Applicable Analysis, 1975, Vol. 5, pp. 81–99

Printed in Great Britain

The Sobolev Equation, II

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Communicated by R. W. Carroll

(*Received June 1, 1971*)

Partial differential equations of the form $MD_tU+LU = f$ are considered in which M and L denote realizations of regular elliptic linear partial differential operators of respective orders $2m$ and $2l$. The abstract existence results of Part I lead to existence of generalized solutions of the equation. Regularity results are obtained by the technique of interpolation theory. The results include the parabolic equation ($m = 0$) and pseudo-parabolic equations $m \geqq l$. The extensive bibliography contains about 36 references to the literature on this class of equations.

This paper is concerned with existence and regularity of solutions of the partial differential equation of Sobolev type

$$\sum_{|\rho|,|\sigma|\leqq m}(-1)^{|\rho|}D^\rho m^{\rho\sigma}(x)D^\sigma D_t u(x,t)+\sum_{|\rho|,|\sigma|\leqq l}(-1)^{|\rho|}D^\rho l^{\rho\sigma}(x)D^\sigma u(x,t) = f(x,t) \qquad \text{(S)}$$

which satisfy linear homogeneous boundary conditions on the walls of a cylinder in $(n+1)$-dimensional Euclidean space and an initial condition at the bottom. These results will be obtained by considering realizations of the differential operators appearing in (S) and combining the existence-regularity theory of such operators with results from Part I on weak and strong solutions of an abstract evolution equation modeled after (S).

The plan of the paper is as follows. Section 1 consists of an announcement of the results of Part I and some remarks on (incidental) applications to equations of parabolic and Schroedinger type. An exposition of the a priori estimates and regularity results for regular elliptic boundary value problems is contained in Section 2, and these immediately yield existence and regularity results for abstract strong solutions of (S) constructed in Part I. Interpolation theory is used in Section 3 to describe an abstract sufficient condition for the weak solution of the homogeneous equation to be essentially strong. Then we discuss in Section 4 the boundary conditions associated with weak solutions and verify that the abstract condition is fulfilled when the domains of the operators appearing in (S) are compatible and $m < l$.

We have chosen the name "Sobolev equation" to designate partial differential equations which contain terms with mixed space and time derivatives or abstract evolution equations with terms containing a (non-trivial) operator acting on a time derivative of the solution. Considerable work on such problems has occurred, though frequently the writers are seemingly not aware of related results of others as well as some interesting applications of their own work. For this reason, this writer has collected in the bibliography all of those published papers on such problems known to him at this time. The abstract equations have been applied to models very different from the one of interest here with correspondingly different hypotheses, perhaps even incompatible. We mention in particular the far-reaching papers of C. Bardos and H. Brezis and of S. Bochner and J. Neumann. Those papers concerned with applications to problems in fluid flow, consolidation and diffusion processes can be identified by title or place of publication. Finally, we note that the applications involve equations like (S) only with $m = 0$ (parabolic) or $m = l$. The case of $0 < m < l$ is of mathematical interest and is the primary objective here, while that of $m \geqq l$ is much easier, and very strong results have been attained [30, 53, 54, 56].

1. THE ABSTRACT PROBLEMS

Let H, W_m and V_l be Hilbert spaces with $V_l \rightarrow W_m \rightarrow H$, where $A \rightarrow B$ means A is a dense subset of B and the injection is continuous. By identifying the space H with its anti-dual H', we obtain by duality $H \rightarrow W'_m \rightarrow V'_l$. The $V_l - V'_l$ duality $\langle \cdot, \cdot \rangle$ agrees with the inner product $(\cdot, \cdot)_H$ on the product space $H \times V_l$ and so also does the $W_m - W'_m$ duality on $H \times W_m$, so we may use $\langle \cdot, \cdot \rangle$ for any one of the three dual pairs without confusion because of the indicated identification.

Let $m(\cdot, \cdot)$ and $l(\cdot, \cdot)$ be continuous sesquilinear forms on W_m and V_l, respectively. These forms can be represented by $m(\phi, \psi) = \langle \mathscr{M}\phi, \psi \rangle$ and $l(\phi, \psi) = \langle \mathscr{L}\phi, \psi \rangle$, where $\mathscr{M} \in \mathscr{L}(W_m, W'_m)$ and $\mathscr{L} \in \mathscr{L}(V_l, V'_l)$. The restriction of $\mathscr{M}$ to the set $D(M) = \{\phi \in W_m : \mathscr{M}\phi \in H\}$ is an unbounded operator M on H, and likewise the restriction of $\mathscr{L}$ to $D(L) = \{\phi \in V_l : \mathscr{L}\phi \in H\}$ is an unbounded operator L on H.

Suppose that $f \in C([0, \infty), V_l')$ and $u_0 \in W_m$ are given. A *weak solution* is a function $u \in C([0, \infty), W_m) \cap C^1((0, \infty), W_m)$ such that $u(0) = u_0$, $u(t) \in V_l$ and

$$\mathscr{M}u'(t) + \mathscr{L}u(t) = f(t) \tag{1.1}$$

for all $t > 0$. We note that (1.1) is an equation in V_l' and is equivalent to requiring that

$$m(u'(t), \phi) + l(u(t), \phi) = \langle f(t), \phi \rangle$$

for all ϕ in V_l.

Let W denote the linear space $D(M)$ with the norm

$$|\phi|_W = (\|\phi\|_m^2 + \|M\phi\|_H^2)^{1/2}. \tag{1.2}$$

Assume that $f \in C([0, \infty), H)$ and $u_0 \in D(M)$ are given. A *strong solution* is a function $u \in C([0, \infty), W) \cap C^1((0, \infty), W)$ such that $u(0) = u_0$, $u(t) \in D(L)$ and (1.1) is satisfied for all $t > 0$. Since $W \subseteq W_m$, $D(L) \subseteq V_l$, and the W-norm is stronger than the W_m-norm, a strong solution is always a weak solution and we may replace $\mathscr{M}$ and $\mathscr{L}$ by M and L, respectively, in (1.1).

