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ON TWO-PHASE STEFAN PROBLEM ARISING FROM A MICROWAVE HEATING PROCESS

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Abstract. In this paper we study a free boundary problem modeling a phase-change process by using microwave heating. The mathematical model consists of Maxwell's equations coupled with nonlinear heat conduction with a phase-change. The enthalpy form is used to characterize the phase-change process in the model. It is shown that the problem has a global solution.

1. **Introduction.** Suppose that a material with solid and liquid phases occupies a bounded domain $\Omega \subset R^3$ with C^1-boundary $\partial\Omega$. If we supply heat to the material by using intense microwaves from the boundary $\partial\Omega$, the solid phase of the material will begin to melt. To describe this physical process, we introduce the electric and magnetic fields $\mathbf{E}(x,t)$ and $\mathbf{H}(x,t)$, respectively, in Ω. Hereafter, a bold letter represents a vector or vector function in three-space dimensions. Then $\mathbf{E}(x,t)$ and $\mathbf{H}(x,t)$ satisfy the following well-known Maxwell equations (see [9]):

$$\varepsilon\mathbf{E}_t + \sigma\mathbf{E} = \nabla\times\mathbf{H}, \qquad (x,t)\in Q_T, \tag{1.1}$$

$$\mu\mathbf{H}_t + \nabla\times\mathbf{E} = 0, \qquad (x,t)\in Q_T, \tag{1.2}$$

where $Q_T = \Omega\times(0,T]$ and $\mathbf{J}(x,t) = \sigma\mathbf{E}$ is used by Ohm's law, ε, μ and σ are the electric permittivity, magnetic permeability and the electric conductivity, respectively.

Let $u(x,t)$ be the temperature in Q_T. During the heating process, the local density of heat source generated by microwaves is equal to $\mathbf{E}\cdot\mathbf{J} = \sigma(x,u)|\mathbf{E}|^2$, where the electric conductivity $\sigma = \sigma(x,u)$ typically depends on u such as (see [12, 13])

$$\sigma(x,u) = \frac{a(x)}{b(x)+c(x)u}, \text{or } \sigma(x,u) = a(x)e^{-b(x)u},$$

where $a(x), b(x)$ and $c(x)$ are positive functions. It also may have a jump discontinuity for the temperature changing from solid phase to liquid phase:

$$\sigma(x,u) = \begin{cases} \sigma_l(x,u), & \text{if } u(x,t) < m, \\ \sigma_s(x,u), & \text{if } u(x,t) > m, \end{cases}$$

where the subscript l or s represents the function in liquid or solid phase and m is the melting temperature.

By using the enthalpy method, we find that the temperature $u(x,t)$ satisfies the following heat equation in the weak sense (see [4, 5] and also Remark 1.1 below),

$$A(u)_t - \nabla\cdot[k(x,u)\nabla u] = \sigma(x,u)|\mathbf{E}|^2, \qquad (x,t)\in Q_T, \tag{1.3}$$

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where

$$A(u) = \begin{cases} u-1, & \text{if } u < m, \\ [m-1, m], & \text{if } u = m, \\ u, & \text{if } u > m, \end{cases}$$

and the coefficient of heat conduction k may be different in solid and liquid phases,

$$k(x,u) = \begin{cases} k_l(x,u), & \text{if } u(x,t) < m, \\ k_s(x,u), & \text{if } u(x,t) > m, \end{cases}$$

Because of the heat source in Eq.(1.3), the interface set $\Gamma_T = \{(x,t) \in Q_T : u(x,t) = m\}$ may have positive area, i.e. a mushy region may exist. In this case one has to define the value of heat conductivity $k(x,u)$ and $\sigma(x,u)$ on Γ_T and Eq.(1.2) is understood as an inclusion ([4]:

$$A(u)_t - \nabla[k(x,u)\nabla u] - \sigma(x,u)|\mathbf{E}|^2 \ni 0, \qquad (x,t) \in Q_T.$$

Define

$$\sigma(x,m) \text{ is between the values } \sigma_l(x,m+) \text{ and } \sigma_s(x,m-) \text{ for any } x \in \Gamma_T,$$
$$k(x,m) \text{ is between the values of } k_l(x,m+) \text{ and } k_s(x,m-) \text{ for any } x \in \Gamma_T,$$

where $\sigma_l(x,m+) = \lim_{u\to m+} \sigma_l(x,u)$ and other quantities are defined similarly.

If $\sigma(x,u)$ and $k(x,u)$ are independent of x. Then one can simply define

$$\sigma(m) \in [\sigma_0, \sigma_1],\ k(m) \in [k_0, k_1],$$

where constants k_0, k_1, σ_0 and σ_1 are defined as follows:

$$k_0 = min\{k_l(m), k_s(m)\}, k_1 = max\{k_l(m), k_s(m)\},$$
$$\sigma_0 = min\{\sigma_l(m), \sigma_s(m)\}, \sigma_1 = max\{\sigma_l(m), \sigma_s(m)\}.$$

To complete the problem, we prescribe the following initial and boundary conditions:

$$\mathbf{N} \times \mathbf{E}(x,t) = \mathbf{N} \times \mathbf{G}(x,t), \qquad (x,t) \in S_T \tag{1.4}$$
$$u_n(x,t) = 0, \qquad (x,t) \in S_T, \tag{1.5}$$
$$\mathbf{E}(x,0) = \mathbf{E}_0(x),\ \mathbf{H}(x,0) = \mathbf{H}_0(x),\ u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.6}$$

where $\mathbf{G}(x,t)$ is given external vector function on $S_T = \partial\Omega \times [0,T]$, $\mathbf{N}$ is the outward normal on $S = \partial\Omega$, $u_n(x,t) := \nabla u \cdot \mathbf{N}$ is the normal derivative on S, $\mathbf{E}_0(x), \mathbf{H}_0(x)$ and $u_0(x)$ are the prescribed initial electric, magnetic fields and initial temperature.

