega;\mathbb{B})}$ is finite. When X $(=[0,T])$ or Ω $(=(0,T))$ represent the indicated time interval, we denote the corresponding *evolution spaces* by $C([0,T];\mathbb{B})$ and $L^2(0,T;\mathbb{B})$, respectively.

The following definitions and results on *two-scale convergence* have been modified to allow for homogenization with a parameter (which we denote by t). These modifications do not affect the proofs from [1] in any essential way, and we refer the reader to [1] for a more thorough discussion.

Definition 1. A function $\psi(t,x,y)$ in $L^2((0,T)\times\Omega;C_\#(Y))$ which satisfies

$$\lim_{\varepsilon\to 0}\int\limits_{(0,T)\times\Omega}\psi\left(t,x,\frac{x}{\varepsilon}\right)^2 dx\,dt=\int\limits_{(0,T)\times\Omega}\int\limits_{Y}\psi(t,x,y)^2\,dy\,dx\,dt,$$

is called an *admissible* test function.

Definition 2. A sequence u^ε in $L^2((0,T)\times\Omega)$ *two-scale converges* to $u_0(t,x,y)$ in $L^2((0,T)\times\Omega\times Y)$ if for any admissible test function $\psi(t,x,y)$,

$$\lim_{\varepsilon\to 0}\int\limits_{(0,T)}\int\limits_{\Omega}u^\varepsilon(t,x)\psi\left(t,x,\frac{x}{\varepsilon}\right)dx\,dt=\int\limits_{(0,T)}\int\limits_{\Omega}\int\limits_{Y}u_0(t,x,y)\psi(t,x,y)\,dy\,dx\,dt.$$

Theorem 3. *If u^ε is a bounded sequence in $L^2((0,T)\times\Omega)$, then there exists a function $u_0(t,x,y)$ in $L^2((0,T)\times\Omega\times Y)$ and a subsequence of u^ε which two-scale converges to u_0. Moreover, this two-scale convergent subsequence converges weakly in $L^2((0,T)\times\Omega)$ to $u(t,x)=\int_Y u_0(t,x,y)\,dy$.*

When the sequence u^ε is bounded in H^1, we get more information.

Theorem 4. *Let u^ε be a bounded sequence in $L^2(0,T;H^1(\Omega))$ that converges weakly to u in $L^2(0,T;H^1(\Omega))$. Then u^ε two-scale converges to u, and there is a function $U(t,x,y)$ in $L^2((0,T)\times\Omega;H^1_\#(Y)/\mathbb{R})$ such that, up to a subsequence, $\nabla_{\boldsymbol{x}}u^\varepsilon$ two-scale converges to $\nabla_{\boldsymbol{x}}u(t,x)+\nabla_{\boldsymbol{y}}U(t,x,y)$ in $L^2((0,T)\times\Omega)^N$.*

Theorem 5. *Let u^ε and $\varepsilon\nabla_{\boldsymbol{x}}u^\varepsilon$ be bounded sequences in $L^2((0,T)\times\Omega)$ and $L^2((0,T)\times\Omega)^N$, respectively. Then there exists a function $U(t,x,y)$ in $L^2((0,T)\times\Omega;H^1_\#(Y)/\mathbb{R})$ such that, up to a subsequence, u^ε and $\varepsilon\nabla_{\boldsymbol{x}}u^\varepsilon$ two-scale converge to $U(t,x,y)$ and $\nabla_{\boldsymbol{y}}U(t,x,y)$ in $L^2((0,T)\times\Omega)$ and $L^2((0,T)\times\Omega)^N$, respectively.*

We review some results for degenerate operators of the type that appear below in our implicit evolution equation. Assume $\mathcal{B}$ is continuous, linear, symmetric and monotone from the Hilbert space V to its dual V'. Then $\langle\mathcal{B}\cdot,\cdot\rangle^{1/2}$ is a seminorm on V; denote the completion of this seminorm space by V_b and the dual Hilbert space by V_b'. Note that $V\hookrightarrow V_b$ is dense and continuous,

$$\|\varphi\|_{V_b}=\mathcal{B}\varphi(\varphi)^{1/2}\leqslant\|\mathcal{B}\|^{1/2}\|\varphi\|,\quad \varphi\in V,$$

and $\mathcal{B}$ has a unique continuous linear extension from V_b onto V_b'. By restriction of functionals we identify $V_b' \subset V'$, and this imbedding is continuous.

Define a seminorm on the range of $\mathcal{B}: V \to V'$ by

$$\|w\|_W = \inf\{\|v\|: v \in V,\ \mathcal{B}v = w\}, \quad w \in \mathrm{Rg}(\mathcal{B}).$$

Since the kernel $\mathrm{Ker}(\mathcal{B})$ is closed this is a norm; the corresponding normed linear space $W = \{\mathrm{Rg}(\mathcal{B}), \|\cdot\|_W\}$ is isomorphic to the quotient $V/\mathrm{Ker}(\mathcal{B})$, and it is therefore a Banach space. Finally, we observe that $W \subset V_b'$ with a continuous inclusion. Specifically, if $w = \mathcal{B}v$ with $v \in V$, then

$$|w(\varphi)| = |\mathcal{B}v(\varphi)| \leqslant \mathcal{B}v(v)^{1/2}\mathcal{B}\varphi(\varphi)^{1/2}, \quad \varphi \in V,$$

so $w \in V_b'$ and $\|w\|_{V_b'} = \mathcal{B}v(v)^{1/2} = \|v\|_{V_b} \leqslant \|\mathcal{B}\|^{1/2}\|v\|$. Taking the infimum and noting that $\mathcal{B}$ is constant on each coset, we obtain

$$\|w\|_{V_b'} \leqslant \|\mathcal{B}\|^{1/2}\|w\|_W, \quad w \in \mathrm{Rg}(\mathcal{B}).$$

The strict homomorphism $\mathcal{B}: V \to W$ has a continuous dual $\mathcal{B}': W' \to V'$ given by

$$\mathcal{B}'g(v) = g(\mathcal{B}v), \quad g \in W',\ v \in V.$$

Using the identification of $V_b \subset V_b'' \subset W'$ we obtain for each $g \in V_b$,

$$\mathcal{B}'g(v) = \mathcal{B}g(v) \leqslant \|g\|_{V_b}\|v\|_{V_b}, \quad v \in V,$$

so $\|\mathcal{B}'g\|_{V_b'} \leqslant \|g\|_{V_b}$. $\mathcal{B}'$ is an extension of $\mathcal{B}: V_b \to V_b'$ and hereafter we denote it too by $\mathcal{B}$. Now V_b' is a Hilbert space whose scalar product satisfies

$$(\mathcal{B}u, \mathcal{B}v)_{V_b'} = \mathcal{B}u(v) = (u, v)_{V_b}, \quad u, v \in V_b.$$

Hence, $(f, w)_{V_b'} = f(v)$ if $w = \mathcal{B}v$, $v \in V_b$, so we obtain for each $f \in V_b'$,

$$\sup\{|(f, w)_{V_b'}|: w \in W,\ \|w\|_W \leqslant 1\} = \sup\{|f(v)|: v \in V,\ \|v\| \leqslant 1\} = \|f\|_{V'}.$$

This shows V' has the norm *dual* to W with respect to V_b', or that V_b' is the *pivot* space between W and V'.

Set $\mathcal{V} = L^2(0, T; V)$. Note that the dual of $\mathcal{V}$ is $\mathcal{V}' = L^2(0, T; V')$.