THEOREM 1 *Let the Hilbert spaces, $V_l \to W_m \to H$, sesquilinear forms, $m(\cdot, \cdot)$ and $l(\cdot, \cdot)$, and the operators M and L be given as above. Assume the following: there is a $k_m > 0$ such that*

$$\operatorname{Re} m(\phi, \phi) \geqq k_m\|\phi\|_m^2 \text{ for all } \phi \in W_m; \tag{1.3}$$

there is a $k_l > 0$ such that

$$\operatorname{Re} l(\phi, \phi) \geqq k_l\|\phi\|_l^2 \text{ for all } \phi \in V_l; \tag{1.4}$$

$$D(L) \subseteq D(M); \text{ and} \tag{1.5}$$

there is a complex cone $K(\theta) = \{z : |\arg z| \leqq \theta\}$ with $\theta \leqq \pi/2$ such that

$$(L\phi, M\phi)_H \in K(\theta) \textit{ for all } \phi \textit{ in } D(L). \tag{1.6}$$

Then there exists a unique strong solution for each $u_0 \in D(L)$ and $f \in C^1([0, \infty), H)$.

COROLLARY *Let the hypotheses of Theorem 1 hold and assume $\theta < \pi/2$ in (1.6). Then there exists a unique strong solution for each $u_0 \in D(M)$ and f Holder continuous from $[0, \infty)$ to H.*

THEOREM 2 *Let the Hilbert spaces $V_l \to W_m \to H$, sesquilinear forms, $m(\cdot, \cdot)$ and $l(\cdot, \cdot)$, and the operators $\mathscr{M}$ and $\mathscr{L}$ be given as above. Assume the following: there is a $k_m > 0$ such that (1.3) holds for all $\phi \in W_m$; there is a $k_l > 0$ such that (1.4) holds for all $\phi \in V_l$; and*

$$m(\phi, \psi) = \overline{m(\psi, \phi)} \text{ for all } \phi, \psi \in W_m.$$

Then there exists a unique weak solution for each $u_0 \in W_m$ and f Holder continuous from $[0, \infty)$ to W_m'.

These last two results were obtained in Part I by showing that the operators $A = -M^{-1}L$ and $\mathscr{A} = -\mathscr{M}^{-1}\mathscr{L}$ generate analytic semigroups on $W = D(M)$ and W_m, respectively.

Finally, consider the following special case. Let H be a Hilbert space, $\alpha \in \mathbb{C}$, and set $m(\phi, \psi) = \alpha(\phi, \psi)_H$, $W_m = H$. Let V_l be a Hilbert space with $V_l \to H$, and $n(\cdot, \cdot)$ a continuous sesquilinear form on V_l which satisfies $n(\phi, \phi) \geqq k\|\phi\|_l^2$ on V_l, $k > 0$. In particular, $n(\cdot, \cdot)$ is a self adjoint form. Let $\beta \in \mathbb{C}$ and set $l(\phi, \psi) = \beta n(\phi, \psi)$ on V_l. If $\mathscr{N}$ is the operator associated with $n(\cdot, \cdot)$, then $\mathscr{L} = \beta\mathscr{N}$ and the Eq. (1.1) is

$$\alpha u'(t) + \beta\mathscr{N}u(t) = f(t). \tag{1.8}$$

The conditions (1.3) and (1.4) are satisfied if $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\beta > 0$, and (1.5) is always true since $D(M) = H \supseteq D(L)$. The condition (1.6) is easy to verify, since

$$(L\phi, M\phi) = \beta\bar{\alpha}\, n(\phi, \phi)$$

and $n(\phi, \phi)$ is real. If $|\arg(\beta\bar{\alpha})| = \pi/2$, then Theorem 1 asserts the existence of a strong solution for u_0 in $D(L)$ and $f \in C^1([0, \infty), H)$. Choosing $\alpha = e^{i\pi/4}$ and $\beta = e^{-i\pi/4}$ we obtain after multiplying by α (and making an appropriate change in f)

$$i\, u'(t) + \mathscr{N}u(t) = f(t),$$

an equation of Schroedinger type. Similarly, we see from the Corollary to Theorem 1 (and also from Theorem 2) that (1.8) is an abstract parabolic equation whenever $|\arg(\beta\bar{\alpha})| < \pi/2$. See [32, 38] for additional results and references.

2. STRONG SOLUTIONS

Let G be a bounded open set in Euclidean n-space with an infinitely differentiable boundary ∂G of dimension $n-1$ with G on one side of ∂G. We shall use the notation $D_j = \partial/\partial x_j$ and $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ for an n-tuple α of nonnegative integers, $|\alpha| = \Sigma\alpha_j$. Similarly, if $x = (x_1, x_2, \dots, x_n)$ is in R^n, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

For integer $k \geqq 0$, H^k denotes the completion of the set C^∞ of infinitely differentiable functions on the closure of G with respect to the norm $\|\phi\|_k = (\phi, \phi)_k^{1/2}$ obtained from the inner product

$$(\phi, \psi)_k = \sum\{\textstyle\int_G D^\alpha\phi \, \overline{D^\alpha\psi} \, dx : |\alpha| \leqq k\}.$$

Then H^k is the Hilbert space of (equivalence classes of) functions ϕ in $H^0 = L^2(G)$ (Lebesgue square-summable) for which $D^\alpha\phi \in L^2(G)$ whenever $|\alpha| \leqq k$, where derivatives are taken in $\mathscr{D}'$, the space of distributions on G [49]. We denote by H_o^k the closure in H^k of C_0^∞, those functions of C^∞ with compact support. See [1, 2, 18, 33, 38, 48, 64] for details.

