Advection-diffusion equation

The conservation equation and flux constitutive equation are

$$c\dot{p} + \boldsymbol{\nabla} \cdot \mathbf{j} = F(x)\,, \quad \mathbf{j} = -a\boldsymbol{\nabla} p + \mathbf{b}\, p\,. \tag{0.1}$$

where $c = c(x)$, $a = a(x)$ and $\mathbf{b} = \mathbf{b}(x)$.

Gravity-driven Fluid Flow. Let p denote *pressure* of a slightly compressible fluid in the porous medium. The stationary weak problem has the form

$$p \in V : \quad \int_G \big(\lambda c p q + (a\boldsymbol{\nabla} p - p\mathbf{b}) \cdot \boldsymbol{\nabla} q\big)\, dx = \int_G F\, q\, dx, \quad q \in V.$$

Introduce the operator $\mathcal{A}^R : V \to V'$ by

$$\mathcal{A}^R p(q) \equiv \int_G \big(a\boldsymbol{\nabla} p \cdot \boldsymbol{\nabla} q - p\mathbf{b} \cdot \boldsymbol{\nabla} q\big)\, dx$$

and note that for smooth functions it is equal to

$$\int_G \big(- \boldsymbol{\nabla} \cdot (a\boldsymbol{\nabla} p - \mathbf{b}p)\big)\, q\, dx + \int_{\partial G} \big(a\boldsymbol{\nabla} p - \mathbf{b}p\big) \cdot \mathbf{n}\, q\, dS\,.$$

This displays the PDE and the complementary boundary conditions of third type (Robin).

Set $V = \{q \in H^1(G) : \ q = 0 \text{ on } \Gamma_D\}$ where Γ_D is a prescribed portion of the boundary, ∂G. In the flow problems, this corresponds to the *drained* portion of the boundary. Then the equation $A^R p = F$ in $L^2(G)$ corresponds to the boundary-value problem

$$-\boldsymbol{\nabla} \cdot (a\boldsymbol{\nabla} p - \mathbf{b}p) = F \text{ in } G, \tag{0.2a}$$

$$p = 0 \text{ on } \Gamma_D, \quad a\frac{\partial p}{\partial n} - \mathbf{b} \cdot \mathbf{n} p = 0 \text{ on } \Gamma_R. \tag{0.2b}$$

The set $\Gamma_R \equiv \partial G - \Gamma_D$ is the *sealed* portion of the boundary.

Estimate $(\lambda c + \mathcal{A}^R)p(p) = \int_G \big(\lambda c p^2 + a|\boldsymbol{\nabla} p|^2 + \frac{1}{2}\boldsymbol{\nabla} \cdot \mathbf{b} p^2\big)dx$
$-\frac{1}{2}\int_{\partial G} p^2 \mathbf{b} \cdot \mathbf{n}\, dS$:

$\mathcal{A}^R p(p) \ge 0$ if $\boldsymbol{\nabla} \cdot \mathbf{b} \ge 0$ in G and $q|_{\partial G} = 0$ where $\mathbf{b} \cdot \mathbf{n} > 0$ $\forall q \in V$.

That is, $\Gamma_D \supset \Sigma$, where the set $\Sigma \equiv \{s \in \partial G : \ \mathbf{b}(s) \cdot \mathbf{n}(s) > 0\}$ is the *outflow region* for the *advective flux*, $p\mathbf{b}$. It corresponds to the *bottom* of the container when $\mathbf{b}$ is the gravity term (pointing downward), and flux may *not* be specified there if A^R is to be accretive.

More generally, let $a_0 \equiv \text{ess inf } a(\cdot) > 0$, and note that

$$|\int_G p\mathbf{b}\cdot\boldsymbol{\nabla} p\,dx| \le \|\mathbf{b}\|_{L^\infty}\|\boldsymbol{\nabla} p\|\|p\|$$
$$\le \|\mathbf{b}\|_{L^\infty}(\frac{a_0}{\|\mathbf{b}\|_{L^\infty}}\|\boldsymbol{\nabla} p\|^2 + \frac{\|\mathbf{b}\|_{L^\infty}}{4a_0}\|p\|^2) \le a_0\|\boldsymbol{\nabla} p\|^2 + \frac{\|\mathbf{b}\|^2_{L^\infty}}{4a_0}\|p\|^2,$$

so it follows that for all λ satisfying $\lambda c \ge \frac{\|\mathbf{b}\|^2_{L^\infty}}{4a_0}$ we have

$$(\lambda c + \mathcal{A}^R)p(p) = \int_G \big(\lambda c p^2 + a|\boldsymbol{\nabla} p|^2 - p\mathbf{b}\cdot\boldsymbol{\nabla} p\big)\,dx$$
$$\ge \int_G \big(\lambda c p^2 + a_0|\boldsymbol{\nabla} p|^2\big)\,dx - |\int_G p\mathbf{b}\cdot\boldsymbol{\nabla} p\,dx| \ge 0.$$

That is, $\lambda c + \mathcal{A}^R$ is accretive if λ is sufficiently large. In fact we can add any first order terms to the bilinear form $\int_G a\boldsymbol{\nabla} p\cdot\boldsymbol{\nabla} q\,dx$ and the sum with $\int_G \lambda pq\,dx$ will be accretive if λ is sufficiently large. Likewise, for λ sufficiently large the bilinear form $\lambda c + \mathcal{A}^R$ is $H^1(G)$-coercive.