Proposition 6. *The Hilbert space $W_2(0, T) \equiv \{u \in \mathcal{V}: (d/dt)\mathcal{B}u \in \mathcal{V}'\}$ is contained in $C([0, T]; V_b)$. Moreover, for each u in $W_2(0, T)$,*

(i) *the function $t \mapsto \mathcal{B}u(t)(u(t))$ is absolutely continuous on $[0, T]$,*
(ii) *$(d/dt)\mathcal{B}u(t)(u(t)) = 2(d/dt)\mathcal{B}u(t)(u(t))$ for a.e. t in $[0, T]$, and*
(iii) *for every u in $W_2(0, T)$, there is a constant C for which*

$$\|u\|_{C([0,T],V_b)} \leqslant C\|u\|_{W_2(0,T)}.$$

Corollary 7. *Given functions u, v in $W_2(0, T)$, the map $t \mapsto \mathcal{B}u(t)(v(t))$ is absolutely continuous on $[0, T]$ and*

$$\frac{d}{dt}\mathcal{B}u(t)(v(t)) = \frac{d}{dt}\mathcal{B}u(t)(v(t)) + \frac{d}{dt}\mathcal{B}v(t)(u(t)) \quad \text{a.e. } t \text{ in } [0, T].$$

Finally, we formulate the *implicit Cauchy problem* for an evolution equation in a Hilbert space in a form that will be convenient for our applications below. Suppose we are given a continuous linear operator $\mathcal{A}: V \to V'$, a vector w_0 in V_b', and a function $f(\cdot)$ in $\mathcal{V}'$. The *Cauchy problem* is to find a function $u(\cdot)$ in $\mathcal{V}$ such that

$$\frac{d}{dt}\big(\mathcal{B}u(\cdot)\big) + \mathcal{A}\big(u(\cdot)\big) = f(\cdot) \quad \text{in } \mathcal{V}', \qquad \mathcal{B}u(0) = w_0 \quad \text{in } V_b'. \tag{1}$$

Implicit in (1) is the fact that $(d/dt)\mathcal{B}u \in \mathcal{V}'$. It follows from Proposition 6 that $\mathcal{B}u(\cdot)$ is continuous into V_b', and the initial condition on $\mathcal{B}u(\cdot)$ is meaningful. The realization of $\mathcal{A}: V \to V'$ as an operator on $\mathcal{V}$ takes on values in $\mathcal{V}'$, and a solution $u \in \mathcal{V}$ to (1) is characterized by the *variational form*

$$u \in \mathcal{V}: \quad -\int_0^T \mathcal{B}u(t)\big(v'(t)\big)\,dt + \int_0^T \mathcal{A}u(t)\big(v(t)\big)\,dt = \int_0^T f(t)v(t)\,dt + w_0\big(v(0)\big)$$
$$\text{for every } v \text{ in } \mathcal{V}, \text{ with } \mathcal{B}v' \in \mathcal{V}' \text{ and } \mathcal{B}v(T) = 0. \tag{2}$$

See Chapter III of [11] for the above and related information on the Cauchy problem. Specifically, we recall the following result.

Proposition 8. *Assume the operators $\mathcal{A}$ and $\mathcal{B}$ are continuous, linear, and symmetric from the Hilbert space V to its dual V', that $\mathcal{B}$ is monotone, and there are numbers λ and $c > 0$ such that*

$$\mathcal{A}v(v) + \lambda\mathcal{B}v(v) \geqslant c\|v\|^2, \quad v \in V.$$

Then for each $f \in L^2(0, T; V')$ and each $w_0 \in V_b'$, there exists a unique solution u of the Cauchy problem (1)*, it is characterized by* (2)*, and it satisfies the a priori estimate*

$$\int_0^T \mathcal{A}u(t)\big(u(t)\big)\,dt + \sup_{0\leqslant t\leqslant T}\big\{\mathcal{B}u(t)\big(u(t)\big)\big\} \leqslant C(f, w_0).$$

2. The highly-heterogeneous micro-model

Let the unit cube Y be given in three complementary parts, Y_1, Y_2 and Y_3, and assume Y_3 *separates* Y_1 from Y_2, so $\partial Y_1 \cap \partial Y_2 = \emptyset$; see Fig. 1.

We denote by $\chi_j(y)$ the characteristic function of Y_j for $j = 1, 2, 3$, extended Y-periodically to all of $\mathbb{R}^N$. Thus, $\chi_1(y) + \chi_2(y) + \chi_3(y) = 1$ for a.e. y in $\mathbb{R}^N$. It is assumed that the sets $\{y \in \mathbb{R}^N\colon\ \chi_j(y) = 1\}$ for $j = 1, 2, 3$, have smooth boundary, but we do not require these sets to be connected. The corresponding ε-periodic characteristic functions are defined by

$$\chi_j^\varepsilon(x) \equiv \chi_j\left(\frac{x}{\varepsilon}\right), \quad x \in \Omega,\ j = 1, 2, 3.$$

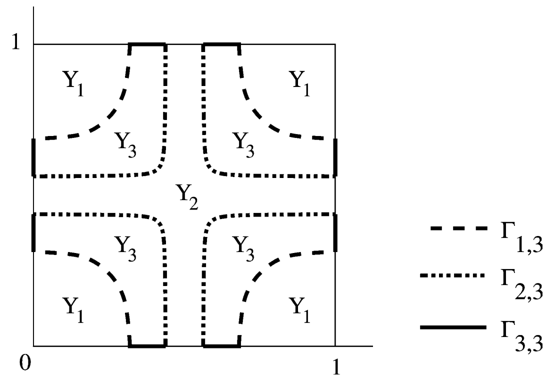

Fig. 1. Two-dimensional representation of the unit cube $Y = (0,1)^N$.

The global domain Ω is thus divided into three sub-domains, Ω_1^ε, Ω_2^ε and Ω_3^ε, which are defined by

$$\Omega_j^\varepsilon \equiv \{x \in \Omega\colon\ \chi_j^\varepsilon(x) = 1\}, \quad j = 1,2,3.$$

We use the characteristic functions as multipliers to denote the *zero-extension* of various functions. For example, given a function w defined on Ω_j^ε, the product $\chi_j^\varepsilon w$ is the zero extension of w to all of Ω. Similarly, if w is given on Y_j then $\chi_j w$ is the corresponding zero-extension to all of Y. Conversely, $H_\#^1(Y_j)$ will be used to denote the restriction to Y_j of functions from $H_\#^1(Y)$. Finally, we denote by $\gamma_j w$ the *trace* or restriction to the boundary ∂Y_j of functions $w \in H^1(Y_j)$, and similarly by $\gamma_j^\varepsilon w$ the trace on $\partial\Omega_j^\varepsilon$ of functions $w \in H^1(\Omega_j^\varepsilon)$.

The two sub-domains Ω_1^ε and Ω_2^ε are the primary *flow path* regions for the model, and it is assumed that their corresponding conductivities μ_j $(j = 1,2)$ are large relative to the conductivity of the third region Ω_3^ε, which we call the *exchange region*. A *radiator* in $\mathbb{R}^3$ is an example of such a medium with the same geometric structure; see Fig. 2. For $j = 1,2$, let $\Gamma_{j,3} \equiv \partial Y_j \cap \partial Y_3 \cap Y$ be that part of the interface between Y_j and Y_3 that is interior to the local cell Y. Then $\Gamma_{j,3}^\varepsilon \equiv \partial\Omega_j^\varepsilon \cap \partial\Omega_3^\varepsilon \cap \Omega$ represents the corresponding interface between Ω_j^ε and Ω_3^ε that is interior to Ω. Likewise, we define $\Gamma_{3,3} \equiv \partial Y_3 \cap \partial Y$ and denote by $\Gamma_{3,3}^\varepsilon$ its periodic extension which forms the artificial interface between those parts of the matrix Ω_3^ε that lie within neighboring εY-cells. A two-dimensional view of the boundaries for the unit cube Y is shown in Fig. 1.

Let $c_j(\cdot), \mu_j(\cdot) \in C_\#(Y)$ be given such that

$$0 \leqslant c_j(y), \quad 0 < c_0 \leqslant \mu_j(y), \quad \text{a.e. } y \in \mathbb{R}^3,\ j = 1,2,3. \tag{3}$$

We note that the heat capacities $c_j(y)$ $(j = 1,2,3)$ are bounded and permitted to vanish. The corresponding ε-periodic coefficients in Ω_j^ε are defined by

$$c_j^\varepsilon(x) \equiv c_j\left(\frac{x}{\varepsilon}\right), \quad \mu_j^\varepsilon(x) \equiv \mu_j\left(\frac{x}{\varepsilon}\right), \quad x \in \Omega_j^\varepsilon,\ j = 1,2,3.$$

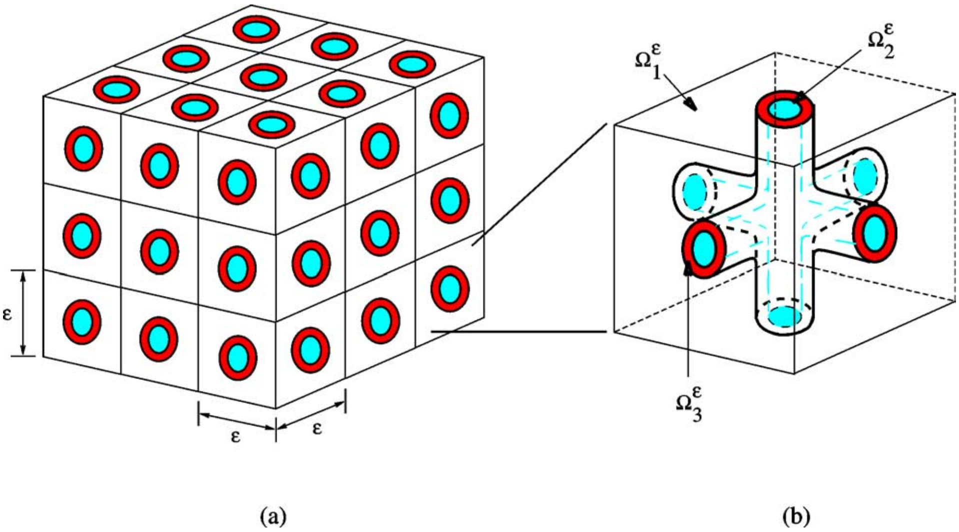

Fig. 2. (a) Three-dimensional representation of periodic structure. (b) An extended view of composite medium over one ε-period.