Let $l \geqq 0$ be an integer and define by

$$l(\phi, \psi) = \sum\{\textstyle\int_G l^{\rho\sigma}(x) \, D^\sigma\phi(x) \, \overline{D^\rho\psi(x)} \, dx : |\rho|, |\sigma| \leqq l\} \tag{2.1}$$

a continuous sesquilinear form on H^l with coefficients in C^∞. If $\langle\cdot, \cdot\rangle$ denotes $\mathscr{D}-\mathscr{D}'$ duality [49], then (2.1) determines an operator $\mathscr{L}: H^l \to \mathscr{D}'$ by

$$\langle \mathscr{L}\phi, \bar{\psi}\rangle = l(\phi, \psi), \qquad \phi \in H^l, \qquad \psi \in C_0^\infty$$

of the form

$$\mathscr{L}\phi = \sum\{(-1)^{|\rho|} D^\rho l^{\rho\sigma}(x) D^\sigma\phi : |\rho|, \quad |\sigma| \leqq l\}. \tag{2.3}$$

The partial differential operator $\mathscr{L}$ is *elliptic* at $x \in G$ if its principal part satisfies

$$\mathscr{L}'(x, \xi) \equiv \sum\{\xi^\rho l^{\rho\sigma}(x)\xi^\sigma : |\rho| = |\sigma| = l\} \neq 0$$

for every $\xi \in R^n$, $\xi \neq 0$. $\mathscr{L}$ is then of order $2l$; it is *properly elliptic* [47] if it is elliptic and for linearly independent $\xi, \eta \in R^n$ the polynomial $z \to \mathscr{L}'(x, \xi + z\eta)$ has l roots with positive imaginary part.

Consider a set of boundary operators

$$L_j\phi = \sum_{|\alpha| \leqq l_j} l_{j,\alpha}(x) \, D^\alpha\phi, \qquad l_{j,\alpha} \in C^\infty, \qquad 1 \leqq j \leqq p. \tag{2.4}$$

These are *normal* if the principal parts satisfy

$$L_j'(x, \xi) \equiv \sum_{|\alpha| = l_j} l_{j,\alpha}(x)\xi^\alpha \neq 0$$

for $x \in \partial G$ and $\xi \neq 0$ normal to ∂G at x, and $l_j \neq l_k$ for $j \neq k$. (That is, they are nowhere characteristic and of distinct orders.) This set is a *Dirichlet system* of order p if it is normal and every $l_j < p$. (Then every order between 0 and $p-1$ appears exactly once in the L_j, $1 \leqq j \leqq p$.)

An important property of Dirichlet systems is the following. Given a Dirichlet system $\{L_j : 1 \leqq j \leqq p\}$ and a set of functions $\{\phi_j : 1 \leqq j \leqq p\}$ in $C^\infty(\partial G)$, there exists a function ϕ in C^∞ such that $L_j\phi = \phi_j$ on ∂G. This is also true for normal systems since these can be complemented to obtain a Dirichlet system [7, 38, 48].

The domain of the realization of $\mathscr{L}$ that we shall describe below will depend on boundary conditions, some of which may be obtained from the following [7, 38]:

Green's Formula Assume that $\mathscr{L}$ is elliptic in G, $0 \leqq p \leqq l$ and that $\{L_j : 1 \leqq j \leqq p\}$ is a normal set of boundary operators of order $< l$. (This set is empty if $p = 0$.) Augment this set by boundary operators $\{K_j : p < j \leqq l\}$ with C^∞ coefficients such that the set $\{L_1, \ldots, L_p, K_{p+1}, \ldots, K_l\}$ is a Dirichlet system of order l. Then there is a Dirichlet system of order $l\{K_1, \ldots, K_p, L_{p+1}, \ldots, L_l\}$ with C^∞ coefficients such that order (K_j)+order $(L_j) - 2l-1$, $1 \leqq j \leqq l$, and for every pair ϕ, ψ in C^∞

$$l(\phi, \psi) = \int_G \mathscr{L}\phi\bar{\psi}\, dx + \int_{\partial G}\Big(\sum_{j=1}^{p} K_j\phi\overline{L_j\psi} + \sum_{j=p+1}^{l} L_j\phi\overline{K_j\psi}\Big)\, ds. \tag{2.5}$$

Suppose now that $\phi, f \in C^\infty$ and

$$l(\phi, \psi) = \int_G f\bar{\psi}\, dx$$

for all ψ in C^∞ such that $L_j\psi = 0$ on ∂G, $1 \leqq j \leqq p$. Since the equality holds for ψ in C_0^∞ it follows from (2.2) that $\mathscr{L}\phi = f$ and hence from (2.5) that

$$\int_{\partial G} \sum_{j=p+1}^{l} L_j\, \phi\overline{K_j\psi}\, ds = 0 \tag{2.6}$$

for all such ψ. But since the set $\{K_{p+1}, \ldots, K_l\}$ is a normal system, (2.6) implies that $L_j\phi = 0$, $p+1 \leqq j \leqq l$. Thus the sesquilinear form (2.1) and the boundary operators (2.4) determine the elliptic boundary value problem

$$\begin{aligned} \mathscr{L}\phi &= f, \quad \text{in } G \\ L_j\phi &= 0, \quad 1 \leqq j \leqq l, \text{ on } \partial G. \end{aligned} \tag{2.7}$$

Those boundary operators of order $< l$ (stable) are specified a priori, and those of order $\geqq l$ (natural) must be obtained from Green's formula by a suitable choice of operators $\{K_j = p < j \leqq l\}$.

LEMMA 2.1 *Define V to be those ϕ in C^∞ for which $L_j\phi = 0$ on ∂G, $1 \leqq j \leqq l$. Assume that the sesquilinear form* (2.1) *satisfies the coercive estimate* (1.4) *for all ϕ in V. Then the $L^2(G)$-closure of $\mathscr{L}: V \to C^\infty$ is precisely the operator L of §1, where $H = L^2(G)$ and V_l is defined below, and $L: H^{2l+r} \cap D(L) \to H^r$ is an isomorphism for every integer $r \geqq 0$.*

Proof Denote the closure of V in H^k for integer $k \geqq 0$ by V_k, and note that (1.4) holds on V_l by continuity. Also, V_k contains those ϕ in C^∞ for which $L_j\phi = 0$ only for those L_j of order $\leqq k-1$ [48, Lemma 4.9], so V_l is determined by the stable boundary operators. Since (1.4) holds on C_0^∞, $\mathscr{L}$ is properly elliptic (in fact, strongly elliptic [2]) on $\bar{G}$. The stable operators in (2.7) are normal, so Green's Theorem shows that the entire set (2.7) is normal. Finally, the estimate (1.4) then implies that these operators cover $\mathscr{L}$ [3, 4] so the boundary value problem (2.7) is regular in the sense indicated below.