The Stefan-type free boundary problems have been studied extensively by many researchers (see monographs [6, 11, 18] and many conference proceedings). The classical enthalpy method is widely used to describe a phase-change process (see [1, 3, 4, 5, 11, 15, 16] etc. for examples). For microwave heating problems without phase-change, some research has been carried out (see [7, 8, 12, 13, 19] etc. and also see recent lecture notes [22] Chapter 6 for the theory). When a phase-change takes place, Coleman [2] studied the microwave melting in one-space dimension and obtained some numerical solutions. In [17], Pangrie et al. used time-harmonic Maxwell's equations and the enthalpy method to model the microwave melting process and obtained the numerical solution for a radially symmetric domain by using finite-difference method. In [20] the author studied a phase-change problem arising from microwave heating processes in one-space dimension, where a kinetic type condition is given on the interface due to the superheating phenomenon. Global existence and uniqueness are established in [20]. We would also like to mention a related work on a phase-change problem for the induction heating ([21]) in which displacement current is neglected and magnetic field is assumed to be time-harmonic. However, none of the previous works deal with the fully coupled system (1.1)-(1.3) with phase-change. One of the difficulties is that there is not much known about the regularity of solutions to Maxwell's equations with variable coefficients. Another difficulty is the nonlinear term $\sigma(x,u)|\mathbf{E}|^2$ which only belongs to $L^1(Q_T)$. Moreover, the electric conductivity may have a jump discontinuity from solid to liquid phase. In this paper we study the phase-change problem (1.1)-(1.6). By using methods from [21] it is shown that under certain conditions on coefficients of the system (1.1)-(1.3) the problem (1.1)-(1.6) has a global weak solution. The global existence is also established for the case when the electric and magnetic fields are assumed to be time-harmonic. Moreover, for one-space dimension we prove the existence of a weak solution for $\sigma(x,t,u)$ with linear growth in the u-variable. In this paper the uniqueness is left out as an open question even for space dimension one.

This paper is organized as follows. In section 2, we prove that the problem (1.1)-(1.6) has a weak solution in Q_T for any $T > 0$. In section 3, we study the problem for time-harmonic electric and magnetic fields and obtain the global solution for the problem. In section 4, we study the problem for one-space dimension and prove the existence of a weak solution for a more general function $\sigma(x,u)$.

2. **Global Existence of Weak Solutions.** In this section we first define weak solution to the problem and then consider an approximate problem by the standard approximation for $A(u)$ and $\sigma(u)$. It is shown that the approximated problem has a unique solution. Moreover, some uniform estimates for the approximate solution are derived. Finally, we establish the global existence to the problem (1.1)-(1.5) by using a compactness argument.

We list some basic assumptions for the coefficients and the known data.

H(2.1): (a)Let $\varepsilon(x)$ and $\mu(x)$ be in $L^\infty(\Omega)$ with a positive lower bound

$$0 < r_0 \le \varepsilon(x), \mu(x) \le R_0, \qquad x \in \Omega,$$

where r_0 and R_0 are positive constants.

(b) $\sigma(x,u)$ is non-negative and is bounded in $\Omega \times [M,\infty)$ for some large $M > 0$ and σ_0:

$$0 \le \sigma(x,u) \le \sigma_0, u\sigma(x,u) \le \sigma_0, \qquad (x,u) \in \Omega \times [M,\infty).$$

(c) The functions $k_l(x,u)$ and $k_s(x,u)$ are of class $C^{1+\alpha}(\Omega\times R)$ and bounded with a positive lower bound:

$$0 < r_0 \le k_s(x,u), k_l(x,u) \le R_0, \qquad (x,u) \in \Omega \times [0,\infty).$$

H(2.2): (a) Let $u_0(x)$ be in $\in L^\infty(\Omega)$ and $\mathbf{E}_0(x), \mathbf{H}_0(x) \in L^2(\Omega)^3$.

(b) Let $\mathbf{G}(x,t) \in C([0,T]; H^{\frac{1}{2}}(S))$.

It is easy to see that the conditions on $\sigma(x,u)$ are satisfied for $\sigma(x,u) = \frac{1}{(1+u)^p}$ with $p \ge 1$ or $\sigma(x,u) = a(x)e^{-u}$. For the reader's convenience, we recall some function spaces associated with Maxwell's equations. Other Sobolev spaces are the same as in [10]. Let

$$\begin{aligned} H(curl,\Omega) &= \{\mathbf{V} \in L^2(\Omega)^3 : \nabla\times\mathbf{V} \in L^2(\Omega)^3\}; \\ H_0(curl,\Omega) &= \{\mathbf{V} \in H(curl,\Omega) : \mathbf{N}\times\mathbf{V} = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

$H(curl,\Omega)$ is a Hilbert space equipped with inner product

$$(\mathbf{V},\mathbf{K}) = \int_\Omega [\mathbf{V}\cdot\mathbf{K} + (\nabla\times\mathbf{V})\cdot(\nabla\times\mathbf{K})]dx.$$

Definition 2.1: A triple of functions $(\mathbf{E}(x,t), \mathbf{H}(x,t), u(x,t))$ is said to be a weak solution to the problem (1.1)-(1.6), if

$$\mathbf{E}(x,t), \mathbf{H}(x,t) \in C([0,T]; L^2(\Omega)),$$

and $u(x,t) \in L^2(0,T;H^1(\Omega)) \bigcap C([0,T];L^2(\Omega))$, and they satisfy the following integral identities:

$$\int_0^T\int_\Omega [-\varepsilon\mathbf{E}\cdot\mathbf{\Psi}_t + \sigma\mathbf{E}\cdot\mathbf{\Psi}]dxdt = \int_0^T\int_\Omega [\mathbf{H}\cdot(\nabla\times\mathbf{\Psi})]dxdt + \int_\Omega \varepsilon\mathbf{E}_0\cdot\mathbf{\Psi}(x,0)dx,$$

$$\int_0^T\int_\Omega [-\mu\mathbf{H}\cdot\mathbf{\Phi}_t + \mathbf{E}\cdot(\nabla\times\mathbf{\Phi})]dxdt = \int_\Omega [\mu\mathbf{H}_0(x)\cdot\mathbf{\Phi}(x,0)]dx,$$

$$\int_0^T\int_\Omega [-A(u)\psi_t + k(x,u)\nabla u\nabla\psi]dxdt = \int_0^T\int_\Omega \sigma(x,u)|\mathbf{E}|^2\psi dxdt + \int_\Omega A(u_0)\psi dx$$

for any test vector functions $\mathbf{\Psi}, \mathbf{\Phi} \in L^2(0,T;H_0(curl,\Omega)) \bigcap C([0,T];L^2(\Omega)^3)$ and any test function $\psi \in H^1(0,T;H^1(\Omega))$ with $\mathbf{\Psi}(x,T) = \mathbf{\Phi}(x,T) = 0$ and $\psi(x,T) = 0$ on Ω.