The temperature at $x \in \Omega$ is denoted by $u(t,x)$, and for $j = 1,2$, the corresponding flux in Ω_j^ε is $-\mu_j(x/\varepsilon)\nabla u^\varepsilon$. In Ω_3^ε, the flux is given by $-\varepsilon^2\mu_3(x/\varepsilon)\nabla u^\varepsilon$. The diffusion of heat within Ω is described by the *exact micro-model*

$$\frac{\partial}{\partial t}\big(c_1^\varepsilon(x)u^\varepsilon(t,x)\big) = \nabla\cdot\big(\mu_1^\varepsilon(x)\nabla u^\varepsilon(t,x)\big), \qquad x\in\Omega_1^\varepsilon,\ t>0, \tag{4a}$$

$$\frac{\partial}{\partial t}\big(c_2^\varepsilon(x)u^\varepsilon(t,x)\big) = \nabla\cdot\big(\mu_2^\varepsilon(x)\nabla u^\varepsilon(t,x)\big), \qquad x\in\Omega_2^\varepsilon,\ t>0, \tag{4b}$$

$$\frac{\partial}{\partial t}\big(c_3^\varepsilon(x)u^\varepsilon(t,x)\big) = \nabla\cdot\big(\varepsilon^2\mu_3^\varepsilon(x)\nabla u^\varepsilon(t,x)\big), \quad x\in\Omega_3^\varepsilon,\ t>0, \tag{4c}$$

$$\gamma_1^\varepsilon u^\varepsilon(t,s) = \gamma_3^\varepsilon u^\varepsilon(t,s), \quad s\in\Gamma_{1,3}^\varepsilon,\ t>0, \tag{4d}$$

$$\gamma_2^\varepsilon u^\varepsilon(t,s) = \gamma_3^\varepsilon u^\varepsilon(t,s), \quad s\in\Gamma_{2,3}^\varepsilon,\ t>0, \tag{4e}$$

$$\mu_1^\varepsilon(s)\nabla u^\varepsilon(t,s)\cdot\boldsymbol{\nu}_1 = \varepsilon^2\mu_3^\varepsilon(s)\nabla u^\varepsilon(t,s)\cdot\boldsymbol{\nu}_1, \quad s\in\Gamma_{1,3}^\varepsilon,\ t>0, \tag{4f}$$

$$\mu_2^\varepsilon(s)\nabla u^\varepsilon(t,s)\cdot\boldsymbol{\nu}_2 = \varepsilon^2\mu_3^\varepsilon(s)\nabla u^\varepsilon(t,s)\cdot\boldsymbol{\nu}_2, \quad s\in\Gamma_{2,3}^\varepsilon,\ t>0, \tag{4g}$$

where $\boldsymbol{\nu}_j$ denotes the unit outward normal on $\partial\Omega_j^\varepsilon$, $j = 1,2,3$. For $j = 1,2$, note that $\boldsymbol{\nu}_j = -\boldsymbol{\nu}_3$ on $\Gamma_{j,3}^\varepsilon$. In the above system, a standard diffusion process takes place in each subregion, and both temperature and flux are continuous across the internal boundaries. Within any small neighborhood, the temperature is expected to be nearly constant in each flow region, e.g., in Y_1 and Y_2. Thus, all essential fine-scale variations in temperature will occur in the exchange region, Y_3, so in Ω_3^ε the flux has been scaled by ε^2 to allow for the steep temperature gradients (of the order $1/\varepsilon$) that necessarily exist within this region. We will show that as $\varepsilon \to 0$ this scale factor has exactly the right order of magnitude to produce a distributed microstructure model that is *fully-coupled*. Since the global boundary conditions on $\partial\Omega$ play no essential role in the development below, we shall assume homogeneous Dirichlet boundary conditions

$$u^\varepsilon(t,s)=0 \quad \text{a.e. } s\in\partial\Omega. \tag{4h}$$

Finally, the initial-boundary-value problem is completed with the initial conditions

$$c_j^\varepsilon(x)u^\varepsilon(0,x)=c_j^\varepsilon(x)u_0(x) \quad \text{a.e. } x\in\Omega,\ j=1,2,3. \tag{4i}$$

As noted above, the model developed by Arbogast et al. [2] for fluid flow in a fissured medium is recovered by setting $Y_2=\emptyset$.

Next we shall develop an equivalent *variational formulation* for our exact micro-model in the *energy space* $V\equiv H_0^1(\Omega)$. Then the continuity of temperature (4d) and (4e) follow from the inclusion $u\in V$. The leading terms in our system are given by the linear degenerate operator $\mathcal{B}^\varepsilon : V\to V'$ defined by

$$\mathcal{B}^\varepsilon u(\varphi)=\int_\Omega c^\varepsilon(x)u(x)\varphi(x)\,dx,\quad u,\varphi\in V,$$

where $c^\varepsilon(x)\equiv\chi_1^\varepsilon(x)c_1^\varepsilon(x)+\chi_2^\varepsilon(x)c_2^\varepsilon(x)+\chi_3^\varepsilon(x)c_3^\varepsilon(x)$. The completion of V with the corresponding semi-scalar product can be characterized as the space V_b^ε of those measurable functions $u(\cdot)$ on the support of $c^\varepsilon(\cdot)$ for which $c^\varepsilon(\cdot)^{1/2}u(\cdot)\in L^2(\Omega)$, and the dual of this space is the Hilbert space $(V_b^\varepsilon)'=\{c^\varepsilon(\cdot)^{1/2}\varphi(\cdot)\colon\ \varphi\in L^2(\Omega)\}$. The symmetric and nonnegative operator $\mathcal{B}^\varepsilon$ is just multiplication by $c^\varepsilon(\cdot)$. The principle operator $\mathcal{A}^\varepsilon : V\to V'$ is defined by

$$\mathcal{A}^\varepsilon u(\varphi)\equiv\int_\Omega \mu^\varepsilon(x)\nabla u(x)\cdot\nabla\varphi(x)\,dx,\quad u,\varphi\in V,$$

where $\mu^\varepsilon(x)\equiv\chi_1^\varepsilon(x)\mu_1^\varepsilon(x)+\chi_2^\varepsilon(x)\mu_2^\varepsilon(x)+\varepsilon^2\chi_3^\varepsilon(x)\mu_3^\varepsilon(x)$. The formal part of this operator in $L^2(\Omega)$ consists of the elliptic parts of (4a)–(4c), and the remaining part contains the flux interface conditions (4f), (4g).

The variational form of the *exact micro-model* given by (4) is the *Cauchy problem* to find $u^\varepsilon(\cdot)\in L^2(0,T;H_0^1(\Omega))$ such that

$$\begin{cases} \frac{d}{dt}\mathcal{B}^\varepsilon u^\varepsilon(t)+\mathcal{A}^\varepsilon u^\varepsilon(t)=0 & \text{in } (L^2(0,T;H_0^1(\Omega)))', \\ c^\varepsilon(\cdot)^{1/2}u^\varepsilon(0)=c^\varepsilon(\cdot)^{1/2}u_0 & \text{in } L^2(\Omega). \end{cases} \tag{5}$$

Here the initial value u_0 could be prescribed in V_b^ε.

Each of the operators $\mathcal{A}^\varepsilon$ is linear, symmetric, and V-elliptic.

Lemma 9. *For each $\varepsilon>0$, there exists a constant $C_\varepsilon>0$ such that*

$$\mathcal{A}^\varepsilon(w)(w)\geqslant C_\varepsilon\|w\|_V^2 \quad \textit{for every } w \textit{ in } V.$$

It follows from Lemma 9 that (5) is well posed. We remark that the coercivity estimate given in this lemma permits even the degenerate *steady-state case* $c(\cdot)=0$. However, we need additional estimates *independent of* ε for the L^2 norm of the solution in our following work.