The boundary value problem (2.7) is called *regular* if $\mathscr{L}$ is properly elliptic, the boundary operators are normal and they *cover* $\mathscr{L}$. That is, for each $x \in \partial G$, each tangent $\xi \neq 0$ at x and each normal $\eta \neq 0$ to ∂G at x, the polynomials $L_j'(x, \xi+z\eta)$, $1 \leqq j \leqq l$, are linearly independent modulo the polynomial $\Pi_{j=1}^l(z-z_j)$, where L_j' is the principle part of L_j and $\{z_j : 1 \leqq j \leqq l\}$ are those roots of $\mathscr{L}'(x, \xi+z\eta) = 0$ with positive imaginary part. For regular boundary value problems we have the following a priori estimate: for every integer $r \geqq 0$ there is a $C_r > 0$ such that

$$\|\phi\|_{2l+r} \leqq C_r(\|\mathscr{L}\phi\|_r + \|\phi\|_0) \tag{2.8}$$

for all ϕ in V. [5, 12, 13, 38, 46].

Denote the L^2-closure of $\mathscr{L}: V \to C^\infty$ by $\mathscr{L}_2$. The estimate (2.8) with $r = 0$ shows that $\mathscr{L}_2$ has domain V_{2l} and closed range $R(\mathscr{L}_2)$ in $L^2(G)$ [12, p. 368]. The estimate (1.4) shows that $\mathscr{L}_2$ is injective. Since (2.2) holds for $\phi \in V_{2l}$ and $\psi \in V_l$ (use (2.5) and extend by continuity) it follows that $\mathscr{L}_2 \subseteq L$ and hence the operators coincide if $\mathscr{L}_2$ is onto $L^2(G)$. But this is true if and only if its adjoint $\mathscr{L}_2^*$ is injective [12, p. 370]. To describe the adjoint, we refer to Green's theorem to obtain

$$l(\phi, \psi) = (\phi, \mathscr{L}^*\psi)_0 + \int_{\partial G}\Big(\sum_{j \leqq p} L_j\phi\overline{K_j'\psi} + \sum_{j>p} K_j\phi\overline{L_j'\psi}\Big)\, ds$$

for ϕ, ψ in C^∞, where

$$\mathscr{L}^*\psi = \sum\{(-1)^{|\sigma|}D^\sigma l^{\rho\sigma}(x)D^\rho\psi : |\rho|, |\sigma| \leqq l\}$$

is the formal adjpint of $\mathscr{L}$. Let $\psi \in C^\infty$. If $(\mathscr{L}\phi, \psi) = (\phi, \mathscr{L}^*\psi)$ for all ϕ in V, then

$$\int_{\partial G}\Big(\sum_{j \leqq p} K_j\phi\overline{L_j\psi} - \sum_{j>p} K_j\phi\overline{L_j'\psi}\Big)\, ds = 0$$

for ϕ in V. Since $\{K_j : 1 \leqq j \leqq l\}$ is a normal set, this shows ψ belongs to the set

$$V' = \{\psi \in C^\infty : L_j\psi = 0, 1 \leqq j \leqq p, L'_j\psi = 0, p < j \leqq l, \text{ on } \partial G\}.$$

As above, $\{\mathscr{L}^*, V'\}$ is a regular boundary value problem, and the L^2-closure of $\mathscr{L}^* : V' \to C^\infty$ is the adjoint of $\mathscr{L}_2$. By (1.4), $\mathscr{L}_2^*$ is injective so $R(\mathscr{L}_2) = L^2(G)$ by [12, p. 370]. The last statement of the lemma follows from (2.8), since $V_{2l+r} = H^{2l+r} \cap D(L)$ for $r \geqq 0$. Q.E.D.

LEMMA 2.2 *Let $\{L_j : 1 \leqq j \leqq l\}$ be a normal set of boundary operators (2.4). Let m be an integer $\leqq l$ and*

$$M_j\phi = \sum_{|\alpha| \leqq m_j} m_{j,\alpha}(x) D^\alpha \phi, \quad m_{j,\alpha} \in C^\infty, \quad 1 \leqq j \leqq m \tag{2.9}$$

a normal set. Let V be the space denoted in Lemma 2.1; define

$$W = \{\phi \in C^\infty : M_j\phi = 0 \text{ on } \partial G, 1 \leqq j \leqq m\}$$

and let W_k denote the closure in H_n of W for each integer $k \geqq 0$. If $V \subseteq W$, then $V_{2l} \subseteq W_{2m}$ and $V_l \subseteq W_m$.

Furthermore, if each L_j with order distinct from the orders of all the M_j's has order $\geqq m$, and if $M_j\phi = 0$ on ∂G for all M_j of order $< m$ implies $L_j\phi = 0$ on ∂G for all L_j of order $< m$ and equal to the order of some M_j, then V_l is dense in W_m.

Proof The first statement follows easily by closure since the norm on V is stronger than the norm on W in each case.

Now, the hypotheses $V \subseteq W$ implies that each M_j has the same order as exactly one of the L_j's [7, p. 306]. Since the $\{M_j\}$ is a normal system, this establishes a bijection $M_j \to L_{j'}$ onto a proper subset of the normal system $\{L_j : 1 \leqq j \leqq l\}$; this bijection is determined by the requirement that order $(M_j) =$ order $(L_{j'})$. We also obtain the fact that the normal system $\{M_j\}$ is *weaker* than the proper subset $\{L_{j'}\}$ so determined. By weaker, we mean that if ϕ is in C^∞ and if every $L_{j'}\phi = 0$ on ∂G, then every $M_j\phi = 0$ on ∂G. These follow from [7, pp. 304–306].

Let $\{L_{j''}\}$ denote those $\{L_j\}$ not included in $\{L_{j'}\}$, so $\{L_j\} = \{L_{j'}\} \cup \{L_{j''}\}$ is a disjoint union. Let V_1 be those ϕ in C^∞ for which all $L_{j'}\phi = 0$. Then we have $V \subseteq V_1 \subseteq W$ from definitions and the preceding paragraph, respectively. Since each $L_{j''}$ has order $\geqq m$ by hypothesis, V is dense in V_1 with respect to the norm of H^m [48, Lemma 4.9]. Similarly, the closures of V_1 and W in H^m depend only on those operators of order $< m$. But the systems $\{L_{j'} : \text{order } L_{j'} < m\}$ and $\{M_j'' \text{ order } M_j < m\}$ are equivalent (each is weaker than the other) by the preceding paragraph and the last hypotheses, so W_m is precisely the closure in H^m of V_1. Hence we have shown that V_l is dense in W_m. Q.E.D.