Since the weak solution $\mathbf{E}(x,t) \in L^2(\Omega)$, we have to specify the boundary condition (1.4) in the weak sense. Note that

$$\mathbf{H}(x,t) = \mathbf{H}_0(x) - \frac{1}{\mu(x)}\int_0^t \nabla\times\mathbf{E}(x,\tau)d\tau = \mathbf{H}_0(x) - \frac{1}{\mu(x)}\nabla\times\mathbf{W}(x,t),$$

where

$$\mathbf{W}(x,t) = \int_0^t \mathbf{E}(x,\tau)d\tau.$$

It follows that $\mathbf{H} \in L^2(\Omega)^3$ implied $\nabla\times\mathbf{W} \in L^2(\Omega)^3$ for each a.e. fixed $t \in [0,T]$. Consequently, the trace $\mathbf{N}\times\mathbf{W}(x,t)$ is well-defined on $\partial\Omega$. We define

$$\mathbf{N}\times(\mathbf{E}(x,t) - \mathbf{G}(x,t)) = 0, \qquad (x,t) \in S_T$$

if and only if

$$\mathbf{N}\times[\mathbf{W}(x,t)-\int_0^t \mathbf{G}(x,\tau)d\tau]=0, \qquad (x,t)\in S_T.$$

Introduce a new function,

$$U(x,t):=K(x,u)=\int_m^u k(x,s)ds, \qquad (x,t)\in Q_T.$$

Then the assumption H(2.1)(b) implies that the inverse function $u(x,t)=K^{-1}(x,U)$ exists. Consequently, Eq. (1.3) can be written in the weak sense as follows:

$$A^*(x,U)_t-\Delta U=\sigma^*(x,U)|\mathbf{E}|^2, \qquad (x,t)\in Q_T,$$

where

$$A^*(x,U)=\begin{cases} K_s^{-1}(x,U)-1, & \text{if } U<0,\\ [-1,0], & \text{if } U=0,\\ K_l^{-1}(x,U), & \text{if } U>0,\end{cases}$$

and $K_s^{-1}(x,U), K_l^{-1}(x,U)$ are the inverse functions of $K_s(x,u), K_l(x,u)$, respectively. Moreover,

$$\sigma^*(x,U)=\begin{cases} \sigma(x,K_s^{-1}(x,U)), & \text{if } U<0,\\ [\sigma_0,\sigma_1], & \text{if } U=0,\\ \sigma(x,K_l^{-1}(x,U)), & \text{if } U>0.\end{cases}$$

¿From now on, instead of using $U(x,t), A^*(x,U)$ and $\sigma^*(x,U)$, we will continue to use notation $u(x,t), A(x,u)$ and $\sigma(x,u)$ for simplicity. By assumption H(2.2), there exists an extension for $\mathbf{G}(x,t)$ such that $\mathbf{G}(x,t)\in C([0,T];H^1(\Omega)^3)$. Moreover, from the assumption H(2.1)(c) there exists a constant $a_0>0$ such that $A'(x,u):=A_u(x,u)\geq a_0$ for all $(x,u)\in\Omega\times R$ whenever $u\neq m$.

Let $A_n(x,u)$ and $\sigma_n(x,u)$ be smooth approximations of $A(x,u)$ and $\sigma(x,u)$, respectively. Moreover, we require that

$$A_n(x,u)=A(x,u), \sigma_n(x,u)=\sigma(x,u), \text{if } |u-m|\geq \tfrac{1}{n},$$
$$A_n'(x,u)\geq \frac{r_0}{2}, A_n(x,u)\to A(x,u), \sigma_n(x,u)\to\sigma(x,u)$$

strongly in $L^2(\Omega\times[-M,M])$ for some large $M>0$ as $n\to\infty$. We also make a smooth approximation of $u_0(x)$, denoted by $u_{0n}(x)$, such that $\nabla u_{0n}(x)=0$ on S and $u_{0n}(x)\to u_0(x)$ strongly in $L^2(Q_T)$.

Consider the following approximate system:

$$\varepsilon(x)\mathbf{E}_t+\sigma_n(x,u)\mathbf{E}=\nabla\times\mathbf{H}, \qquad (x,t)\in Q_T, \tag{2.1}$$
$$\mu(x)\mathbf{H}_t+\nabla\times\mathbf{H}=0, \qquad (x,t)\in Q_T, \tag{2.2}$$
$$A_n(x,u)_t-\Delta u=\sigma_n(x,u)|\mathbf{E}|^2, \qquad (x,t)\in Q_T, \tag{2.3}$$
$$\mathbf{N}\times\mathbf{E}(x,t)=\mathbf{N}\times\mathbf{G}(x,t), \qquad (x,t)\in S_T, \tag{2.4}$$
$$u_n(x,t)=0, \qquad (x,t)\in S_T, \tag{2.5}$$
$$\mathbf{E}(x,0)=\mathbf{E}_0(x), \mathbf{H}(x,0)=\mathbf{H}_0(x), u(x,0)=u_{0n}(x), x\in\Omega. \tag{2.6}$$

¿From Theorem 2.1 ([19]), the problem (2.1)-(2.6) has a unique weak solution

$$(\mathbf{E}_n(x,t),\mathbf{H}_n(x,t))\in C([0,T];H_0(curl,\Omega))\times C([0,T];H(curl,\Omega))$$

and

$$u_n(x,t)\in C([0,T];L^2(\Omega))\bigcap L^2(0,T;H^1(\Omega)).$$

Moreover,

$$\mathbf{E}_n(x,t)-\mathbf{G}(x,t)\in L^\infty(0,T;H_0(curl,\Omega)).$$

Furthermore, since $u_0(x)\in L^\infty(\Omega)$ and $\sigma_n(x,u)|\mathbf{E}|^2\geq 0$, it follows from the maximum principle that there exists a constant $M_0>0$ independent of n such that $u_n(x,t)\geq -M_0$ on Q_T.

