CHAPTER I

Linear Problems ... an Introduction

I.1. Boundary-Value Problems in 1-D

We begin by considering two classical boundary value problems. Let $H \equiv L^2(a,b)$, the space of (equivalence classes of) square-summable functions on the real interval (a,b). By $u' \in H$ we mean that u is an anti-derivative of a function in $L^2(a,b)$, hence, it is absolutely continuous and the classical derivative $u'(x)$ exists at a.e. point x in (a,b). We shall consider the two following boundary-value problems. Let $c \in \mathbb{R}$ and $F \in H$ be given. The *Dirichlet problem* is to find

$$u \in H : -u'' + cu = F \text{ in } H, \qquad u(a) = u(b) = 0 \ ,$$

and the *Neumann problem* is to find

$$u \in H : -u'' + cu = F \text{ in } H, \qquad u'(a) = u'(b) = 0 \ .$$

An implicit requirement of each of these classical formulations is that $u'' \in H$. This smoothness condition can be relaxed: multiply the equation by $v \in H$ and integrate. If also $v' \in H$ we obtain the following by an integration-by-parts. Let $V_0 \equiv \{v \in H : v' \in H \text{ and } v(a) = v(b) = 0\}$; a solution of the Dirichlet problem is characterized by

$$u \in V_0 \text{ and } \int_a^b (u'v' + cuv)\, dx = \int_a^b Fv\, dx \ , \qquad v \in V_0 \ .$$

Similarly, a solution of the Neumann problem satisfies

$$u \in V_1 \text{ and } \int_a^b (u'v' + cuv)\, dx = \int_a^b Fv\, dx \ , \qquad v \in V_1 \ ,$$

where $V_1 \equiv \{v \in H : v' \in H\}$. These are the corresponding *weak formulations* of the respective problems. We shall see directly that they are actually equivalent to their respective classical formulations. Moreover we see already the primary ingredients of the variational theory:

(a) *Functionals*. Each function, e.g., $F \in H$, is identified with a functional, $\widetilde{F} : H \to \mathbb{R}$, defined by $\widetilde{F}(v) \equiv \int_a^b Fv\, dx$, $v \in H$. This identification is achieved by way of the L^2 scalar product. For a pair $u \in V_1$, $v \in V_0$ an integration by parts shows $\widetilde{u}'(v) = -\widetilde{u}(v')$. Thus, for this identification of functions with functionals to be consistent with the usual differentiation of functions, it is *necessary* to define the *generalized derivative* of a functional f by $\partial f(v) \equiv -f(v')$, $v \in V_0$.

(b) *Function Spaces*. From $L^2(a,b)$ and the generalized derivative ∂ we construct the Sobolev space $H^1(a,b) \equiv \{v \in L^2(a,b) : \partial v \in L^2(a,b)\}$. We shall see directly that $H^1(a,b)$ is a Hilbert space with the scalar product

$$(u,v)_{H^1} \equiv \int_a^b (\partial u \partial v + uv)\, dx$$

and corresponding norm $\|u\|_{H^1} = (u,u)_{H_1}^{1/2}$, and each of its members is absolutely continuous, hence, $u(x) - u(y) = \int_y^x \partial u$ for $u \in H^1(a,b)$, $a < y < x < b$. This space arises naturally above in the Neumann problem; we denote by $H_0^1(a,b)$ the subspace $\{v \in H^1(a,b) : v(a) = v(b) = 0\}$ which occurs in the Dirichlet problem.

(c) *Forms*. Each of our weak formulations is phrased as

$$u \in V : a(u,v) = f(v) \ , \qquad v \in V \ ,$$

where V is the appropriate Hilbert space (either H_0^1 or H^1), $f = \widetilde{F}$ is a continuous linear functional on V, and $a(\cdot,\cdot)$ is the bilinear form on V defined by

$$a(u,v) = \int_a^b (\partial u \partial v + cuv)\, dx \ , \qquad u,v \in V \ .$$

This form is bounded or *continuous* on V: there is a $C > 0$ such that

$$|a(u,v)| \le C\|u\|_V \|v\|_V \ , \qquad u,v \in V \ . \tag{1.1}$$

Moreover, it is V-*coercive*, i.e., there is a $c_0 > 0$ for which

$$|a(v,v)| \ge c_0 \|v\|_V^2 \ , \qquad v \in V \tag{1.2}$$

in the case of $V = H^1(a,b)$ if (and only if!) $c > 0$ and in the case of $V = H_0^1(a,b)$ for any $c > -2/(b-a)^2$. (This last inequality follows from (1.3) below but it is not the optimal constant.) We shall see that the weak formulation constitutes a well-posed problem whenever the bilinear form is bounded and coercive.

First we focus on the notion of a generalized derivative of functions and, even more generally, of functionals. These notions are fundamental to the construction of Sobolev spaces. A non-standard aspect of our presentation is that we shall refer to any linear functional (not necessarily continuous) on test functions as a *distribution*. Since all analysis is done in Hilbert subspaces of such functionals, no topological notions are needed for the whole space of functionals.

