

ADVECTION-DIFFUSION EQUATION

The conservation equation and flux constitutive equation are

$$c\dot{u} + \nabla \cdot \mathbf{j} = F(x), \quad \mathbf{j} = -k\nabla u + \mathbf{b}u.$$

where $c = c(x)$, $k = k(x)$ and $\mathbf{b} = \mathbf{b}(x)$.

Gravity-driven Fluid Flow. Let u denote either pressure or density in the porous medium. The stationary weak problem has the form

$$u \in V : \int_G (\lambda c u v + (k\nabla u - u\mathbf{b}) \cdot \nabla v) dx = \int_G F v dx, \quad v \in V.$$

Introduce the operator $\mathcal{A}^R : V \rightarrow V'$ by

$$\begin{aligned} \mathcal{A}^R u(v) &\equiv \int_G (k\nabla u \cdot \nabla v - u\mathbf{b} \cdot \nabla v) dx \\ &= \int_G (-\nabla \cdot (k\nabla u - \mathbf{b}u)) v dx + \int_{\partial G} (k\nabla u - \mathbf{b}u) \cdot \mathbf{n} v dS \end{aligned}$$

This displays the PDE and the boundary conditions.

Set $V = \{v \in H^1(G) : v = 0 \text{ on } \Gamma_D\}$ where Γ_D is a prescribed portion of the boundary, ∂G . In the flow problems, this corresponds to the *drained* portion of the boundary. Then the equation $\mathcal{A}^R u = F$ in $L^2(G)$ corresponds to the boundary-value problem

$$(0.1a) \quad -\nabla \cdot (k\nabla u - \mathbf{b}u) = F \text{ in } G,$$

$$(0.1b) \quad u = 0 \text{ on } \Gamma_D, \quad k \frac{\partial u}{\partial n} - \mathbf{b} \cdot \mathbf{n} u = 0 \text{ on } \partial G - \Gamma_D.$$

The set $\partial G - \Gamma_D$ is the *sealed* portion of the boundary.

Estimate $(\lambda c + \mathcal{A}^R)u(u) = \int_G (\lambda c u^2 + k|\nabla u|^2 + \frac{1}{2}\nabla \cdot \mathbf{b}u^2) dx - \frac{1}{2} \int_{\partial G} u^2 \mathbf{b} \cdot \mathbf{n} dS :$

$\mathcal{A}^R u(u) \geq 0$ if $\nabla \cdot \mathbf{b} \geq 0$ in G and $v|_{\partial G} = 0$ where $\mathbf{b} \cdot \mathbf{n} > 0 \forall v \in V$.

That is, $\Gamma_D \supset \Sigma$, where the set $\Sigma \equiv \{s \in \partial G : \mathbf{b}(s) \cdot \mathbf{n}(s) > 0\}$ is the *outflow region* for the *advective flux*, $u\mathbf{b}$. It corresponds to the *bottom* of the container when \mathbf{b} is the gravity term (pointing downward).

More generally, let $k_0 \equiv \text{ess inf } k(\cdot) > 0$, and note that

$$\begin{aligned} \left| \int_G u \mathbf{b} \cdot \nabla u \, dx \right| &\leq \|\mathbf{b}\|_{L^\infty} \|\nabla u\| \|u\| \\ &\leq \|\mathbf{b}\|_{L^\infty} \left(\frac{k_0}{\|\mathbf{b}\|_{L^\infty}} \|\nabla u\|^2 + \frac{\|\mathbf{b}\|_{L^\infty}}{4k_0} \|u\|^2 \right) \leq k_0 \|\nabla u\|^2 + \frac{\|\mathbf{b}\|_{L^\infty}^2}{4k_0} \|u\|^2, \end{aligned}$$

so it follows that for all λ satisfying $\lambda c \geq \frac{\|\mathbf{b}\|_{L^\infty}^2}{4k_0}$ we have

$$\begin{aligned} (\lambda c + \mathcal{A}^R)u(u) &= \int_G (\lambda c u^2 + k |\nabla u|^2 - u \mathbf{b} \cdot \nabla u) \, dx \\ &\geq \int_G (\lambda c u^2 + k_0 |\nabla u|^2) \, dx - \left| \int_G u \mathbf{b} \cdot \nabla u \, dx \right| \geq 0. \end{aligned}$$

That is, $\lambda c + \mathcal{A}^R$ is accretive if λ is sufficiently large. In fact we can add any first order terms to the bilinear form $\int_G k \nabla u \cdot \nabla v \, dx$ and the sum with $\int_G \lambda u v \, dx$ will be accretive if λ is sufficiently large. Likewise, for λ sufficiently large the bilinear form $\lambda c + \mathcal{A}^R$ is $H^1(G)$ -coercive.