Now we derive some uniform estimates.

Lemma 2.1: There exists constant C_1 such that

$$\sup_{0\leq t\leq T}\int_\Omega[|\mathbf{E}_n|^2+|\mathbf{H}_n|^2]dx\leq C_1,$$

where C_1 depends only on the known data.
Proof: For simplicity, we shall drop the subscript n for the solution $(\mathbf{E}_n, \mathbf{H}_n, u_n)$ whenever without causing confusion. To derive the estimate, we take the inner product by $\mathbf{E}(x,t) - \mathbf{G}$ to Eq. (2.1) and by $\mathbf{H}(x,t)$ to Eq. (2.2), respectively, to obtain:

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\int_\Omega [\varepsilon|\mathbf{E}|^2]dx + \int_\Omega \sigma(x,u)|\mathbf{E}|^2 dx \\ &= \int_\Omega \varepsilon \mathbf{E}\cdot \mathbf{G}dx + \int_\Omega [\sigma \mathbf{E}\cdot \mathbf{G}]dx + \int_\Omega [\nabla\times \mathbf{H}\cdot(\mathbf{E}-\mathbf{G})]dx, \\ &\frac{d}{dt}\frac{1}{2}\int_\Omega [\mu|\mathbf{H}|^2]dx + \int_\Omega [\nabla\times\mathbf{E}\cdot\mathbf{H}]dx = 0. \end{aligned}$$

We add up the above equations and use the fact,

$$\int_\Omega \nabla\times\mathbf{H}\cdot(\mathbf{E}-\mathbf{G})dx = \int_\Omega \mathbf{H}\cdot[\nabla\times(\mathbf{E}-\mathbf{G})]dx,$$

to obtain

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\int_\Omega [\varepsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2]dx + \int_\Omega \sigma(x,u)|\mathbf{E}|^2 dx \\ &\le C\int_\Omega [|\mathbf{E}|^2 + |\mathbf{H}|^2]dx + C\int_\Omega [|\mathbf{G}|^2 + |\nabla\times\mathbf{G}|^2]dx, \end{aligned}$$

where C depends only on L^∞-bounds of $\varepsilon(x), \mu(x)$ and $\sigma(x,u)$, but independent of n.

Gronwall's inequality yields the desired estimate.

Q.E.D.

Lemma 2.2: There exists a constant C_2 such that

$$\sup_{0\le t\le T}\int_\Omega |u_n|^2 dx + \int\int_{Q_T} |\nabla u_n|^2 dxdt \le C_2,$$

where C_2 depends only on known data.
Proof: Since $A_n'(x,u) \ge \frac{r_0}{2}$, the inverse of the function for $v(x,t) := A_n(x,u)$ exists, denoted by $u = B_n(x,v)$. Then

$$\begin{aligned} &\int_0^t\int_\Omega A_n(u)_t u dxdt = \int_0^t [\frac{d}{dt}\int_\Omega\int_0^v B_n(x,s)dsdx]dt \\ &= \int_\Omega\int_0^v B_n(x,s)dsdx - \int_\Omega\int_0^{A_n(u_0)} B_n(x,s)dsdx \\ &\ge b_0\int_\Omega u^2dx - C, \end{aligned}$$

for some $b_0 > 0$ since $k_0 s \le B_n(x,s) \le k_1(s+1)$ from the assumption H(2.1)(c).

On the other hand, it is clear that

$$-\int_0^t\int_\Omega (\Delta u)u dxdt = \int_0^t\int_\Omega |\nabla u|^2 dxdt.$$

Moreover, by Lemma 2.1 and the assumption H(2.1) we have

$$\int_\Omega \sigma(x,u)u|\mathbf{E}|^2 dx \le C,$$

where C depends only on known data.

We sum up the above estimates to obtain

$$\int_\Omega u^2 dx + \int_0^t\int_\Omega |\nabla u|^2 dxdt \le C_2,$$

where C_2 depends only on known data.

Q. E. D.

To prove the existence of a weak solution for the problem (1.1)-(1.6), we need the following lemma from [19].

Lemma 2.3: Suppose $\sigma_n(x,t) \to \sigma(x,t)$ strongly in $L^2(Q_T)$. Let $(\mathbf{E}_n(x,t), \mathbf{H}_n(x,t))$ be the solution of the Maxwell equations:

$$\begin{aligned} &\varepsilon \mathbf{E}_t + \sigma_n(x,t)\mathbf{E} = \nabla \times \mathbf{H}, && (x,t) \in Q_T, \\ &\mu \mathbf{H}_t + \nabla \times \mathbf{E} = 0, && (x,t) \in Q_T, \\ &\mathbf{N} \times \mathbf{E} = \mathbf{N} \times \mathbf{G}, && (x,t) \in \partial\Omega \times (0,T], \\ &\mathbf{E}(x,0) = \mathbf{E}_0(x), \mathbf{H}(x,0) = \mathbf{H}_0(x), && x \in \Omega. \end{aligned}$$

Let $(\mathbf{E}(x,t), \mathbf{H}(x,t))$ be the solution of the above Maxwell equations where $\sigma_n(x,t)$ is replaced by $\sigma(x,t)$. Then $(\mathbf{E}_n(x,t), \mathbf{H}_n(x,t))$ converges to $(\mathbf{E}(x,t), \mathbf{H}(x,t))$ strongly in $L^2(Q_T)$.

Q.E.D.

Theorem 2.4: The problem (1.1)-(1.6) possesses at least one weak solution in Q_T for any $T > 0$.
Proof: From Lemma 2.1-2.2 and the weak compactness, we know, after extracting a subsequence if necessary, that

$$\begin{aligned} &\mathbf{E}_n(x,t) \to \mathbf{E}(x,t), \mathbf{H}_n(x,t) \to \mathbf{H}(x,t) \text{ in weak-* } L^\infty(0,T; L^2(\Omega)^3), \\ &u_n(x,t) \to u(x,t) \text{ weakly in } L^2(0,T; H^1(\Omega)). \end{aligned}$$

Moreover, by applying the result of Lemma 5.1 from [15] we see that $u_n(x,t)$ converges to $u(x,t)$ strongly in $L^2(Q_T)$ and almost everywhere in Q_T.