Denote the *rescaling* of the functions $\mu^\varepsilon(\cdot)$ and $c^\varepsilon(\cdot)$ to $\varepsilon = 1$ by

$$\mu(y) \equiv \chi_1(y)\mu_1(y) + \chi_2(y)\mu_2(y) + \chi_3(y)\mu_3(y)$$

and

$$c(y) \equiv \chi_1(y)c_1(y) + \chi_2(y)c_2(y) + \chi_3(y)c_3(y),$$

respectively. Then we have the following

Lemma 10. *If $\int_Y c(y)\,dy > 0$, then there exists a constant $K > 0$, such that*

$$\int_Y \mu(y)\left|\nabla w(y)\right|^2 dy + \int_Y c(y)\left|w(y)\right|^2 dy \geqslant K\|w\|^2_{L^2(Y)} \tag{6}$$

for all w in $H^1(Y)$.

This follows by a compactness argument; see Proposition II.5.2 in [11].

Lemma 11. *There exists a constant $K > 0$, such that*

$$\int_\Omega \mu^\varepsilon(x)\left|\nabla w(x)\right|^2 dx + \int_\Omega c^\varepsilon(x)\left|w(x)\right|^2 dx \geqslant K\|w\|^2_{L^2(\Omega)} \tag{7}$$

for all w in $H^1(\Omega)$ and $0 < \varepsilon \leqslant 1$.

Proof. Recall that $Y^\varepsilon = \varepsilon Y = \{x = \varepsilon y\colon\ y \in Y\}$. For a given $w \in H^1(Y^\varepsilon)$, the change of variable $y = x/\varepsilon$ yields

$$\begin{aligned}
&\int_{Y_1^\varepsilon} \mu_1^\varepsilon(x)\left|\nabla w(x)\right|^2 dx + \int_{Y_2^\varepsilon} \mu_2^\varepsilon(x)\left|\nabla w(x)\right|^2 dx \\
&\quad + \int_{Y_3^\varepsilon} \mu_3^\varepsilon(x)\left|\varepsilon\nabla w(x)\right|^2 dx + \int_{Y^\varepsilon} c^\varepsilon(x)\left|w(x)\right|^2 dx \\
&\geqslant \varepsilon^N\Bigg[\int_{Y_1} \mu_1(y)\left|\nabla_y w(y)\right|^2 dy + \int_{Y_2} \mu_2(y)\left|\nabla_y w(y)\right|^2 dy \\
&\quad + \int_{Y_3} \mu_3(y)\left|\nabla_y w(y)\right|^2 dy + \int_Y c(y)\left|w(y)\right|^2 dy\Bigg] \\
&\geqslant \varepsilon^N K\|w\|^2_{L^2(Y)}.
\end{aligned} \tag{8}$$

The first inequality in (8) is a result of the change of variable $y = x/\varepsilon$, which leads to integration over the fixed domain Y, and the second inequality comes from applying our first “Poincare-type” inequality obtained in (6). Using $x = \varepsilon y$ to change back to the micro-variable yields

$$\int_{Y^\varepsilon} \mu^\varepsilon(x)\left|\nabla w(x)\right|^2 dx + \int_{Y^\varepsilon} c^\varepsilon(x)\left|w(x)\right|^2 dx \geqslant K\|w\|^2_{L^2(Y^\varepsilon)}. \tag{9}$$

The domain Ω is covered by a regular mesh of size ε: each cell Y_i^ε is a translate of the ε-cell, Y^ε. Denote by $\tilde{\Omega}_\varepsilon$ the union of all these cells, that is, $\tilde{\Omega}_\varepsilon = \bigcup_{i=1}^{N(\varepsilon)} Y_i^\varepsilon$, where the number of cells is $N(\varepsilon) = |\Omega|\varepsilon^{-N}[1 + o(1)]$. We note that $\Omega \subset \tilde{\Omega}_\varepsilon$ and hence by zero-extensions we have $H_0^1(\Omega) \subset H_0^1(\tilde{\Omega}_\varepsilon)$.

Now, let $w \in H_0^1(\Omega)$. By summing the inequalities in (9) over all of the Y_i^ε-cells, we get

$$\int_{\tilde{\Omega}_\varepsilon} \mu^\varepsilon(x)\left|\nabla w(x)\right|^2 dx + \int_{\tilde{\Omega}_\varepsilon} c^\varepsilon(x)\left|w(x)\right|^2 dx \geqslant K\|w\|_{L^2(\tilde{\Omega}_\varepsilon)}^2.$$

Replace $\tilde{\Omega}_\varepsilon$ by Ω in the domain of integration to obtain the desired estimate. □

Theorem 12. *Assume* (3), *that* $\int_Y c(y)\,dy > 0$, *and* $u_0 \in L^2(\Omega)$. *Then for each* $0 < \varepsilon \leqslant 1$ *there exists a unique solution* u^ε *of the Cauchy problem* (5), *and these satisfy the uniform estimate* $\|u^\varepsilon\|_{L^2((0,T)\times\Omega)} \leqslant C$.

Proof. Given w in V^ε,

$$\begin{aligned}
&\mathcal{A}^\varepsilon(w)(w) + \mathcal{B}^\varepsilon(w)(w)\\
&\quad\geqslant \frac{1}{2}\left[\int_\Omega \mu^\varepsilon(x)\left|\nabla w(x)\right|^2 dx\right] + \frac{1}{2}\left[\int_\Omega \mu^\varepsilon(x)\left|\nabla w(x)\right|^2 dx + \int_\Omega c^\varepsilon(x)\left|w(x)\right|^2 dx\right]\\
&\quad\geqslant \frac{1}{2}\int_\Omega \mu^\varepsilon(x)\left|\nabla w(x)\right|^2 dx + \frac{1}{2}K\|w\|_{L^2(\Omega)}^2\\
&\quad\geqslant \frac{c_0\varepsilon}{2}\|\nabla w\|_{L^2(\Omega)}^2 + \frac{1}{2}K\|w\|_{L^2(\Omega)}^2. \quad \square
\end{aligned}$$

3. The two-scale limit

Let us begin this section by giving some preliminary convergence results. In the following, we shall denote the gradient in the x-variable by ∇, the gradient in the y-variable by ∇_y, and we use the symbol "$\xrightarrow{2}$" to denote two-scale convergence.

Lemma 13. *For each* $\varepsilon > 0$, *let* $u^\varepsilon(\cdot)$ *denote the unique solution to the Cauchy problem* (5). *Then there exist*

(i) *a pair of functions* u_j *in* $L^2(0,T;H_0^1(\Omega))$, $j = 1, 2$,
(ii) *a triple of functions* U_j *in* $L^2((0,T)\times\Omega; H_\#^1(Y_j)/\mathbb{R})$,

and a subsequence of $u^\varepsilon(\cdot)$, *hereafter denoted by* u^ε, *which two-scale converges as follows*:

$$\chi_j^\varepsilon u^\varepsilon \xrightarrow{2} \chi_j(y)u_j(t,x), \quad j=1,2, \tag{10a}$$

$$\chi_j^\varepsilon \nabla u^\varepsilon \xrightarrow{2} \chi_j(y)\big[\nabla u_j(t,x)+\nabla_y U_j(t,x,y)\big], \quad j=1,2, \tag{10b}$$

$$\chi_3^\varepsilon u^\varepsilon \xrightarrow{2} \chi_3(y)U_3(t,x,y), \tag{10c}$$

$$\chi_3^\varepsilon \varepsilon\nabla u^\varepsilon \xrightarrow{2} \chi_3(y)\nabla_y U_3(t,x,y), \tag{10d}$$

$$\chi_j^\varepsilon \mu_j^\varepsilon \nabla u^\varepsilon \xrightarrow{2} \chi_j(y)\mu_j(y)\big[\nabla u_j(t,x)+\nabla_y U_j(t,x,y)\big], \quad j=1,2, \tag{10e}$$

$$\chi_3^\varepsilon \varepsilon\mu_3^\varepsilon \nabla u^\varepsilon \xrightarrow{2} \chi_3(y)\mu_3(y)\nabla_y U_3(t,x,y). \tag{10f}$$