These lemmas combine with Theorem 1 and its corollary to give the following result.

THEOREM 3 *Let the sesquilinear form* (2.1) *and a normal set of boundary operators* (2.4) *of order* $< l$ *be given. Let* $\mathscr{L}$ *be the partial differential operator determined by* (2.1) *through* (2.2) *and let* (2.7) *denote the boundary value problem determined by* (2.1) *and* (2.4) *through* (2.5).

Let m *be an integer* $< l$ *and define a sesquilinear form*

$$m(\phi, \psi) = \sum\{\int_G m^{\rho\sigma}(x) D^\sigma \phi(x) \overline{D^\rho \psi(x)}\, dx : |\rho|, |\sigma| \leqq m\} \tag{2.10}$$

with C^∞ *coefficients. Assume that the normal set of boundary operators* (2.9) *is determined as above by* (2.10), *the stable operators in* (2.9) *and Green's theorem. Let* $\mathscr{M}$ *be the partial differential operator determined by* $m(\phi, \psi) = \langle \mathscr{M}\phi, \bar{\psi}\rangle$ *for all* ϕ *and* ψ *in the set* W *defined in Lemma 2.2. We shall assume the following:*

i) (1.3) holds for all ϕ in W;

ii) (1.4) holds for all ϕ in V;

iii) $V \subseteq W$;

iv) if the order of L_j is $< m$, then there is an M_i with order $(M_i) =$ order (L_j);

v) if $M_j\phi = 0$ on ∂G for all M_j of order $< m$, then $L_j\phi = 0$ on ∂G whenever order $L_j < m$;

vi) there is a complex cone $K(\theta) = \{z : |\arg z| \leq \theta\}$, $0 \leq \theta < \pi/2$, such that for all ϕ in V,

$$\int_G \mathscr{L}\phi(x)\overline{\mathscr{M}\phi(x)}\, dx \in K(\theta).$$

Then for each $u_0 \in W_{2m}$ *and each holder continuous* $L^2(G)$*-valued function* $f(t) = f(\cdot, t)$ *defined on* $[0, \infty)$:

$$\|f(t) - f(\tau)\|_{L^2(G)} \leqq K|t-\tau|^\gamma, \quad K > 0,\ \gamma > 0,$$

there is a unique W_{2m}*-valued function* $u(t) = u(\cdot, t)$ *continuous on* $[0, \infty)$ *and continuously differentiable on* $(0, \infty)$ *such that* $u(0) = u_0$, $u(t) \in V_{2l}$ *and* $\mathscr{M}u'(t) + \mathscr{L}u(t) = f(t)$ *in* $L^2(G)$ *for each* $t > 0$.

Proof We need only note that Lemmas 2.1 and 2.2 allow us to apply the corollary of Theorem 1 with V_l and W_m defined as above, $H = L^2(G)$, $D(L) = V_{2l}$ and $D(M) = W_{2m}$. Q.E.D.

COROLLARY 1 *If $\theta = \pi/2$, the above is true when $u_0 \in V_{2l}$ and f is continuously differentiable on $[0, \infty)$.*

COROLLARY 2 *If f is C^∞ (analytic) on $[0, \infty)$, then u is C^∞ (analytic) on $(0, \infty)$.*

COROLLARY 3 *If $f = 0$, then for $t > 0$, $u(t)$ is (equal a.e. to a function $u(x, t)$ which is) C^∞ in x, analytic in t, satisfies (S) in G and*

$$L_j u(x, t) = 0, 1 \leq j \leq l$$

on $\partial G x\, [0, \infty)$.

The first two corollaries follow from Theorem 1 and known regularity results for evolution equations [14]. The third follows from Sobolev's lemma [2, 14, 18, 44] and the observation that the generator of the resolving semigroup, $A = -M^{-1}$, is a bijection of $V_{2l} \cap H^{2l+r}$ onto $W_{2m} \cap H^{2m+r}$ $r \geq 0$, and hence A^{-k} maps W_{2m} continuously into $V_{2l} \cap H^{2m+2k(l-m)}$, $k \geq 0$. The argument is similar to that following Theorem 4 in the next section.

Remarks 1) Since we have restricted attention to those boundary value problems for which the Garding-type estimates (1.3) and (1.4) are true, the class of boundary conditions is delimited considerably as compared to those solvable by other coercive estimates. See [30, 38, 47] for further discussion.

2) The hypothesis (iii) of Theorem 3 is equivalent to saying the operators $\{M_j\}$ are weaker than the operators $\{L_j\}$. See [7, 304–306; 47, 468–470] for this and other equivalent conditions on systems of boundary operators, some of which we used in the proof of Lemma 2.2.

3. ABSTRACT REGULARITY OF THE WEAK SOLUTION

Consider the weak solution constructed in Theorem 2 of §1 for the homogeneous Eq. (1.1) ($f \equiv 0$) and initial condition u_0 in W_m. This weak solution cannot be a strong solution unless u_0 is in $D(M)$. (This is a consequence of the definitions.) More significant is the fact that in the special case of $V_l = W_m$ and $D(L) = D(M)$, one can show that $u(t_0) \in D(M)$ for some $t_0 \geq 0$

if and only if $u(t) \in D(M)$ for all $t \geqq 0$ [53, 54].) Hence even if all the hypotheses of Theorems 1 and 2 hold, a weak solution might not be regular in the sense of belonging to $D(M)$. We shall use some results on interpolation theory and fractional powers of operators to describe an abstract sufficient condition for the weak solution to belong to (at least) $C([0, \infty], W_m) \cap C^1((0, \infty), D(M))$. Then it is essentially a strong solution on $(0, \infty)$ but takes the initial condition in W_m.