Now let's assume the given function p_0 is positive on Γ_D, and then set $K \equiv \{q \in H^1(G) : q = p_0 \text{ on } \Gamma_D,\ q \le 0 \text{ on } \Gamma_U$, where Γ_D and Γ_U are disjoint subsets of the boundary ∂G. Denote the remainder by $\Gamma_N = \partial G - \Gamma_D - \Gamma_U$. The corresponding variational inequality is

$$p \in K : \int_G \big(\lambda c p(q-p) + (a\boldsymbol{\nabla} p - p\mathbf{b})\cdot\boldsymbol{\nabla}(q-p)\big)\,dx \ge \int_G F\,(q-p)\,dx,\ q \in K.$$

A solution p is characterized by the *unilateral boundary-value problem*

(0.3a) $$\lambda c p - \boldsymbol{\nabla}\cdot(a\boldsymbol{\nabla} p - \mathbf{b}p) = F \text{ in } G,$$

(0.3b) $$p = p_0 \text{ on } \Gamma_D, \quad a\frac{\partial p}{\partial n} - \mathbf{b}\cdot\mathbf{n}p = 0 \text{ on } \Gamma_N, \text{ and}$$

(0.3c) $$p \le 0,\ a\frac{\partial p}{\partial n} - \mathbf{b}\cdot\mathbf{n}p \le 0, \big(a\frac{\partial p}{\partial n} - \mathbf{b}\cdot\mathbf{n}p\big)\,p = 0 \text{ on } \Gamma_U\,.$$

In the flow problem, Γ_D is the boundary exposed to the water and $p_0(s)$ is the *depth*, Γ_N is the part that's sealed, so there is no flow, and on the remaining part exposed to air (at pressure 0) the water pressure is non-negative, any flow across the boundary must be outward, and at every point either the pressure or the flow is null. The *seepage surface* $\{s \in \Gamma_U : a\frac{\partial p}{\partial n}(s) - \mathbf{b}\cdot\mathbf{n}p(s) < 0$ is *unknown*, and finding it would reduce the problem to a much simpler Dirichlet-Neumann boundary-value problem.

Transport of Concentration. Suppose that we know the velocity $\mathbf{b}(x)$ of a fluid flowing through the porous medium, and that fluid is carrying a chemical of concentration $u(x)$. We write the weak form of the corresponding transport equation as

$$u \in V: \quad \int_G \big(\lambda c u v + a\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v + \boldsymbol{\nabla} \cdot (u\mathbf{b})v\big)\, dx = \int_G F\, v\, dx, \quad v \in V.$$

Note that the advection (first order) terms are not integrated.

Introduce the operator $\mathcal{A}^N : V \to V'$, where the space V is to be chosen as before.

$$\mathcal{A}^N u(v) = \int_G \big(a\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v + \boldsymbol{\nabla} \cdot (u\mathbf{b})v\big)\, dx$$

$$= \int_G \big(- \boldsymbol{\nabla} \cdot (a\boldsymbol{\nabla} u - \mathbf{b}u)\big)\, v\, dx + \int_{\partial G} a\boldsymbol{\nabla} u \cdot \mathbf{n}\, v\, dS$$

Then the equation $A^N u = F$ in $L^2(G)$ corresponds to the boundary-value problem

$$-\boldsymbol{\nabla} \cdot (a\boldsymbol{\nabla} u - \mathbf{b}u) = F \text{ in } G, \tag{0.4a}$$

$$u = 0 \text{ on } \Gamma_D, \quad a\frac{\partial u}{\partial n} = 0 \text{ on } \partial G - \Gamma_D. \tag{0.4b}$$

This is the same PDE but the boundary condition is of second type (Neumann): it contains no advection.

Estimate $\mathcal{A}^N u(u) = \int_G \big(a|\boldsymbol{\nabla} u|^2 + \frac{1}{2}\boldsymbol{\nabla} \cdot \mathbf{b}u^2\big)dx + \frac{1}{2}\int_{\partial G} u^2\mathbf{b} \cdot \mathbf{n}\, dS$:
$\mathcal{A}^N u(u) \geq 0$ if $\boldsymbol{\nabla} \cdot \mathbf{b} \geq 0$ in G and $v|_{\partial G} = 0$ where $\mathbf{b} \cdot \mathbf{n} < 0$ $\forall v \in V$

This is the *inflow region*. Here the *stability* estimate results from fixing the concentration where the fluid is entering the medium. Of course, in any case we have $\lambda c + \mathcal{A}^N$ is accretive if λ is sufficiently large.

Note that

$$\mathcal{A}^N u(v) = \mathcal{A}^R u(v) + \int_{\partial G} \mathbf{b} \cdot \mathbf{n} u v\, dS.$$