We multiply Eq.(2.1) and Eq.(2.2) by test functions $\mathbf{\Psi}(x,t)$ and $\mathbf{\Phi}(x,t)$, respectively, in $H^1(0,T; H_0(curl,\Omega))$ to obtain

$$\begin{aligned} &\int_0^T \int_\Omega [-\varepsilon \mathbf{E}_n \cdot \mathbf{\Psi}_t + \sigma_n(x,u_n)\mathbf{E}_n \cdot \mathbf{\Psi}] dxdt \\ &= \int_0^T \int_\Omega [\mathbf{H}_n \cdot (\nabla \times \mathbf{\Psi}) dxdt + \int_\Omega \varepsilon \mathbf{E}_0(x) \cdot \mathbf{\Psi}(x,0) dx, \\ &\int_\Omega [-\mu \mathbf{H}_n \cdot \mathbf{\Phi}_t + \mathbf{E}_n \cdot (\nabla \times \mathbf{\Phi})] dxdt = \int_\Omega [\mu \mathbf{H}_0(x) \cdot \mathbf{\Phi}(x,0)] dx. \end{aligned}$$

After taking the limit as $n \to \infty$, we see that $(\mathbf{E}, \mathbf{H})$ satisfies the integral identities in Definition 2.1 if we can prove $\mathbf{J}_n(x,t) := \sigma_n(x,u_n)\mathbf{E}_n$ converges to $\mathbf{J}(x,t) = \sigma(x,u)\mathbf{E}(x,t)$ weakly in $L^2(Q_T)$. We omit the proof here since it can be done by using the same technique as for a more complicated term $\sigma_n(x,u_n)|\mathbf{E}_n|^2$ below (see detailed proof below). Moreover, since $\mathbf{E}_n(x,t) - \mathbf{G}(x,t) \in L^\infty(0,T; H_0(curl,\Omega))$ and $\mathbf{H}_n(x,t) \in L^\infty([0,T]; L^2(\Omega))$, it follows that $\nabla \times \mathbf{W}_n(x,t) \in L^\infty([0,T], L^2(\Omega))$, where

$$\mathbf{W}_n(x,t) = \int_0^t \mathbf{E}_n(x,\tau) d\tau.$$

Thus, the trace $\mathbf{N} \times \mathbf{W}_n$ is well defined on $\partial\Omega$. Since $\mathbf{N} \times \mathbf{W}_n = \mathbf{N} \times \int_0^t \mathbf{G}(x,\tau)d\tau$ and $\mathbf{N} \times [\mathbf{E}_n - \mathbf{G}] = 0$ is equivalent to $\mathbf{N} \times [\mathbf{W}_n - \int_0^t \mathbf{G}(x,\tau)d\tau] = 0$. It follows that the boundary condition (1.4) holds.

For any small $\gamma > 0$ by Egorof's theorem there exists a subset $Q \subset Q_T$ with $|Q_T \backslash Q| < \gamma$ such that $u_n(x,t)$ converges to $u(x,t)$ uniformly on Q. Set $Q_\gamma = \{(x,t) \in Q : |u(x,t) - m| > \gamma\}$. Then, for $(x,t) \in Q_\gamma$, if n is sufficiently large,

$$|u_n(x,t) - m| \geq \frac{\gamma}{2} \geq \frac{1}{n}.$$

On the other hand, for any $(x,t) \in Q \backslash Q_\gamma$

$$|u_n - u| \leq |u_n - m| + |u - m| \leq 2\gamma,$$

provided that n is large enough.

Let ϕ be a smooth test vector function.

$$\begin{aligned} &\int\int_{Q_T} [\sigma_n(x,u_n)|\mathbf{E}_n|^2 - \sigma(x,u)|\mathbf{E}|^2]\phi dxdt \\ &= \int\int_Q [\sigma_n(x,u_n)|\mathbf{E}_n|^2 - \sigma(x,u)|\mathbf{E}|^2]\phi dxdt + \\ &\int\int_{Q_T \backslash Q} [\sigma_n(x,u_n)|\mathbf{E}_n|^2 - \sigma(x,u)|\mathbf{E}|^2]\phi dxdt \\ &:= I_1 + I_2. \end{aligned}$$

It is clear that $I_2 \to 0$ as $\gamma \to 0$ since $|Q_T \backslash Q| < \gamma$.

$$\begin{aligned} I_1 &:= \int\int_{Q_\gamma} [\sigma(x,u_n)|\mathbf{E}_n|^2 - \sigma(x,u)|\mathbf{E}|^2]\phi dxdt \\ &+ \int\int_{Q\backslash Q_\gamma} [\sigma_n(x,u_n)|\mathbf{E}_n|^2 - \sigma(x,u)|\mathbf{E}|^2]\phi dxdt := J_1 + J_2. \end{aligned}$$

$$\begin{aligned} |J_1| &\leq |\int\int_{Q_\gamma} \sigma(x,u_n)[|\mathbf{E}_n|^2 - |\mathbf{E}|^2]|\phi| dxdt| + \int\int_{Q_\gamma} |\sigma(x,u_n) - \sigma(x,u)|\mathbf{E}|^2]|\phi| dxdt \\ &:= J_{11} + J_{12}. \end{aligned}$$

It is clear that $J_{11} \to 0$ since $\mathbf{E}_n$ converges to $\mathbf{E}$ strongly in $L^2(Q_T)$ by Lemma 2.3 and $\sigma(x,u_n)$ is bounded. On the other hand,

$$Q_\gamma = [Q_\gamma \bigcap \{(x,t) \in Q : u(x,t) \geq m+\gamma\}] \bigcup [Q_\gamma \bigcap \{(x,t) \in Q : u(x,t) \leq m-\gamma\}].$$

Since $u_n \to u(x,t)$ a.e. on Q_T and uniformly in Q, it follows that on $Q_\gamma \bigcap \{(x,t) : u(x,t) \geq m+\gamma\}$, $\sigma(x,u_n) = \sigma_l(x,u_n) \to \sigma_l(x,u)$ a.e. and on $Q_\gamma \bigcap \{(x,t) : u(x,t) \leq m-\gamma\}$, $\sigma(x,u_n) = \sigma_s(x,u_n) \to \sigma_s(x,u)$ a.e. as $n \to \infty$, by dominated convergence theorem we see $J_{12} \to 0$ as $n \to \infty$.