Now let's assume the given function p_0 is positive on Γ_D , and then set $K \equiv \{v \in H^1(G) : v = p_0 \text{ on } \Gamma_D, v \leq 0 \text{ on } \Gamma_U, \text{ where } \Gamma_D \text{ and } \Gamma_U \text{ are disjoint subsets of the boundary } \partial G. \text{ Denote the remainder by } \Gamma_N = \partial G - \Gamma_D - \Gamma_U. \text{ The corresponding variational inequality is}$

$$u \in K : \int_G (\lambda c u(v-u) + (k \nabla u - u \mathbf{b}) \cdot \nabla (v-u)) \, dx \geq \int_G F(v-u) \, dx, \quad v \in K.$$

A solution u is characterized by the *unilateral boundary-value problem*

$$(0.2a) \quad \lambda c u - \nabla \cdot (k \nabla u - \mathbf{b}u) = F \text{ in } G,$$

$$(0.2b) \quad u = p_0 \text{ on } \Gamma_D, \quad k \frac{\partial u}{\partial n} - \mathbf{b} \cdot \mathbf{n}u = 0 \text{ on } \Gamma_N, \text{ and}$$

$$(0.2c) \quad u \leq 0, \quad k \frac{\partial u}{\partial n} - \mathbf{b} \cdot \mathbf{n}u \leq 0, \quad (k \frac{\partial u}{\partial n} - \mathbf{b} \cdot \mathbf{n}u) u = 0 \text{ on } \Gamma_U.$$

In the flow problem, Γ_D is the boundary exposed to the water and $p_0(s)$ is the *depth*, Γ_N is the part that's sealed, so there is no flow, and on the remaining part exposed to air (at pressure 0) the water pressure is non-negative, any flow across the boundary must be outward, and at every point either the pressure or the flow is null. The *seepage surface* $\{s \in \Gamma_U : k \frac{\partial u}{\partial n}(s) - \mathbf{b} \cdot \mathbf{n}u(s) < 0\}$ is *unknown*, and finding it would reduce the problem to a much simpler Dirichlet-Neumann boundary-value problem.

Transport of Concentration. Suppose that we know the velocity $\mathbf{b}(x)$ of a fluid flowing through the porous medium, and that fluid is carrying a chemical of concentration $u(x)$. We write the weak form of the corresponding transport equation as

$$u \in V : \int_G (\lambda c u v + k \nabla u \cdot \nabla v + \nabla \cdot (u \mathbf{b}) v) dx = \int_G F v dx, \quad v \in V.$$

Note that the advection (first order) terms are not integrated.

Introduce the operator $\mathcal{A}^N : V \rightarrow V'$, where the space V is to be chosen as before.

$$\begin{aligned} \mathcal{A}^N u(v) &= \int_G (k \nabla u \cdot \nabla v + \nabla \cdot (u \mathbf{b}) v) dx \\ &= \int_G (-\nabla \cdot (k \nabla u - \mathbf{b}u)) v dx + \int_{\partial G} k \nabla u \cdot \mathbf{n} v dS \end{aligned}$$

Then the equation $\mathcal{A}^N u = F$ in $L^2(G)$ corresponds to the boundary-value problem

$$(0.3a) \quad -\nabla \cdot (k \nabla u - \mathbf{b}u) = F \text{ in } G,$$

$$(0.3b) \quad u = 0 \text{ on } \Gamma_D, \quad k \frac{\partial u}{\partial n} = 0 \text{ on } \partial G - \Gamma_D.$$

This is the same PDE but a modified boundary condition: it contains no advection.

$$\text{Estimate } \mathcal{A}^N u(u) = \int_G (k |\nabla u|^2 + \frac{1}{2} \nabla \cdot \mathbf{b} u^2) dx + \frac{1}{2} \int_{\partial G} u^2 \mathbf{b} \cdot \mathbf{n} dS :$$

$$\mathcal{A}^N u(u) \geq 0 \text{ if } \nabla \cdot \mathbf{b} \geq 0 \text{ in } G \text{ and } v|_{\partial G} = 0 \text{ where } \mathbf{b} \cdot \mathbf{n} < 0 \forall v \in V$$

This is the *inflow region*. Here the *stability* estimate results from fixing the concentration only where the fluid is entering the medium. Of course, in any case we have $\lambda c + \mathcal{A}^N$ is accretive if λ is sufficiently large.

Note that

$$\mathcal{A}^N u(v) = \mathcal{A}^R u(v) + \int_{\partial G} \mathbf{b} \cdot \mathbf{n} u v dS.$$