Proof. Apply the evolution equation (5) to the solution $u^\varepsilon(t)$. Integrating with respect to t, we obtain

$$\frac{1}{2}\mathcal{B}u^\varepsilon(t)\big(u^\varepsilon(t)\big)-\frac{1}{2}\mathcal{B}u^\varepsilon(0)\big(u^\varepsilon(0)\big)+\int_0^t \mathcal{A}^\varepsilon u^\varepsilon(s)\big(u^\varepsilon(s)\big)\,ds=0.$$

With the assumptions given in (3), in particular $0<c_0\leqslant\mu_j$, $j=1,2,3$, we have

$$\begin{aligned}
&\frac{1}{2}\big\|(c^\varepsilon)^{1/2}u^\varepsilon(t)\big\|^2_{L^2(\Omega)}\\
&\qquad+c_0\int_0^t\Big(\big\|\chi_1^\varepsilon\nabla u^\varepsilon\big\|^2_{L^2(\Omega)}+\big\|\chi_2^\varepsilon\nabla u^\varepsilon\big\|^2_{L^2(\Omega)}+\big\|\varepsilon\chi_3^\varepsilon\nabla u^\varepsilon\big\|^2_{L^2(\Omega)}\Big)\,ds\\
&\quad\leqslant\frac{1}{2}\big\|(c^\varepsilon)^{1/2}u^0\big\|^2_{L^2(\Omega)},\quad t\in[0,T].
\end{aligned}$$

This estimate shows that $(c^\varepsilon)^{1/2}u^\varepsilon(\cdot)$ is bounded in $L^\infty(0,T;L^2(\Omega))$. In addition, the gradients $\chi_1^\varepsilon\nabla u^\varepsilon$, $\chi_2^\varepsilon\nabla u^\varepsilon$ and $\varepsilon\chi_3^\varepsilon\nabla u^\varepsilon$ are bounded in $L^2(0,T;L^2(\Omega)^N)$. Statements (10a) and (10b) follow from Theorem 4. Likewise, statements (10c) and (10d) follow from Theorem 5. Finally, the flux terms $\chi_j^\varepsilon\mu_j^\varepsilon(x)\nabla u^\varepsilon(t,x)$, $j=1,2$, and $\chi_3^\varepsilon\varepsilon\mu_3^\varepsilon(x)\nabla u^\varepsilon(t,x)$ are bounded in $L^2(0,T;L^2(\Omega)^N)$. By Theorem 3, $\chi_1^\varepsilon\mu_1^\varepsilon(x)\nabla u^\varepsilon(t,x)$, $\chi_2^\varepsilon\mu_2^\varepsilon(x)\nabla u^\varepsilon(t,x)$, and $\varepsilon\chi_3^\varepsilon\mu_3^\varepsilon(x)\nabla u^\varepsilon(t,x)$ converge as stated. □

By writing $u^\varepsilon=\chi_1^\varepsilon u^\varepsilon+\chi_2^\varepsilon u^\varepsilon+\chi_3^\varepsilon u^\varepsilon$, we find that Lemma 13 implies

$$\begin{aligned}
&u^\varepsilon\xrightarrow{2}\chi_1(y)u_1(t,x)+\chi_2(y)u_2(t,x)+\chi_3(y)U_3(t,x,y),\\
&\varepsilon\nabla u^\varepsilon\xrightarrow{2}\chi_3(y)\nabla_y U_3(t,x,y),
\end{aligned}$$

and for any test function $\boldsymbol{\varphi}$ in $C_0^\infty(\Omega;[C_\#^\infty(Y)]^N)$,

$$\begin{aligned}
&\int_\Omega\varepsilon\nabla u^\varepsilon(t,x)\cdot\boldsymbol{\varphi}\left(x,\frac{x}{\varepsilon}\right)dx\\
&\quad=-\int_\Omega u^\varepsilon(t,x)\left[\varepsilon\nabla\cdot\boldsymbol{\varphi}\left(x,\frac{x}{\varepsilon}\right)+\nabla_y\cdot\boldsymbol{\varphi}\left(x,\frac{x}{\varepsilon}\right)\right]dx.
\end{aligned}$$

By taking two-scale limits on both sides of this last equation, we obtain

$$\int_\Omega \int_Y \chi_3(y)\nabla_{\boldsymbol y} U_3(t,x,y)\cdot \boldsymbol\varphi(x,y)\,dy\,dx$$
$$= -\int_\Omega \int_Y \bigl[\chi_1(y)u_1(t,x) + \chi_2(y)u_2(t,x) + \chi_3(y)U_3(t,x,y)\bigr]\nabla_{\boldsymbol y}\cdot \boldsymbol\varphi(x,y)\,dy\,dx. \tag{11}$$

The divergence theorem shows that the left side of (11) is simply

$$\int_\Omega \int_{Y_3} \nabla_{\boldsymbol y} U_3(t,x,y)\cdot \boldsymbol\varphi(x,y)\,dy\,dx$$
$$= -\int_\Omega \int_{Y_3} U_3(t,x,y)\nabla_{\boldsymbol y}\cdot \boldsymbol\varphi(x,y)\,dy\,dx + \int_\Omega \int_{\partial Y_3} U_3(t,x,s)\boldsymbol\varphi(x,s)\cdot \boldsymbol\nu_3\,ds\,dx,$$

while the right side of (11) can be written as

$$-\int_\Omega \int_{Y_1} u_1(t,x)\nabla_{\boldsymbol y}\cdot \boldsymbol\varphi(x,y)\,dy\,dx - \int_\Omega \int_{Y_2} u_2(t,x)\nabla_{\boldsymbol y}\cdot \boldsymbol\varphi(x,y)\,dy\,dx$$
$$-\int_\Omega \int_{Y_3} U_3(t,x,y)\nabla_{\boldsymbol y}\cdot \boldsymbol\varphi(x,y)\,dy\,dx.$$

Combining these last two results, (11) yields

$$\int_\Omega \int_{\partial Y_3} U_3(t,x,s)\boldsymbol\varphi(x,s)\cdot \boldsymbol\nu_3\,ds\,dx$$
$$= -\int_\Omega \int_{Y_1} u_1(t,x)\nabla_{\boldsymbol y}\cdot \boldsymbol\varphi(x,y)\,dy\,dx - \int_\Omega \int_{Y_2} u_2(t,x)\nabla_{\boldsymbol y}\cdot \boldsymbol\varphi(x,y)\,dy\,dx$$
$$= -\int_\Omega \int_{\partial Y_1} u_1(t,x)\boldsymbol\varphi(x,s)\cdot \boldsymbol\nu_1\,ds\,dx - \int_\Omega \int_{\partial Y_2} u_2(t,x)\boldsymbol\varphi(x,s)\cdot \boldsymbol\nu_2\,ds\,dx.$$

Given the periodicity of U_3 and $\boldsymbol\varphi$ on $\Gamma_{3,3}$ we obtain the following constraint in the limit.

Lemma 14. *The two-scale limits* $[u_1, u_2, U_3]$ *obtained in Lemma* 13 *satisfy the continuity condition*

$$u_j(t,x) = \gamma_3 U_3(t,x,s), \quad s \in \Gamma_{j,3},\ j = 1,2,\ \text{a.e. } x \in \Omega.$$

This shows that the temperature in the local cell Y_3 matches the corresponding macro-temperatures across its respective internal boundaries.

Next, we develop the variational problem satisfied by these two-scale limits. Choose smooth test functions

(i) φ_j in $L^2(0,T;H_0^1(\Omega))$, $j=1,2$, and
(ii) Φ_j in $L^2((0,T)\times\Omega;H_\#^1(Y_j)/\mathbb{R})$, $j=1,2,3$,

such that $\partial\varphi_j/\partial t$ is in $L^2(0,T;[H_0^1(\Omega)]')$ for $j=1,2$, $\partial\Phi_3/\partial t$ is in $L^2((0,T)\times\Omega;[H_\#^1(Y_3)/\mathbb{R}]')$, $\Phi_3(T)=0$, and $\gamma_3\Phi_3=\varphi_j$ on $\Gamma_{j,3}$, $j=1,2$. For each $j=1,2$, we can assume that the extensions of Φ_j into Y_3 have no common support. Also, we use the notation $(\cdot)_{,t}$ to represent the time derivative $(\partial/\partial t)(\cdot)$. If we apply (5) to the triple $[\varphi_1(t,x)+\varepsilon\Phi_1(t,x,x/\varepsilon),\varphi_2(t,x)+\varepsilon\Phi_2(t,x,x/\varepsilon),\Phi_3^\varepsilon(t,x,x/\varepsilon)]$ in $L^2(0,T;V^\varepsilon)$, where the function $\Phi_3^\varepsilon(t,x,y)$ is defined by