Next we prove $J_2 \to 0$ as $n \to \infty$. Without loss of generality, we assume

$$\sigma_s(x,m) \leq \sigma_l(x,m), \qquad x \in \Omega.$$

Then the approximation sequence $\sigma_n(x,u)$ can be assumed to be increasing in u-vaiable in a neighborhood of $u = m$. On $Q\backslash Q_\gamma$,

$$m - \gamma \leq u_n(x,t) \leq m + \gamma,$$

As $\sigma_n(x,u)$ is increasing in a neighborhood of $u = m$, we see

$$\sigma_n(x, m-\gamma) \leq \sigma_n(x,u_n) \leq \sigma_n(x, m+\gamma).$$

If n is sufficiently large,

$$\sigma_n(x,m-\gamma) = \sigma_s(x,m-\gamma), \sigma_n(x,m+\gamma) = \sigma_l(m+\gamma).$$

By the weak convergence and the definition of $\sigma(x,m)$, when $\gamma \to 0, n \to \infty$ we have

$$\sigma_s(x,m-) \leq \sigma(x,m) \leq \sigma(x,m+), \qquad a.e. x \in \Omega.$$

It follows that

$$\begin{aligned} |J_2| &\leq |\int\int_{Q\backslash Q_\gamma} [\sigma_n(x,u_n)(|\mathbf{E}_n|^2 - |\mathbf{E}|^2)|\phi| dxdt| \\ &+ |\int\int_{Q\backslash Q_\gamma} (\sigma_n(x,u_n) - \sigma(x,u)|\mathbf{E}|^2]\phi) dxdt| \to 0 \end{aligned}$$

as $n \to \infty, \gamma \to 0$.

Next we consider the convergence for $A_n(x,u_n)$. The weak compactness implies that there exists a function $\beta(x,t) \in L^2(Q_T)$ such that $A_n(u_n) \to \beta(x,t)$ weakly in $L^2(Q_T)$. We need to prove that the graph of $\beta(x,t) \in A(u)$ a.e. on Q_T. From the construction of A_n and convergence of u_n we see

$$\beta(x,t) = A(u) \text{ a.e. on } Q_\gamma.$$

On $Q_T \backslash Q_\gamma$,

$$m - \frac{2}{n} \leq u_n \leq m + \frac{2}{n}.$$

The monotonicity of $A(u)$ implies

$$A_n(m - \frac{2}{n}) \leq A_n(u_n) \leq A_n(m + \frac{2}{n}),$$

which is the same as

$$A(m - \frac{2}{n}) \leq A_n(u_n) \leq A(m + \frac{2}{n}).$$

The weak convergence yields that for a.e. $(x,t) \in Q_T \backslash Q_\gamma$

$$m - 1 \leq \beta(x,t) \leq m.$$

Now we multiply Eq. (2.3) by any test function $\psi \in H^1(0,T;H^1(\Omega))$ with $\psi(x,T)=0$ and then integrate over Q_T to obtain

$$\begin{aligned}&\int_0^T\int_\Omega[-A_n(x,u_n)\psi_t+\nabla u_n\nabla\psi]dx\\&=\int_0^T\int_\Omega\sigma(x,u_n)|\mathbf{E}_n|^2\psi dxdt+\int_\Omega A(x,u_{0n}(x))\psi(x,0)dx.\end{aligned}$$

Finally, we take limit as $n\to\infty$ to see that $(\mathbf{E}(x,t),\mathbf{H}(x,t),u(x,t))$ is indeed a weak solution of the problem (1.1)-(1.6). Q.E.D.

3. **Global Existence in Time-Harmonic Fields.** For some industrial applications (see [12, 13]), the time scale for electromagnetic field and the heat conduction is quite different. It is often to assume that the electric and magnetic fields are time-harmonic. This leads to the following problem:

$$i\mu\omega\mathbf{H}+\nabla\times\mathbf{E}=0,\qquad x\in\Omega, \tag{3.1}$$

$$(i\varepsilon\omega+\sigma)\mathbf{E}=\nabla\times\mathbf{H},\qquad x\in\Omega, \tag{3.2}$$

where i represents the complex unit and ω is the frequency.

For many applied problems, it is often convenient to use a unified approach by assuming that (see [12]):

$$\varepsilon(x)=\varepsilon_1(x)+i\varepsilon_2(x),\mu(x)=\mu_1(x)-i\mu_2(x),$$

where $\varepsilon_1(x),\varepsilon_2(x),\mu_1(x)$ and $\mu_2(x)$ are positive functions.

It is clear that the system (3.1)-(3.2) is equivalent to the following one:

$$\nabla\times[\gamma(x)\nabla\times\mathbf{E}]+r(x,u)\mathbf{E}=0,\qquad x\in\Omega, \tag{3.3}$$

where

$$\begin{aligned}\gamma(x)\quad&:=\quad\frac{1}{\mu(x)}=\frac{\mu_1}{\sqrt{|\mu_1|^2+|\mu_2|^2}}+i\frac{\mu_2}{\sqrt{|\mu_1|^2+|\mu_2|^2}},\\ r(x,u)\quad&:=\quad i\omega(i\varepsilon(x)\omega+\sigma(x,u)).\end{aligned}$$

Consider the phase-change problem:

$$\nabla\times[\gamma(x)\nabla\times\mathbf{E}]+r(x,u)\mathbf{E}=0,\qquad x\in\Omega, \tag{3.4}$$

$$A(x,u)_t-\Delta u=\sigma(x,u)|\mathbf{E}|^2,\qquad (x,t)\in Q_T, \tag{3.5}$$

$$\mathbf{N}\times\mathbf{E}(x)=\mathbf{N}\times\mathbf{G}(x),\qquad x\in\partial\Omega, \tag{3.6}$$

$$u_n(x,t)=0,\qquad (x,t)\in\partial\Omega\times(0,T], \tag{3.7}$$

$$u(x,0)=u_0(x),\qquad x\in\Omega. \tag{3.8}$$

Note that the coefficient $\sigma(x,u)$ depends on t since $u(x,t)$ is a function of t. The solution $\mathbf{E}$ is also a function of t. However, this time variable for the heat conduction is different from the time-variable in electromagnetic field.

A weak solution to (3.4)-(3.8) can be defined as follows.