Let A_0 and A_1 be Banach spaces and assume $A_0 \to A_1$. Consider the space $W(2, \alpha, A_0, A_1)$ of (equivalence classes of) functions $u(t)$ such that $t^\alpha u(t) \in L^2(0, \infty; A_0)$, $t^\alpha u'(t) \in L^2(0, \infty; A_1)$, where we use the Bochner integral on strongly measurable functions and derivatives are taken in the sense of distributions. If $-1/2 < \alpha < 1/2$, then $W(2, \alpha, A_0, A_1)$ is a Banach space with norm

$$\max\{\|t^\alpha u(t)\|_{L^2(0,\infty;A_0)}, \|t^\alpha u'(t)\|_{L^2(0,\infty;A_1)}\}.$$

Each $u \in W(2, \alpha, A_0, A_1)$ uniquely determines an element $u(0) \in A_1$ called the trace of u. Let $T(2, \alpha, A_0, A_1)$ be the space of traces $u(0)$ obtained from functions u in $W(2, \alpha, A_0, A_1)$ with the (quotient) norm of an element $a \in T(2, \alpha, A_0, A_1)$ given by

$$\inf\{\|u\|_{W(2,\alpha,A_0,A_1)} : u(0) = a\}.$$

Then $T(2, \alpha, A_0, A_1)$ is a Banach space which we hereafter denote by $[A_0, A_1, \theta] = T(2, \alpha, A_0, A_1)$, $1/2+\alpha = \theta$. Also define $[A_0, A_1; 0] = A_0$ and $[A_0, A_1; 1] = A_1$.

We shall use the following properties of these spaces [34–37]:

(reiteration): $[[A_0, A_1; \theta_0], [A_0, A_1; \theta_1]; \theta] = [A_0, A_1; (1-\theta)\theta_0 + \theta\theta_1]$. (3.1)

(duality): $[A_0, A_1; \theta]' = [A_1', A_0'; 1-\theta]$
whenever A_0 and A_1 are reflexive. (3.2)

(interpolation): If $\{B_0, B_1\}$ is a second pair of Banach spaces with $B_0 \to B_1$, then $\mathscr{L}(A_0, B_0) \cap \mathscr{L}(A_1, B_1) \subseteq \mathscr{L}([A_0, A_1; \theta], [B_0, B_1; \theta])$ for every $\theta, 0 \leqq \theta \leqq 1$. (3.3)

Many other constructions and points of view have been developed which lead to similar results and we refer to the bibliographies of [14, 38] for references to interpolation methods. Our interest here concerns the results for Hilbert spaces where most of these methods are equivalent.

Consider the unbounded operator L introduced in §1 by the pair of spaces $V_l \to H$ and sesquilinear form $l(\cdot, \cdot)$ on V_l. We shall assume that

(1.4) holds; this makes L a regularly accretive operator [25, 28]. Thus, fractional powers, L^θ, $0 \leqq \theta \leqq 1$, can be defined in a natural way, since L is closed and maximal accretive, and each resulting L^θ is closed and maximal accretive. Results analogous to (3.1–3) have been demonstrated for the domains $D(L^\theta)$ [26, 27] and furthermore one has

$$D(L^{1-\theta}) = [D(L), H; \theta], \quad 0 \leqq \theta \leqq 1, \tag{3.4}$$

and the norms are equivalent [37]. A similar result holds for the operator M of §1 when we assume the estimate (1.3).

Hereafter we shall assume the hypotheses of Theorem 2 and also that $f = 0$. Then both M and L are regularly accretive on H, and M is self adjoint. From the coercive estimate (1.4) it follows that $\mathscr{L}: V_l \to V_l'$ and $L: D(L) \to H$ are isomorphisms, where $D(L)$ has the graph norm and is dense in V_l, hence dense in H [32, pp. 9–12].

The densely defined $L: D(L) \to H$ has an adjoint $L^*: D(L^*) \to H$ which is precisely the operator obtained from the form $l^*(\phi, \psi) = \overline{l(\psi, \phi)}$ on V_l. That is, $l(\phi, \psi) = (\phi, L^*\psi)_H$ for $\phi \in V_l$ and $\psi \in D(L^*)$, and we have $D(L^*)$ dense in V_l and $L^*: D(L^*) \to H$ is an isomorphism. Here $D(L^*)$ has the graph norm and the $D(L^*)' - D(L^*)$ antiduality is denoted by $[\cdot, \cdot]_*$. Since $D(L^*)$ is dense in V_l, we can identify V_l' as a subspace of $D(L^*)'$ by restricting a functional in V_l' to $D(L^*)$. Thus we have $[f, \phi]_* = \langle f, \phi\rangle$ for f in V_l' and ϕ in $D(L^*)$.

By taking the continuous dual of the (Banach space) isomorphism $L^*: D(L^*) \to H$, we obtain the isomorphism $(L^*)': H \to D(L^*)'$ defined by $[(L^*)'h, \phi]_* = (h, L^*\phi)_H$ for h in H and ϕ in $D(L^*)$. From the identifications $H \subseteq V_l' \subseteq D(L^*)'$ we have $[(L^*)'h, \phi]_* = l(h, \phi) = \langle \mathscr{L}h, \phi\rangle = [\mathscr{L}h, \phi]_*$ for h in V_l and ϕ in $D(L^*)$. This implies $(L^*)'h \in V_l'$ and $(L^*)' = \mathscr{L}$ on V_l. Denote this extension of $\mathscr{L}: V_l \to V_l'$ by $\mathscr{L}_1 = (L^*)': H \to D(L^*)'$.

Similarly, we obtain an extension $\mathscr{M}_1: H \to D(M^*)'$ of $\mathscr{M}$ by taking the (continuous) dual of $M^*: D(M^*) \to H$. Since M is self adjoint, we have additionally $D(M) = D(M^*)$.

Let $\{u(t): t \geqq 0\}$ be the weak solution obtained in Theorem 2 in the form $u(t) = S(t)u_0$; $\{S(t): t \geqq 0\}$ is the analytic semigroup of bounded operators in W_m generated by $\mathscr{A} = -\mathscr{M}^{-1}\mathscr{L}$. Since this semigroup is analytic, we have the following abstract regularity result. [23, 31, 63, 64].

LEMMA 3.1 *For each $t > 0$, and integer $N \geqq 1$, $\mathscr{A}^N S(t) \in \mathscr{L}(W_m)$, $S^{(N)}(t)$ exists in the uniform topology of $\mathscr{L}(W_m)$ and $S^{(N)}(t) = \mathscr{A}^N S(t)$. In particular, $S(t)$ is a continuous operator from W_m into $D(\mathscr{A}^N)$.*

Our objective in the following is to find a sufficient condition to assure that $D(\mathscr{A}^N) \subseteq D(L)$ for some (large) integer N. This will place $u(t)$ in $D(L)$ for $t > 0$.