Let $-\infty \le a < b \le +\infty$. The *support* of a function $\varphi : (a,b) \to \mathbb{R}$ is the closure in (a,b) of the set $\{x \in (a,b) : \varphi(x) \ne 0\}$. Then $C_0^\infty(a,b)$ is the linear space of those infinitely differentiable functions $\varphi : (a,b) \to \mathbb{R}$ each of which has compact support in (a,b). An example is given by

$$\varphi(x) = \begin{cases} \exp[-1/(1-|x|^2)] \ , & |x| < 1 \\ 0 \ , & |x| \ge 1 \ . \end{cases}$$

A linear functional, $T : C_0^\infty(a,b) \to \mathbb{R}$, is called a *distribution* on (a,b); the linear space of all distributions is the algebraic dual $C_0^\infty(a,b)^*$ of $C_0^\infty(a,b)$. We shall refer to $C_0^\infty(a,b)$ as the space of *test functions* on (a,b).

A measurable function $u : (a,b) \to \mathbb{R}$ is *locally integrable* on (a,b) if for every compact set $K \subset (a,b)$, we have $\int_K |u|\,dx < \infty$. The space of all such (equivalence classes of) functions is denoted by $L^1_{\text{loc}}(a,b)$. Suppose u is (a representative of) an element of $L^1_{\text{loc}}(a,b)$. Then we define a corresponding distribution $\widetilde{u}$ by

$$\widetilde{u}(\varphi) = \int_a^b u(x)\varphi(x)\,dx\ , \qquad \varphi \in C_0^\infty(a,b)\ .$$

Note that $\widetilde{u}$ is independent of the representative and that the function $u \mapsto \widetilde{u}$ is linear from L^1_{loc} to $C_0^{\infty *}$.

LEMMA 1.1. *If $\widetilde{u} = 0$ then $u = 0$.*

SKETCH OF PROOF. We have $\int u\varphi = 0$ for all $\varphi \in C_0^\infty$. Extend this to hold for all continuous functions with compact support by a convolution approximation. (See II.3.) Then extend to all bounded measurable functions with compact support by Lebesgue theory. Finally, choose $\varphi_\lambda(x) = u(x)$ for $|x| \le \lambda$ and $|u(x)| \le \lambda$, set $\varphi_\lambda(x) = \lambda$ for $u_\lambda(x) \ge \lambda$, $\varphi_\lambda(x) = -\lambda$ for $u(x) \le -\lambda$, and $\varphi_\lambda(x) = 0$ for $|x| \ge \lambda$. Then let $\lambda \to \infty$. □

PROPOSITION 1.1. *The mapping $u \mapsto \widetilde{u}$ of $L^1_{\text{loc}}(a,b)$ into $C_0^\infty(a,b)^*$ is linear and one-to-one.*

PROOF. The kernel of this map is the zero element. □

We call $\{\widetilde{u} : u \in L^1_{\text{loc}}(G)\}$ the *regular distributions*. Two examples in $C_0^\infty(\mathbb{R})^*$ are the *Heaviside functional*

$$\widetilde{H}(\varphi) = \int_0^\infty \varphi\ , \qquad \varphi \in C_0^\infty(\mathbb{R})\ ,$$

obtained from the *Heaviside function*: $H(x) = 1$ if $x > 0$ and $H(x) = 0$ for $x < 0$, and the constant functional

$$T(\varphi) = \int_{\mathbb{R}} \varphi\ , \qquad \varphi \in C_0^\infty(\mathbb{R})$$

given by $T = \widetilde{1}$. An example of a non-regular distribution is the *Dirac functional* given by

$$\delta(\varphi) = \varphi(0)\ , \qquad \varphi \in C_0^\infty(\mathbb{R})\ .$$

According to Proposition 1.1 the space of distributions is so large that it contains all functions with which we shall be concerned, i.e., it contains L^1_{loc}. Such a large space was constructed by taking the dual of the "small" space C_0^∞. Next we shall take advantage of the linear differentiation operator on C_0^∞ to construct a corresponding generalized differentiation operator on the dual space of distributions. Moreover, we shall define the derivative of a distribution in such a way that it is consistent with the classical derivative on functions. Let D denote the classical derivative, $D\varphi = \varphi'$, when it is defined at a.e. point of the domain of φ. As we observed above, if we want to define a generalized derivative ∂T of a distribution T so that for each $u \in C^\infty(a,b)$ we have $\partial\widetilde{u} = \widetilde{(Du)}$, that is,

$$\partial\widetilde{u}(\varphi) = -\int_a^b u \cdot D\varphi = -\widetilde{u}(D\varphi)\ , \qquad \varphi \in C_0^\infty(a,b)\ ,$$

then we must define ∂ as follows.