Definition 3.1: A pair functions $(\mathbf{E}(x,t),u(x,t))$ is called a weak solution of (3.4)-(3.8) if $\mathbf{E}(x,t)\in H(curl,\Omega),u(x,t)\in L^2(0,T;H^1(\Omega))$ with $\mathbf{N}\times(\mathbf{E}-\mathbf{G})\in H_0(curl,\Omega)$ and the following integral identities hold:

$$\int_\Omega[\gamma(\nabla\times\mathbf{E})\cdot(\nabla\times\mathbf{\Psi})+r(x,u)\mathbf{E}\cdot\mathbf{\Psi}]dx=0,$$

$$\int\int_{Q_T}[-A(x,u)\psi_t+\nabla u\nabla\psi]dxdt=\int\int_{\Omega_T}\sigma(x,u)|\mathbf{E}|^2\psi dxdt+\int_\Omega A(x,u_0)\psi(x,0)dx$$

for any test functions $\mathbf{\Psi},\in H_0(curl,\Omega)$ and $\psi\in H^1(0,T;H^1(\Omega))$ with $\psi(x,T)=0$ on Ω.

H(3.1): (a) Let $\varepsilon_1(x),\varepsilon_2(x),\mu_1(x)$ and $\mu_2(x)$ be nonnegative and of class $L^\infty(\Omega)$ with $\varepsilon_1\geq r_0,\varepsilon_2\geq 0$. Let

$$0\leq\sigma(x,u)\leq\sigma_0,\ u\sigma(x,u)\leq\sigma_1,u\in[M,\infty).$$

Moreover, there exists a constant σ_1 such that

$$\sigma(x,u)-|\varepsilon_2|_{L^\infty(\Omega)}\geq\sigma_1>0.$$

(b) Let $\mathbf{G}(x) \in H(curl, \Omega)$.
H(3.2): Let $A(x,u)$ be defined as in section 2 and satisfy the same condition as in H(2.1)(c). Moreover, $u_0(x) \in L^2(\Omega)$ and nonnegative.

Theorem 3.1 Under the assumptions H(3.1)-(3.3), the problem (3.4)-(3.8) has a global weak solution.
Proof: As the proof is quite similar to that of Theorem 2.4, we only give an outline.
Step 1: Approximating the problem. By constructing smooth approximation for σ and $A(x,u)$, we consider the approximation problem:

$$\nabla \times [\gamma(x)\nabla \times \mathbf{E}] + r_n(x,u)\mathbf{E} = 0, \qquad x \in \Omega, \tag{3.9}$$
$$A_n(x,u)_t - \Delta u = \sigma_n(x,u)|\mathbf{E}|^2, \qquad (x,t) \in Q_T, \tag{3.10}$$
$$\mathbf{N} \times \mathbf{E} = \mathbf{N} \times \mathbf{G}, \qquad x \in \partial\Omega, \tag{3.11}$$
$$u_n(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T], \tag{3.12}$$
$$u(x,0) = u_0(x), \qquad x \in \Omega. \tag{3.13}$$

This problem has at least one weak solution $(\mathbf{E}_n(x,t), u_n(x,t))$ ([21]).
Step 2: Deriving uniform estimates.
There exist constants C_1 and C_2 independent of n such that

$$\int_\Omega |\nabla \times \mathbf{E}_n|^2 dx + \int_\Omega |\mathbf{E}_n|^2 dx \le C_1,$$
$$\sup_{0\le t\le T} \int_\Omega u_n^2 dx + \int\int_{Q_T} |\nabla u_n|^2 dxdt \le C_2.$$

To prove the first estimate, we take the inner product by $(\mathbf{E} - \mathbf{G})^*$, the complex conjugate of $\mathbf{E} - \mathbf{G}$, to Eq.(3.9) to obtain

$$\int_\Omega \gamma(\nabla \times \mathbf{E}) \cdot [\nabla \times (\mathbf{E} - \mathbf{G})^*] + r_n(x,u)\mathbf{E} \cdot (\mathbf{E} - \mathbf{G})^*]dx = 0. \tag{3.14}$$

We first take the imaginary part of the above equation to obtain

$$\int_\Omega \frac{\mu_2}{\sqrt{\mu_1^2 + \mu_2^2}} |\nabla \times \mathbf{E}|^2 dx + \int_\Omega (\sigma - \varepsilon_2)|\mathbf{E}|^2 dx \le C,$$

where the constant C depends only on known data. By H(3.1), we obtain

$$\int_\Omega |\mathbf{E}|^2 dx \le \frac{C}{\sigma_1}.$$

Now we take the real part of Eq. (3.15) and use the assumption H(3.1) again to obtain

$$\int_\Omega |\nabla \times \mathbf{E}|^2 dx \le C,$$

where C depends only on known data.
The second estimate is the same as Lemma 2.3.
Step 3: Taking the limit. This step is almost identical to that of Theorem 2.4, we omit it here.
Q.E.D.
Remark 3.1: The assumption H(3.1) is only one type of sufficient condition to ensure the global existence of a unique weak solution to time-harmonic Maxwell's equations. Other types of sufficient conditions can be found in [14, 22].

4. **One-dimensional Problem.** In this section we study the problem (1.1)-(1.5) in one space dimension and prove that the weak solution exists globally for $\sigma(x,t,u)$ with linear growth.

Let $Q_T = \{(x,t) : 0 < x < 1, 0 < t < T\}$. For one-space dimension, we assume $\mathbf{E}(x,t) = \{0, e(x,t), 0\}, \mathbf{H}(x,t) = \{0,0,h(x,t)\}$. Then the system (1.1)-(1.3) becomes the following form:

$$\varepsilon(x)e_t + \sigma(x,t,u)e = -h_x, \qquad (x,t) \in Q_T, \tag{4.1}$$
$$\mu(x)h_t + e_x = 0, \qquad (x,t) \in Q_T, \tag{4.2}$$
$$A(x,u)_t - u_{xx} = \sigma(x,t,u)|e(x,t)|^2, \qquad (x,t) \in Q_T, \tag{4.3}$$

where $A(x,u)$ is the same as in Section 2.