$$\Phi_3^\varepsilon(t,x,y)\equiv\Phi_3(t,x,y)+\varepsilon\sum_{j=1}^{2}\Phi_j(t,x,y)\quad\text{in } Y_j,\ \text{for } j=1,2,$$

and integrate by parts in t, we obtain

$$\begin{aligned}
&-\sum_{j=1}^{2}\int_0^T\int_\Omega \chi_j^\varepsilon(x)c_j^\varepsilon(x)u^\varepsilon(t,x)\left(\varphi_{j,t}(t,x)+\varepsilon\Phi_{j,t}\left(t,x,\frac{x}{\varepsilon}\right)\right)dx\,dt\\
&\quad-\int_0^T\int_\Omega \chi_3^\varepsilon(x)c_3^\varepsilon(x)u^\varepsilon(t,x)\Phi_{3,t}^\varepsilon\left(t,x,\frac{x}{\varepsilon}\right)dx\,dt\\
&\quad-\sum_{j=1}^{2}\int_\Omega \chi_j^\varepsilon(x)c_j^\varepsilon(x)u_0(x)\left(\varphi_j(0,x)+\varepsilon\Phi_j\left(0,x,\frac{x}{\varepsilon}\right)\right)dx\\
&\quad-\int_\Omega \chi_3^\varepsilon(x)c_3^\varepsilon(x)u_0(x)\Phi_3^\varepsilon\left(0,x,\frac{x}{\varepsilon}\right)dx\\
&\quad+\sum_{j=1}^{2}\int_0^T\int_\Omega \chi_j^\varepsilon(x)\mu_j^\varepsilon(x)\nabla u^\varepsilon(t,x)\cdot\nabla\left(\varphi_j(t,x)+\varepsilon\Phi_j\left(t,x,\frac{x}{\varepsilon}\right)\right)dx\,dt\\
&\quad+\int_0^T\int_\Omega \chi_3^\varepsilon(x)\mu_3^\varepsilon(x)\varepsilon\nabla u^\varepsilon(t,x)\cdot\varepsilon\nabla\Phi_3^\varepsilon\left(t,x,\frac{x}{\varepsilon}\right)dx\,dt=0. \qquad (12)
\end{aligned}$$

Taking two-scale limits in (12) yields the following result.

Theorem 15. *Assume the coefficients satisfy* (3) *and* $\int_Y c(y)\,dy>0$, *and that the function* $u_0(\cdot)$ *is in* $L^2(\Omega)$. *Then the two-scale limits* $[u_1,u_2,U_1,U_2,U_3]$ *established in Lemma* 13 *satisfy the two-scale limit system*

(i) u_j *in* $L^2(0,T;H_0^1(\Omega))$ *for* $j=1,2$,
(ii) U_j *in* $L^2((0,T)\times\Omega;H_\#^1(Y_j)/\mathbb{R})$ *for* $j=1,2,3$,
(iii) $\gamma_3U_3(t,x,s)=u_j(t,x)$ *on* $\Gamma_{j,3}$ *for* $j=1,2$,

such that

$$
\begin{aligned}
&-\sum_{j=1}^{2}\int_0^T\int_\Omega\int_Y \chi_j(y)c_j(y)u_j(t,x)\varphi_{j,t}(t,x)\,dy\,dx\,dt \\
&\quad-\int_0^T\int_\Omega\int_Y \chi_3(y)c_3(y)U_3(t,x,y)\Phi_{3,t}(t,x,y)\,dy\,dx\,dt \\
&\quad-\sum_{j=1}^{2}\int_\Omega\int_Y \chi_j(y)c_j(y)u_0(x)\varphi_j(0,x)\,dy\,dx \\
&\quad-\int_\Omega\int_Y \chi_3(y)c_3(y)u_0(x)\Phi_3(0,x,y)\,dy\,dx \\
&\quad+\sum_{j=1}^{2}\int_0^T\int_\Omega\int_Y \chi_j(y)\mu_j(y)\big(\nabla u_j(t,x)+\nabla_{\boldsymbol{y}}U_j(t,x,y)\big) \\
&\quad\times\big(\nabla\varphi_j(t,x)+\nabla_{\boldsymbol{y}}\Phi_j(t,x,y)\big)\,dy\,dx\,dt \\
&\quad+\int_0^T\int_\Omega\int_Y \chi_3(y)\mu_3(y)\nabla_{\boldsymbol{y}}U_3(t,x,y)\cdot\nabla_{\boldsymbol{y}}\Phi_3(t,x,y)\,dy\,dx\,dt=0
\end{aligned}
\tag{13}
$$

for all

(i) φ_j *in* $L^2(0,T;H_0^1(\Omega))$, $j=1,2$,
(ii) Φ_j *in* $L^2((0,T)\times\Omega;H_\#^1(Y_j)/\mathbb{R})$, $j=1,2,3$,

for which

(iii) $\partial\varphi_j/\partial t\in L^2(0,T;[H_0^1(\Omega)]')$, $j=1,2$,
(iv) $\partial\Phi_3/\partial t\in L^2((0,T)\times\Omega;[H_\#^1(Y_j)/\mathbb{R}]')$, $\Phi_3(T)=0$ *in* $L^2(\Omega\times Y_3)$, *and*
(v) $\gamma_3\Phi_3=\varphi_j$ *on* $\Gamma_{j,3}$, $j=1,2$.

Note that conditions (iv) and (v) together imply that each $\varphi_j(T)=0$ in $L^2(\Omega)$. Uniqueness of u_1, u_2, and U_3 will follow from the proof of Theorem 18 below. Moreover, U_1 and U_2 are determined to within a constant for each $t\in(0,T)$, so each of these is unique up to a corresponding function of t.

4. The homogenized system

We shall eliminate the variables U_1 and U_2 and obtain a closed system for the remaining three unknowns. This is possible since there are no constraints between these variables and the remaining unknowns. Thus, for each $k=1,2$, consider the *cell problem*:

Find U_k in $L^2((0,T)\times\Omega; H^1_\#(Y_k)/\mathbb{R})$:

$$\int_0^T\int_\Omega\int_{Y_k}\mu_k(y)\big(\nabla u_k(t,x)+\nabla_{\boldsymbol{y}}U_k(t,x,y)\big)\cdot\nabla_{\boldsymbol{y}}\Phi(t,x,y)\,dy\,dx\,dt=0$$

for every Φ in $L^2((0,T)\times\Omega; H^1_\#(Y_k)/\mathbb{R})$.

This is obtained by setting each $\varphi_j=0$ and $\Phi_3=0$ in (13). The input to this problem is the gradient $\nabla u_k(t,x)$, which we can write in terms of the orthonormal basis vectors as

$$\nabla u_k(t,x)=\sum_{i=1}^N\frac{\partial u_k}{\partial x_i}(t,x)\boldsymbol{e}_i.$$

This is independent of y, so the functions $U_k(t,x,y)$ can be represented with *separated variables* by

$$U_k(t,x,y)=\sum_{i=1}^N\frac{\partial u_k}{\partial x_i}(t,x)W_i^k(y),$$

where the functions $W_i^k(y)$ are solutions of the following *coefficient cell problems*.

Definition 16. For each $k=1,2$ and $1\leqslant i\leqslant N$, let the function $W_i^k(y)\in H^1_\#(Y_k/\mathbb{R})$ satisfy

$$\int_{Y_k}\mu_k(y)\big[\boldsymbol{e}_i+\nabla_{\boldsymbol{y}}W_i^k(y)\big]\cdot\nabla_{\boldsymbol{y}}\Phi(y)\,dy=0\quad\text{for }\Phi\in H^1_\#(Y_k)/\mathbb{R}.$$

The corresponding strong forms are the Neumann-periodic boundary-value problems

$$\begin{cases}\nabla_{\boldsymbol{y}}\cdot(\mu_k(y)[\boldsymbol{e}_i+\nabla_{\boldsymbol{y}}W_i^k(y)])=0 & \text{in } Y_k,\\ \mu_k(s)[\boldsymbol{e}_i+\nabla_{\boldsymbol{y}}W_i^k(s)]\cdot\boldsymbol{\nu_k}=0 & \text{on } \Gamma_{k,3},\\ W_i^k(y)\text{ and }\mu_k(s)[\nabla_{\boldsymbol{y}}W_i^k(s)]\cdot\boldsymbol{\nu_k}\text{ are } Y\text{-periodic.}\end{cases}\tag{14}$$

Note that each of the coefficient cell functions W_i^k is determined up to a constant.

