Remarks on Cauchy Problem

Let A denote a linear operator on a Hilbert space H, with domain $D(L) \subset H$ a subspace. (Note that we do *not* make any continuity assumptions on A.) Suppose the operator satisfies

- $(Av, v)_H \geq 0$ for all $v \in D(L)$,
- $I + A$ maps $D(L)$ *onto* H.

If the operator satisfies a V-elliptic type assumption, then the first condition will follow easily, and then the Lax-Milgram theorem can be used to show that the second condition also holds. Moreover, these conditions hold for much more general classes of operators than those arising from *elliptic* problems.

We will show later that if an operator satisfies these two conditions, then there is a unique function $u : [0, \infty) \to H$ which satisfies the *initial-value problem*

$$cu'(t) + Au(t) = 0, \quad u(0) = u_0.$$

Here we assume $c > 0$ and $u_0 \in H$ are given. Also, in this situation it will follow that the corresponding non-homogeneous equation is likewise solvable.

Example Let $H = L^2(0, 1)$, $D(L) = \{v \in H^1(0, 1) : v(0) = 0\}$, and $A = \frac{d}{dx}$.

The two conditions are a bit restrictive, but we can relax them considerably with an elementary observation. Suppose that u is a solution of the initial-value problem above, and let $\lambda \in \mathbb{R}$. Define $w(t) = e^{-\lambda t}u(t)$ for $t \geq 0$. Then it is easy to check that $w(t)$ is a solution of the problem

$$cw'(t) + (\lambda cI + A)w(t) = 0, \quad w(0) = u_0,$$

and, conversely, $w(t)$ is a solution of this problem only if $u(t) = e^{\lambda t}w(t)$ is a solution of the original problem. Thus, the initial-value problem is well-posed if

- $((\lambda cI + A)v, v)_H \geq 0$ for all $v \in D(L)$,
- $(1 + \lambda c)I + A$ maps $D(L)$ *onto* H,

for *some* $\lambda \in \mathbb{R}$. Frequently this is satisfied for large enough λ.

Suppose that $u(t)$ is the solution under these more general hypotheses. Then we find that

$$\tfrac{d}{dt}c\|u(t)\|_H^2 = -2(Au(t), u(t))_H \leq 2\lambda c\|u(t)\|_H^2,$$

so we conclude that

$$c\|u(t)\|_H^2 \leq e^{2\lambda t}c\|u_0\|_H^2, \ t \geq 0.$$

This *stability estimate* shows it is worthwhile to know how small the number λ can be taken.