By solving $h(x,t)$ from Eq.(4.2), we see

$$h(x,t) = h_0(x) - \frac{1}{\mu(x)} w_x(x,t), \qquad (x,t) \in Q_T,$$

where

$$w(x,t) = \int_0^t e(x,\tau)d\tau.$$

For simplicity, we assume $h_0(x) = 0$ on Q_T. It follows that Eq.(4.1)-(4.3) is equivalent to the following system:

$$\varepsilon(x)w_{tt} - (\gamma(x)w_x)_x + \sigma(x,t,u)w_t = 0, \tag{4.4}$$

$$A(x,u)_t - u_{xx} = \sigma(x,t,u)|w_t(x,t)|^2, \qquad (x,t) \in Q_T. \tag{4.5}$$

The initial and boundary conditions are as follows:

$$w(0,t) = f_1(t), w(1,t) = f_2(t), u_x(0,t) = u_x(1,t) = 0, t \in [0,T], \tag{4.6}$$

$$w(x,0) = 0, w_t(x,0) = e_0(x), u(x,0) = u_0(x), \qquad 0 < x < 1. \tag{4.7}$$

H(4.1): (a) Let $\varepsilon(x), \gamma(x)$ satisfies H(2.1)(a). Let $\sigma(x,t,u)$ satisfies

$$0 \le \sigma(x,t,u) \le \sigma_0(1+u), (x,t,u) \in Q_T \times [0,\infty).$$

(b) Let $A(x,u)$ be defined as in section 2 and satisfy the same condition as in H(2.1)(c).
(c) Let $f_1(t), f_2(t) \in H^1(0,T)$ with $f_1(0) = f_2(0) = 0$ and $e_0(x), u_0(x) \in L^2(0,1)$.

Theorem 4.1: Under the assumption H(4.1), the problem (4.4)-(4.7) has a weak solution globally.
Proof: As for $n-$dimensional case, we make a smooth approximation for $\sigma(x,t,u)$ and $A(x,u)$ and then consider the following approximate problem:

$$\varepsilon(x)w_{tt} - (\gamma(x)w_x)_x + \sigma_n(x,t,u)w_t = 0, \qquad (x,t) \in Q_T, \tag{4.8}$$

$$A_n(x,u)_t - u_{xx} = \sigma_n(x,t,u)|w_t(x,t)|^2, \qquad (x,t) \in Q_T, \tag{4.9}$$

$$w(0,t) = f_1(t), w(1,t) = f_2(t), u_x(0,t) = u_x(1,t) = 0, t \in [0,T], \tag{4.10}$$

$$w(x,0) = 0, w_t(x,0) = e_0(x), u(x,0) = u_0(x), \qquad 0 < x < 1. \tag{4.11}$$

This problem has a unique weak solution ([19])

$$(w_n(x,t), u_n(x,t)) \in H^1(0,T; H^1(0,1)) \times L^2(0,T; H^1(0,1)).$$

Again we will omit the subscript n. Now we derive some uniform estimates. First of all, set

$$g(x,t) = (1-x)f_1(t) + xf_2(t), (x,t) \in Q_T.$$

We multiply Eq.(4.8) by $w_t - g_t(x,t)$ and then integrate over Q_T. Using the assumption H(4.1) and the growth condition for $\sigma(x,t,u)$, we obtain, after some routine calculations, that

$$\sup_{0\le t\le T} \int_0^1 [w_t^2 + w_x^2]dx + \int_0^T \int_0^1 \sigma(x,t,u)w_t^2 dxdt$$
$$\le C_1 + C_2 \int_0^T \int_0^1 |u|dxdt,$$

where the constants C_1 and C_2 depend only on known data, but not on n.

On the other hand, we use an estimate from the paper [3] to obtain

$$\int_0^1 |u|dx \le C_2 + C_4 \int_0^T \int_0^1 \sigma(x,t,u)|w_t|^2 dxdt,$$

where the constants C_3 and C_4 depend only on known data.

It follows that

$$\int_0^1 |u|dx \le C_3 + C_4[C_1 + C_2 \int_0^T \int_0^1 |u|dxdt].$$

The above estimate holds if we replace T by any $T' \in [0,T]$, we can apply Gronwall's inequality to obtain

$$\int_0^1 |u|dx \le C_5,$$

where C_5 depends only on known data.

Next, we multiply Eq. (4.9) by u and then integrate over $Q_{T'}$ with $T' \in (0, T]$. Using the same technique as in Section 2, we see

$$\int\int_{Q_{T'}} A_n(x,u)_t u dx dt \geq b_0 \int_\Omega u^2 dx - C_6,$$
$$\int\int_{Q_{T'}} u_{xx} u dx dt = -\int\int_{Q_{T'}} u_x^2 dx dt$$

where $b_0 > 0$ and C_6 depend only on known data.

It follows that

$$\int_0^1 u^2(x,T')dx + \int_0^{T'}\int_0^1 u_x^2 dx dt \leq C + \int_0^T\int_0^1 |u|\sigma(x,t,u)w_t^2 dx dt$$
$$\leq C + C\int_0^T ||u||^2_{L^\infty(0,1)} dt.$$

Now, by Sobolev's embedding ([10]),

$$\begin{aligned} ||u||^2_{L^\infty(0,1)} &\leq C||u||_{W^{1,2}(0,1)}||u||_{L^2(0,1)} \\ &\leq \delta\int_0^1 [u^2 + u_x^2]dx + C(\delta)\int_0^1 u^2 dx. \end{aligned}$$

We combine the above estimates and choose δ sufficiently small to obtain

$$\int_0^1 u^2 dx + \int_0^{T'}\int_0^1 u_x^2 dx dt \leq C + C\int_0^{T'}\int_0^1 u^2 dx dt.$$

Again Gronwall's inequality yields

$$\int_0^1 u^2 dx + \int_0^{T'}\int_0^1 u_x^2 dx dt \leq C_8,$$

where the constant C_8 depends only on known data.

With those uniform estimates, as for Theorem 2.4 we can extract a subsequence from $(w_n(x,t), u_n(x,t))$ and then take the limit to obtain a weak solution to the problem (4.4)-(4.7). We shall not repeat these steps here.

Q.E.D.

Remark 4.1: It would be interesting to show that the temperature is continuous over Q_T (see [1, 4]).

Remark 4.2: The uniqueness is an open question, even for one-space dimension.

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