Substitute $U_k(t,x,y)$ into (13) along with

$$\Phi_k(t,x,y)=\sum_{j=1}^N\frac{\partial\varphi_k}{\partial x_j}(t,x)W_j^k(y)\quad\text{for }k=1,2.$$

Then (13) yields the variational *homogenized problem*:

Find a triple of functions

(i) u_j in $L^2(0,T;H^1_0(\Omega))$ for $j=1,2$, and
(ii) U_3 in $L^2((0,T)\times\Omega; H^1_\#(Y_3))$ with
(iii) $\gamma_3U_3(t,x,s)=u_j(t,x)$ on $\Gamma_{j,3}$ for $j=1,2$,

such that

$$
\begin{aligned}
&-\sum_{k=1}^{2} \tilde{c}_k \left(\int_0^T \int_\Omega u_k(t,x)\varphi_{k,t}(t,x)\,dx\,dt + \int_\Omega u_0(x)\varphi_k(0,x)\,dx \right) \\
&\quad + \int_0^T \int_\Omega \sum_{i,j=1}^{N} A_{i,j}^k \frac{\partial u_k}{\partial x_i}(t,x) \frac{\partial \varphi_k}{\partial x_j}(t,x)\,dx\,dt \\
&\quad - \int_0^T \int_\Omega \int_Y \chi_3(y)c_3(y)U_3(t,x,y)\Phi_{3,t}(t,x,y)\,dy\,dx\,dt \\
&\quad - \int_\Omega \int_Y \chi_3(y)c_3(y)u_0(x)\Phi_3(0,x,y)\,dy\,dx \\
&\quad + \int_0^T \int_\Omega \int_Y \chi_3(y)\mu_3(y)\nabla_{\boldsymbol{y}} U_3(t,x,y)\cdot \nabla_{\boldsymbol{y}}\Phi_3(t,x,y)\,dy\,dx\,dt = 0 \qquad (15)
\end{aligned}
$$

for all

(i) φ_j in $L^2(0,T;H_0^1(\Omega))$, $j=1,2$,
(ii) Φ_3 in $L^2((0,T)\times\Omega;H_\#^1(Y_3))$,

for which

(iii) $\partial\varphi_j/\partial t \in L^2(0,T;[H_0^1(\Omega)]')$, $j=1,2$,
(iv) $\partial\Phi_3/\partial t \in L^2((0,T)\times\Omega;H_\#^1(Y_3)')$, $\Phi_3(T)=0$ in $L^2(\Omega\times Y_3)$, and
(v) $\gamma_3\Phi_3=\varphi_j$ on $\Gamma_{j,3}$, $j=1,2$.

The *homogenized coefficients* in (15) are the constants $\tilde{c}_k$ and A_{ij}^k for $k=1,2$, $1 \leqslant i,j \leqslant N$, defined by

$$
\begin{aligned}
\tilde{c}_k &\equiv \int_{Y_k} c_k(y)\,dy, \\
A_{ij}^k &\equiv \int_{Y_k} \mu_k(y)\big(\boldsymbol{e}_i + \nabla_{\boldsymbol{y}} W_i^k(y)\big)\cdot\big(\boldsymbol{e}_j + \nabla_{\boldsymbol{y}} W_j^k(y)\big)\,dy. \qquad (16)
\end{aligned}
$$

The specific heats $\tilde{c}_k$ are the indicated simple averages, and the *non-isotropic* diffusion coefficients $\{A_{ij}^k\}$ are constructed directly from the coefficient cell functions $\{W_i^k\}$.

Lemma 17. *For $k=1,2$, the matrix of homogenized coefficients $\{A_{ij}^k\}$ is symmetric and positive-definite.*

Proof. The symmetry of $\{A^k_{ij}\}$ is clear from the definition. For any $\xi \in \mathbb{R}^N$ we have

$$\sum_{i,j=1}^{N} A^k_{ij}\xi_i\xi_j = \int_{Y_k} \mu_k(y)\boldsymbol{\nabla}_{\boldsymbol{y}} \sum_{i=1}^{N} \xi_i\big(W^k_i(y)+y_i\big)\cdot \boldsymbol{\nabla}_{\boldsymbol{y}} \sum_{j=1}^{N} \xi_j\big(W^k_j(y)+y_j\big)\,dy \geqslant 0,$$

and if this is zero, then $\sum_{i=1}^{N}\xi_i(W^k_i(y)+y_i)$ is constant. From here we find $\sum_{i=1}^{N}\xi_i y_i$ must satisfy the boundary conditions in (14), hence, $\xi = 0$. □

The strong form of (15) can now be determined by making appropriate choices for the remaining test functions. By setting $\varphi_j(t,x)=0$ for $j=1,2$ we obtain from Eq. (15) the following mixed Dirichlet-periodic *local cell problem* for each $x \in \Omega$:

$$\begin{cases} c_3(y)\frac{\partial}{\partial t}U_3(t,x,y) - \boldsymbol{\nabla}_{\boldsymbol{y}}\cdot[\mu_3(y)\boldsymbol{\nabla}_{\boldsymbol{y}}U_3(t,x,y)] = 0 \quad \text{in } Y_3, \\ U_3(t,x,y) \text{ and } \mu_3(y)\boldsymbol{\nabla}_{\boldsymbol{y}}U_3(t,x,y)\cdot\boldsymbol{\nu} \text{ are } Y\text{-periodic on } \Gamma_{3,3}, \\ U_3(t,x,s) = u_j(t,x) \quad \text{on } \Gamma_{j,3},\ j=1,2. \end{cases} \tag{17a}$$

Now letting φ_1 in $C_0^\infty(0,T;H_0^1(\Omega))$ and choosing $\varphi_2=0$, and Φ_3 in $L^2((0,T)\times\Omega;H^1_\#(Y_3))$ as above, we obtain the first *macro-diffusion* equation

$$\begin{aligned} &\tilde{c}_1\frac{\partial}{\partial t}u_1(t,x) - \sum_{i,j=1}^{N} A^1_{ij}\frac{\partial}{\partial x_j}\left(\frac{\partial u_1}{\partial x_i}(t,x)\right) \\ &\quad + \int_{\Gamma_{1,3}} \mu_3(s)\boldsymbol{\nabla}_{\boldsymbol{y}}U_3(t,x,s)\cdot\boldsymbol{\nu_3}\,ds = 0. \end{aligned} \tag{17b}$$

Similarly, letting φ_2 in $C_0^\infty(0,T;H_0^1(\Omega))$ and choosing $\varphi_1=0$, and Φ_3 in $L^2((0,T)\times\Omega;H^1_\#(Y_3))$ as above, we obtain the second *macro-diffusion* equation

$$\begin{aligned} &\tilde{c}_2\frac{\partial}{\partial t}u_2(t,x) - \sum_{i,j=1}^{N} A^2_{ij}\frac{\partial}{\partial x_j}\left(\frac{\partial u_2}{\partial x_i}(t,x)\right) \\ &\quad + \int_{\Gamma_{2,3}} \mu_3(s)\boldsymbol{\nabla}_{\boldsymbol{y}}U_3(t,x,s)\cdot\boldsymbol{\nu_3}\,ds = 0. \end{aligned} \tag{17c}$$

The inclusions in $L^2(0,T;H_0^1(\Omega))$ given by (i) yield the pair of *global boundary conditions*

$$u_k(t,s) = 0 \quad \text{on } \partial\Omega,\ k=1,2. \tag{17d}$$

Finally, we have also the *initial conditions*

$$\begin{aligned} &\tilde{c}_k u_k(0,x) = \tilde{c}_k u_0(x), \quad k=1,2, \\ &c_3(y)U_3(0,x,y) = c_3(y)u_0(x) \quad \text{for a.e. } y\in Y_3,\ x\in\Omega. \end{aligned} \tag{17e}$$

It follows from Lemma 17 that the elliptic operators in (17b) and (17c) are *strongly-elliptic*.

4.1. The main result

We summarize and complete the proof of the preceding results.

