Remarks on Cauchy Problem

Let A denote a linear operator on a Hilbert space H, with domain $D(L) \subset H$ a subspace. (Note that we do *not* make any continuity assumptions on A.) Suppose the operator satisfies

- $(Av, v)_H \geq 0$ for all $v \in D(L)$,
- $I + A$ maps $D(L)$ *onto* H.

If the operator satisfies a V-elliptic type assumption, then the first condition will follow easily, and then the Lax-Milgram theorem can be used to show that the second condition also holds. Moreover, these conditions hold for much more general classes of operators than those arising from *elliptic* problems.

We will show later that if an operator satisfies these two conditions, then there is a unique function $u : [0, \infty) \to H$ which satisfies the *initial-value problem*

$$cu'(t) + Au(t) = 0, \quad u(0) = u_0.$$

Here we assume $c > 0$ and $u_0 \in H$ are given. Also, in this situation it will follow that the corresponding non-homogeneous equation is likewise solvable.

Example Let $H = L^2(0,1)$, $D(L) = \{v \in H^1(0,1) : v(0) = 0\}$, and $A = \frac{d}{dx}$.

The two conditions are a bit restrictive, but we can relax them considerably with an elementary observation. Suppose that u is a solution of the initial-value problem above, and let $\lambda \in \mathbb{R}$. Define $w(t) = e^{-\lambda t}u(t)$ for $t \geq 0$. Then it is easy to check that $w(t)$ is a solution of the problem

$$cw'(t) + (\lambda cI + A)w(t) = 0, \quad w(0) = u_0,$$

and, conversely, $w(t)$ is a solution of this problem only if $u(t) = e^{\lambda t}w(t)$ is a solution of the original problem. Thus, the initial-value problem is well-posed if

- $((\lambda cI + A)v, v)_H \geq 0$ for all $v \in D(L)$,
- $(1 + \lambda c)I + A$ maps $D(L)$ *onto* H,

for *some* $\lambda \in \mathbb{R}$. Frequently this is satisfied for large enough λ.

Suppose that $u(t)$ is the solution under these more general hypotheses. Then we find that

$$\tfrac{d}{dt}c\|u(t)\|_H^2 = -2(Au(t), u(t))_H \leq 2\lambda c\|u(t)\|_H^2,$$

so we conclude that

$$c\|u(t)\|_H^2 \leq e^{2\lambda t}c\|u_0\|_H^2, \; t \geq 0.$$

This *stability estimate* shows it is worthwhile to know how small the number λ can be taken.

Advection-diffusion equation

The conservation equation and flux constitutive equation are

$$c\dot{u} + \boldsymbol{\nabla} \cdot \mathbf{j} = F(x)\,, \quad \mathbf{j} = -k\boldsymbol{\nabla} u + \mathbf{b}\,u\,.$$

where $c = c(x)$, $k = k(x)$ and $\mathbf{b} = \mathbf{b}(x)$.

Gravity-driven Fluid Flow. Assume $\lambda \geq 0$. Let V be a subspace of $H^1(G)$ that contains $H_0^1(G)$. and consider the boundary-value problem

$$u \in V: \quad \int_G \big(\lambda c u v + (k\boldsymbol{\nabla} u - u\mathbf{b}) \cdot \boldsymbol{\nabla} v\big)\, dx = \int_G F\, v\, dx, \quad v \in V.$$

Define the corresponding operator $\mathcal{A}_\lambda^R : V \to V'$ by

$$\mathcal{A}_\lambda^R u(v) = \int_G \big(\lambda c u v + k\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v - u\mathbf{b} \cdot \boldsymbol{\nabla} v\big)\, dx$$

$$= \int_G \big(\lambda c u - \boldsymbol{\nabla} \cdot (k\boldsymbol{\nabla} u - \mathbf{b}u)\big)\, v\, dx + \int_{\partial G} \big(k\boldsymbol{\nabla} u - \mathbf{b}u\big) \cdot \mathbf{n}\, v\, dS$$

This displays the PDE and the boundary conditions.

Estimates:

$$\mathcal{A}_\lambda^R u(u) = \int_G \big(\lambda c u^2 + k|\boldsymbol{\nabla} u|^2 + \tfrac{1}{2}\boldsymbol{\nabla} \cdot \mathbf{b} u^2\big) dx$$

$$-\tfrac{1}{2} \int_{\partial G} u^2 \mathbf{b} \cdot \mathbf{n}\, dS$$

$$\mathcal{A}_\lambda^R u(u) \geq 0 \text{ if } \boldsymbol{\nabla} \cdot \mathbf{b} \geq 0 \text{ in G and } v|_{\partial G} = 0 \text{ where } \mathbf{b} \cdot \mathbf{n} > 0\ \forall v \in V$$

This is the *outflow region.* This problem is appropriate for models of *fluid flow* where $\mathbf{b}$ is the gravity term.

Transport of Concentration.

$$u \in V: \quad \int_G \big(\lambda c u v + k\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v + \boldsymbol{\nabla} \cdot (u\mathbf{b}) v\big)\, dx = \int_G F\, v\, dx, \quad v \in V.$$

Define the operator $\mathcal{A}_\lambda^N : V \to V'$ by

$$\mathcal{A}_\lambda^N u(v) = \int_G \big(\lambda c u v + k\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v + \boldsymbol{\nabla} \cdot (u\mathbf{b}) v\big)\, dx$$

$$= \int_G \big(\lambda c u - \boldsymbol{\nabla} \cdot (k\boldsymbol{\nabla} u - \mathbf{b}u)\big)\, v\, dx + \int_{\partial G} k\boldsymbol{\nabla} u \cdot \mathbf{n}\, v\, dS$$

This leads to the same PDE but with a modified boundary condition: there is no advection in the boundary condition.

Estimates:

$$\mathcal{A}_\lambda^N u(u) = \int_G \big(\lambda c u^2 + k|\boldsymbol{\nabla} u|^2 + \tfrac{1}{2}\boldsymbol{\nabla} \cdot \mathbf{b} u^2\big) dx$$

$$+\tfrac{1}{2} \int_{\partial G} u^2 \mathbf{b} \cdot \mathbf{n}\, dS$$

$$\mathcal{A}_\lambda^N u(u) \geq 0 \text{ if } \boldsymbol{\nabla} \cdot \mathbf{b} \geq 0 \text{ in G and } v|_{\partial G} = 0 \text{ where } \mathbf{b} \cdot \mathbf{n} < 0\ \forall v \in V$$

This is the *inflow region.* Such problems arise as models of *concentration transport* where $\mathbf{b}$ is the velocity term. Note that

$$\mathcal{A}_\lambda^N u(v) = \mathcal{A}_\lambda^R u(v) + \int_{\partial G} \mathbf{b} \cdot \mathbf{n} u v\, dS$$

so it is easier for $\mathcal{A}_\lambda^N$ to be coercive than for $\mathcal{A}_\lambda^R$.