Since M and L are regularly accretive and M is self-adjoint, we have the following. [25, 37].

LEMMA 3.2 *$D(M^\phi) = D(M^{*\phi})$ for real ϕ, $0 \leqq \phi \leqq 1/2$; $D(L^\phi) = D(L^{*\phi})$ for real ϕ, $0 \leqq \phi < 1/2$.*

Remark The full strength of self-adjointness of M is not used, but only the equality $D(M^{1/2}) = D(M^{*1/2}) = W_m$. This holds if M arises from a regular boundary value problem [37].

LEMMA 3.3 *For real ϕ, $0 \leqq \phi \leqq 1/2$, $[H, D(M)'; \phi] = [H, D(M^*)'; \phi]$. Also $[H, D(L)'; \phi] = [H, D(L^*)'; \phi]$ whenever $0 \leqq \phi < 1/2$.*

Proof From (3.4) and Lemma 3.2 we have $[D(M), H; 1-\phi] = [D(M^*), H; 1-\phi]$ and (3.2) yields the result for $0 \leqq \phi \leqq 1/2$. The last statement is proved similarly. Q.E.D.

LEMMA 3.4 *Assume that for some real θ, $0 \leqq \theta \leqq 1$, $[D(L), H; \theta] \subseteq D(M)$. Then for each ϕ, $0 \leqq \phi \leqq 1$, we have $[H, D(M)'; \phi] \subseteq [H, D(L)'; \phi(1-\theta)]$.*

Proof By hypotheses, we have $[D(M), H; 1-\phi] \supseteq [[D(L), H; \theta]; H; 1-\phi]$ and this last space is precisely $[D(L), H; 1-\phi+\phi\theta]$ by (3.1). The result follows from (3.2). Q.E.D.

LEMMA 3.5 *Assume that for some real θ, $0 < \theta \leqq 1$, $[D(L), H; \theta] \subseteq D(M)$. Let ϕ be real and $0 \leqq \phi \leqq 1/2$. Then $\mathscr{L}^{-1}\mathscr{M}$ maps $[D(M), H; \phi]$ into $[D(L), H; \phi(1-\theta)]$.*

Proof The operator $\mathscr{M}_1$ is an isomorphism of $[D(M), H; \phi]$ onto $[H, D(M^*)'; \phi]$ by (3.3). But $[H, D(M^*)'; \phi]$ is precisely $[H, D(M)'; \phi]$ by Lemma 3.3; this is contained in $[H, D(L)'; \phi(1-\theta)]$. But we assumed $\theta > 0$, so $\phi(1-\theta) < 1/2$ and Lemma 3.3 implies $[H, D(L)'; \phi(1-\theta)] = [H, D(L^*)'; \phi(1-\theta)]$. Thus we have shown that $\mathscr{M}_1$ maps $[D(M), H; \phi]$ into $[H, D(L^*)'; \phi(1-\theta)]$. Since $\mathscr{L}_1$ is an isomorphism of $D(L)$ onto H and of H onto $D(L^*)'$, we have by (3.3) that $\mathscr{L}_1^{-1}$ is an isomorphism of $[H, D(L^*)'; \phi(1-\theta)]$ onto $[D(L), H; \phi(1-\theta)]$. The desired result follows by tracing the composite operator $\mathscr{L}_1^{-1}\mathscr{M}_1$. Q.E.D.

LEMMA 3.6 *Assume the hypotheses of Lemma 3.5. Then there is an integer N such that $(\mathscr{L}^{-1}\mathscr{M})^N$ maps W_m into $D(L)$.*

Proof Since we have $[D(M), H; 1/2] = W_m$, $\mathscr{M}_1 : W_m \to W'_m \subseteq V'_l$ and $\mathscr{L}_1^{-1} : V'_l \to V_l$, we replace $\mathscr{M}_1$ and $\mathscr{L}_1$ by their respective restrictions, $\mathscr{M}$ and $\mathscr{L}$. From Lemma 3.5 and an easy induction we see that $(\mathscr{L}^{-1}\mathscr{M})^{N-1}$ maps W_m into $[D(L), H; (1\theta)^{N-1}/2]$ and this is contained in $D(M)$ if N is chosen so large that $(1-\theta)^{N-1}/2 \leq \theta$. Then $(\mathscr{L}^{-1}\mathscr{M})^N$ maps W_m into $D(L)$.

Q.E.D.

Remark For $\theta > 0$ but very small, we may decrease the number of iterations necessary to carry W_m into $D(L)$ by the following observation. Lemma 3.5 showed that $\mathscr{L}^{-1}\mathscr{M}$ maps $[D(M), H; \phi]$ into $[D(L), H; \phi(1-\theta)]$ if $0 \leq \phi \leq 1/2$. Now if $\phi(1-\theta) \leq \theta$, then the latter space is contained in $D(M)$ and one more iteration suffices. Otherwise, $\phi(1-\theta) > \theta$ and we have $[D(L), H; \phi(1-\theta)] = [D(M), H; \phi-(\theta/1-\theta)]$. Hence, the interpolation variable decreases by the amount $\theta/(1-\theta)$ at each iteration until it is $\leq \theta$. Thus $(1-\theta)/2\theta$ iterations will suffice.