Theorem 18. *Assume that the coefficients in the exact micro-model* (4) *satisfy the conditions* (3) *and* $\int_Y c(y)\,dy > 0$, *and that the initial function* $u_0(\cdot)$ *is in* $L^2(\Omega)$. *For* $k = 1, 2$, $1 \leqslant i \leqslant N$, *let the functions* $W_i^k(\cdot) \in H^1_\#(Y_k/\mathbb{R})$ *be the solutions to the coefficient cell problems* (14), *and define the homogenized coefficients by* (16). *For each* $\varepsilon > 0$, *let* u^ε *denote the unique solution to the initial-boundary-value problem* (4). *Then there exist a triple of functions* u_j *in* $L^2(0, T; H^1_0(\Omega))$, $j = 1, 2$, U_3 *in* $L^2((0, T) \times \Omega; H^1_\#(Y_j)/\mathbb{R})$, *and a subsequence of* u^ε, *likewise denoted by* u^ε, *for which we have two-scale convergence*

$$\chi_j^\varepsilon u^\varepsilon \xrightarrow{2} \chi_j(y)u_j(t, x), \quad j = 1, 2, \qquad \chi_3^\varepsilon u^\varepsilon \xrightarrow{2} \chi_3(y)U_3(t, x, y),$$

and these two-scale limits u_1, u_2, U_3 *are the unique solution of the homogenized system* (17). *Furthermore, the coefficients in this system satisfy*

$$\tilde{c}_j \geqslant 0, \quad j = 1, 2, \qquad c_3(y) \geqslant 0, \quad y \in Y_3, \qquad \tilde{c}_1 + \tilde{c}_2 + \int_{Y_3} c_3(y)\,dy > 0,$$

$$\text{and } \left\{A^k_{ij}\right\} \text{ is symmetric and positive-definite}, \quad k = 1, 2.$$

Proof. It remains only to prove the *uniqueness* of the solution of (15). We shall show that it is just the variational form (2) of a well-posed Cauchy problem (1) for an appropriate evolution equation in Hilbert space. Define the *energy space*

$$\begin{aligned} V \equiv \big\{&[\varphi_1, \varphi_2, \Phi_3] \in H^1_0(\Omega) \times H^1_0(\Omega) \times L^2\big(\Omega; H^1_\#(Y_3)\big): \\ &\gamma_3\Phi_3(x, y) = \varphi_j(x) \text{ for } y \in \Gamma_{j,3},\ j = 1, 2\big\}, \end{aligned}$$

and the operators $\mathcal{A}$ and $\mathcal{B}: V \to V'$ by

$$\begin{aligned} \mathcal{A}u(\varphi) \equiv{}& \sum_{k=1}^{2} \int_\Omega \sum_{i,j=1}^{N} A^k_{i,j} \frac{\partial u_k}{\partial x_i}(x) \frac{\partial \varphi_k}{\partial x_j}(x)\,dx \\ &+ \int_\Omega \int_{Y_3} \mu_3(y) \nabla_{\boldsymbol{y}} U_3(x, y) \cdot \nabla_{\boldsymbol{y}} \Phi_3(x, y)\,dy\,dx, \\ \mathcal{B}u(\varphi) \equiv{}& \sum_{k=1}^{2} \int_\Omega \tilde{c}_k u_k(x)\varphi_k(x)\,dx + \int_\Omega \int_{Y_3} c_3(y)U_3(x, y)\Phi_3(x, y)\,dy\,dx \end{aligned}$$

for $u = [u_1, u_2, U_3]$, $\varphi = [\varphi_1, \varphi_2, \Phi_3]$ in V. It is easy to see that $\mathcal{A}$ and $\mathcal{B}$ satisfy the conditions of Proposition 8. Specifically, it follows from Lemma 17 that $\mathcal{A}$ is V-elliptic. Furthermore, it is straightforward to check that the system (15) is precisely the weak form (2) of the Cauchy problem (1), so the solution u is unique. Moreover, this establishes independently the existence of this weak solution to the homogenized system (17). □

4.2. Quasi-static exchange

Consider the case of no heat storage in the exchange region, that is, let $c_3(\cdot) = 0$. Then the function $U_3(t, x, y)$ satisfies the local boundary-value problem

$$\begin{cases} -\nabla_{\boldsymbol{y}} \cdot [\mu_3(y)\nabla_{\boldsymbol{y}} U_3(t, x, y)] = 0 \quad \text{in } Y_3, \\ U_3(t, x, s) = u_j(t, x) \quad \text{on } \Gamma_{j,3},\ j = 1, 2, \\ U_3(t, x, s) \text{ and } \mu_3(s)\nabla_{\boldsymbol{y}} U_3(t, x, s) \cdot \boldsymbol{\nu} \text{ are } Y\text{-periodic on } \Gamma_{3,3}. \end{cases}$$

In order to calculate the exchange flux, we exploit the linear dependence of the solution of this elliptic partial differential equation on the boundary conditions. Thus, the solution to the local cell problem will be represented in terms of the solution $U(\cdot)$ to the *exchange cell problem*

$$\begin{cases} \nabla_{\boldsymbol{y}} \cdot [\mu_3(y)\nabla_{\boldsymbol{y}} U(y)] = 0 \quad \text{in } Y_3, \\ U(y) = 1 \quad \text{on } \Gamma_{1,3}, \\ U(y) = 0 \quad \text{on } \Gamma_{2,3}, \\ U(y) \text{ and } \mu_3\nabla_{\boldsymbol{y}} U \cdot \nu_3 \text{ are periodic on } \Gamma_{3,3}. \end{cases}$$

The solution $U(\cdot)$ is the *characteristic flow potential* in Y_3, and we see that $1 - U(\cdot)$ is the solution to the corresponding problem with 1 and 0 interchanged in the boundary conditions on $\Gamma_{j,3}$, $j = 1, 2$. Since each $u_j(t, x)$ is independent of y, the solution to the local boundary-value problem is given by

$$U_3(t, x, y) = u_1(t, x)U(y) + u_2(t, x)\bigl(1 - U(y)\bigr).$$

Using the divergence theorem, we compute the flux terms $q_{j,3}$ across the corresponding boundaries into Y_3,

$$\int_{Y_3} \nabla_{\boldsymbol{y}} \cdot (\mu_3\nabla_{\boldsymbol{y}} U_3)\, dy = \int_{\Gamma_{1,3}} \mu_3\nabla_{\boldsymbol{y}} U_3 \cdot \nu_3\, ds + \int_{\Gamma_{2,3}} \mu_3\nabla_{\boldsymbol{y}} U_3 \cdot \nu_3\, ds \equiv q_{1,3} + q_{2,3}.$$

This shows for the quasi-static case that $q_{1,3} + q_{2,3} = 0$, and furthermore we have

$$-q_{2,3} = q_{1,3} \equiv \int_{\Gamma_{1,3}} \mu_3\nabla_{\boldsymbol{y}} U_3 \cdot \nu_3\, ds = \bigl[u_1(t, x) - u_2(t, x)\bigr] \int_{\Gamma_{1,3}} \mu_3\nabla_{\boldsymbol{y}} U \cdot \nu_3\, ds.$$

This yields the heat exchange in the form

$$\Gamma(u_1, u_2)(t, x) = \kappa\bigl[u_1(t, x) - u_2(t, x)\bigr],$$

which is exactly the exchange term given in the Rubinstein–Barenblatt systems. We note that all the effects of the microstructure geometry are contained in the characteristic flow potential $U(\cdot)$. Moreover, the maximum principle implies $\nabla_{\boldsymbol{y}} U \cdot \nu_3 > 0$ on $\Gamma_{1,3}$, so the exchange coefficient satisfies

$$\kappa \equiv \int_{\Gamma_{1,3}} \mu_3\nabla_{\boldsymbol{y}} U \cdot \nu_3\, ds > 0. \tag{18}$$

In summary, the quasi-static case of the decoupled homogenized system (17) takes the form

$$\tilde{c}_1 \frac{\partial u_1}{\partial t}(t,x) - \sum_{i,j=1}^{N} A_{ij}^1 \frac{\partial}{\partial x_j}\left(\frac{\partial u_1}{\partial x_i}(t,x)\right) + \kappa\big(u_1(t,x) - u_2(t,x)\big) = 0, \tag{19a}$$

$$\tilde{c}_2 \frac{\partial u_2}{\partial t}(t,x) - \sum_{i,j=1}^{N} A_{ij}^2 \frac{\partial}{\partial x_j}\left(\frac{\partial u_2}{\partial x_i}(t,x)\right) + \kappa\big(u_2(t,x) - u_1(t,x)\big) = 0, \tag{19b}$$

which is precisely the *double-diffusion* model of Rubinstein [10] and Barenblatt [3]. We have shown that the system (19) is the two-scale limit of the exact micro-model (4) in the case of *quasi-static* diffusion in the exchange region. Moreover, in this micro-model the temperature and flux are continuous across all internal boundaries, and we have found an explicit representation (18) for the heat transfer coefficient κ. Finally, we note that the diffusion coefficients in the component equations are necessarily *non-isotropic*.

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