THEOREM 4 *Let the hypotheses of Theorem 2 of §1 hold. Assume furthermore that $[D(L), H; \theta] \subseteq D(M)$ for some real θ, $0 < \theta \leq 1$. Then for each $u_0 \in W_m$, the weak solution $\{u(t) : t \geq 0\}$ of the homogeneous form of Eq. (1.1) is in $C^\infty((0, \infty), D(L))$.*

Proof The solution is obtained from the semigroup by $u(t) = S(t)u_0$. Choose N as in Lemma 3.6 and let $t > 0$. Lemma 3.1 shows that $u(t) \in D(L)$. Regarding the differentiability of the function $u : (0, \infty) \to D(L)$, we have for $\delta \neq 0$ sufficiently small

$$(u(t+\delta)-u(t))/\delta = \mathscr{A}^{-N}\{(S(t/2+\delta)-S(t/2))/\delta\}\mathscr{A}^N S(t/2)u_0. \tag{3.5}$$

This follows from the semigroup property, the commutativity of $\mathscr{A}$ with each $S(\tau)$, and the fact that $\mathscr{A}^N S(t/2)u_0$ is in W_m (Lemma 3.1). Since $S(\cdot)$ is strongly-differentiable in W_m at $t/2$, the expression in (3.5) to the right of $\mathscr{A}^{-N}$ converges in W_m as $\delta \to 0$. Hence the left side of (3.5) converges in $D(L)$. (Note that $\mathscr{A}^{-N}$ is continuous from W_m into $D(L)$; this follows from Lemma 3.6 and the closed-graph theorem [53, 54].)

This argument can be repeated to show u is infinitely differentiable.

Q.E.D.

Remark The above proofs apply if $D(M^{1/2}) = D(M^{*1/2})$. The result of Theorem 4 also follows if we drop this requirement but assume the following: $u_0 \in [D(M), H; \phi]$ for some real ϕ, $0 \leq \phi < 1/2$. However, the existence of weak solutions for non-self-adjoint M is still open.

4. WEAK SOLUTIONS

We shall apply the abstract results of Sections 1 and 3 to obtain and describe weak solutions of the Sobolev equation. Lemma 2.1 and the following result (similar to Lemma 2.2) will be used to obtain the realizations of the differential operators and boundary conditions that occur.

LEMMA 4.1 *Let $\{M_j\}$ and $\{L_j\}$ be normal systems of boundary operators as in Lemma 2.1. ($\{M_j\}$ contains m operators, each of order $< 2m$, etc.) Let the systems be indexed such that $\{M_1, \ldots, M_p\}$ and $\{L_1, \ldots, L_q\}$ denote all of the operators of order $< m$. (Then $0 \leqq p \leqq m$ and the indicated set is empty if $p = 0$.) Assume the following: For every $\phi \in C^\infty$, we have $M_j\phi = 0$ on ∂G, $1 \leqq j \leqq p$, if and only if $L_j\phi = 0$ on ∂G, $1 \leqq j \leqq q$. Then V_l is a dense subset of W_m.*

The proof is similar to that of Lemma 2.2, so we shall not give it. As a corollary one finds that $p = q$ and the M_j and L_j with $1 \leqq j \leqq p$ are in one-to-one correspondence by means of their orders.

The fundamental Theorem 2, Lemma 2.1 and Lemma 4.1 yield the following result.

THEOREM 5 *Assume that the two regular boundary value problems $\{\mathscr{L}; L_j, 1 \leqq j \leqq l\}$ and $\{\mathscr{M}; M_j, 1 \leqq j \leqq m\}$ have been determined by sesquilinear forms* (2.1) *and* (2.10), *respectively, and the stable boundary operators of each problem as in Section* 2. *Assume the following*: (i) For any ϕ in C^∞, we have $M_j\phi = 0$ on ∂G for those M_j of order $< m$ if and only if $L_j\phi = 0$ on ∂G for those L_j of order $< m$. (ii) There is a $k_m > 0$ such that (1.3) and (1.7) hold for every ϕ in W. There is a $k_l > 0$ such that (1.4) holds for ϕ in V. *Then for each u_0 in W_m) and Holder-continuous f:$[0, \infty) \to L^2(G)$, there exists a unique function $u \in C([0, \infty), W_m) \cap C^1((0, \infty), W_m)$ such that $u(0) = u_0$ and, for $t > 0$, $u(t) \in V_l$ and*

$$m(u'(t), \phi) + l(u(t), \phi) = \langle f(t), \bar{\phi} \rangle \tag{4.1}$$

for all ϕ in V_l.

Remarks 1) The Eq. (4.1) is more natural for our concrete weak problems than is (1.1) in V_l' since V_l' is not necessarily a space of distributions. Since (4.1) holds for all ϕ in C_0^∞, it follows that (1.1) is satisfied in $\mathscr{D}'$.

2) More general f can be used but the structure of W_m' is often difficult to describe.

3) These results are also valid when $m = l$.

In order to verify in our example the hypothesis of Theorem 4 involving the interpolation subspaces, we shall need the next result.

LEMMA 4.2 *Let $p > q \geqq 0$ be integers and $r = (1-\theta)p+\theta q$ be integer, where $0 \leqq \theta \leqq 1$. Then $[V_p, V_q; \theta] \subseteq V_r$.*

Proof First note that the identity is continuous from V_p into H^p and from V_q into H^q, hence by interpolation from $[V_p, V_q; \theta]$ into $[H^p, H^q; \theta]$. But it is well known [38, p. 49] that $[H^p, H^q; \theta] = H^r$, so $[V_p, V_q; \theta]$ is continuously imbedded in H^r.

Now, let $f \in [V_p, V_q; \theta]$. Since V_p is dense in $[V_p, V_q; \theta]$, there is a sequence $f_n \in V_p$ such that $f_n \to f$ in $[V_p, V_q; \theta]$. But by the above paragraph, this sequence is Cauchy in H^r, hence converges to some $g \in V_r$. By uniqueness of limits in H, we have $f = g \in V_r$. Q.E.D.

THEOREM 6 *In addition to the hypotheses of Theorem 5, assume $V \subseteq W$, $m < l$ and $f \equiv 0$. Then the function u is, for each $t > 0$, equal a.e. to a function $u(\cdot, t)$ in V. For each $x \in G$, $u(x, \cdot)$ is analytic on $(0, \infty)$, and $u(x, t)$ satisfies (S) on $Gx(0, \infty)$.*

Proof By the method of proof of Theorem 4, we need only show that $[V_{2l}, V_0; \theta] \subseteq W_{2m}$ for some $\theta > 0$. From Lemma 4.2 we have $[V_{2l}, V_0; 1/2l] \subseteq V_{2l-1}$. But $l > m$, so $2l-1 > 2m$ and from the hypotheses $V \subseteq W$ we obtain $V_{2l-1} \subseteq W_{2m}$. Hence we have $[D(L), H; 1/2l] \subseteq D(M)$.

Q.E.D.

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Date Communicated: June 1, 1971