Definition. For each distribution $T \in C_0^\infty(a,b)^*$ the derivative $\partial T \in C_0^\infty(a,b)^*$ is defined by

$$\partial T(\varphi) = -T(D\varphi) \ , \qquad \varphi \in C_0^\infty(a,b) \ .$$

Note that $D : C_0^\infty(a,b) \to C_0^\infty(a,b)$ and $T : C_0^\infty(a,b) \to \mathbb{R}$ are both linear, so $\partial T : C_0^\infty(a,b) \to \mathbb{R}$ is clearly linear. Also ∂ is just the negative of the dual D^* of the linear map D, $\partial = -D^* : C_0^\infty(a,b)^* \to C_0^\infty(a,b)^*$. Since ∂ is defined on all distributions, it follows that every distribution has derivatives of all orders. Specifically, every $u \in L^1_{\text{loc}}$ has derivatives in $C_0^\infty(a,b)^*$ of all orders.

Example 1.a. Let f be continuously differentiable on $\mathbb{R}$. Then we have

$$\partial \widetilde{f}(\varphi) = -\widetilde{f}(D\varphi) = -\int f D\varphi \, dx = \int Df\varphi = \widetilde{(Df)}(\varphi)$$

for $\varphi \in C_0^\infty(\mathbb{R})$. The third equality follows from integration-by-parts and all other equalities are definitions. Thus the generalized derivative coincides with the classical derivative on smooth functions. Of course the definition was rigged to make this occur.

Example 1.b. Let $r(x) = xH(x)$ where $H(x)$ is given above. For this piecewise-differentiable function we have

$$\partial \widetilde{r}(\varphi) = -\int_0^\infty x D\varphi(x) \, dx = \int_0^\infty \varphi(x) \, dx = \widetilde{H}(\varphi) \ , \qquad \varphi \in C_0^\infty(\mathbb{R}) \ ,$$

so $\partial \widetilde{r} = \widetilde{H}$ even though $Dr(0)$ does not exist.

Example 1.c. For the piecewise-continuous function H we have

$$\partial \widetilde{H}(\varphi) = -\int_0^\infty D\varphi(x) \, dx = \varphi(0) = \delta(\varphi) \ , \qquad \varphi \in C_0^\infty(\mathbb{R}) \ ,$$

so $\partial \widetilde{H} = \delta$, the non-regular Dirac functional. More generally, let $f : \mathbb{R} \to \mathbb{R}$ be absolutely continuous in a neighborhood of each $x \neq 0$ and have one-sided limits $f(0^+)$ and $f(0^-)$ from the right and left, respectively, at 0. Then we obtain

$$\begin{aligned} \partial \widetilde{f}(\varphi) = -\int_0^\infty f D\varphi - \int_{-\infty}^0 f D\varphi = \int_0^\infty (Df)\varphi + f(0^+)\varphi(0) \\ + \int_{-\infty}^0 (Df)\varphi - f(0^-)\varphi(0) = \widetilde{(Df)}(\varphi) + \sigma_0(f)\varphi(0), \qquad \varphi \in C_0^\infty \ , \end{aligned}$$

where $\sigma_0(f) = f(0^+) - f(0^-)$ is the jump in f at 0. That is, $\partial \widetilde{f} = \widetilde{Df} + \sigma_0(f)\delta$, and this formula can be repeated if Df satisfies the preceding conditions on f:

$$\partial^2 \widetilde{f} = \widetilde{(D^2 f)} + \sigma_0(Df)\delta + \sigma_0(f)\partial\delta \ .$$

For example we have

$$\partial(H \cdot \sin) = H \cdot \cos \ , \quad \partial(H \cdot \cos) = -H \cdot \sin + \delta \ .$$

Before discussing further the interplay between ∂ and D we note that a distribution T on $\mathbb{R}$ is constant if and only if $T = \widetilde{c}$ for some $c \in \mathbb{R}$, i.e.,

$$T(\varphi) = c \int_{\mathbb{R}} \varphi \ , \qquad \varphi \in C_0^\infty \ .$$

This occurs exactly when T depends only on the mean value of each φ. This observation is the key to the description of primitives or anti-derivatives of a given distribution. Suppose we are given a distribution S on $\mathbb{R}$; does there exist a primitive, a distribution T such that $\partial T = S$? That is, do we have a distribution T for which

$$T(D\psi) = -S(\psi) \ , \qquad \psi \in C_0^\infty(\mathbb{R}) \ ?$$

Lemma 1.2.

(a) $\{D\psi : \psi \in C_0^\infty(\mathbb{R})\} = \{\zeta \in C_0^\infty(\mathbb{R}) : \int \zeta = 0\}$ *and the correspondence is given by* $\psi(x) = \int_{-\infty}^x \zeta$.

(b) *Denote the space in (a) by* $\mathcal{H}$ *and let* $\varphi_0 \in C_0^\infty(\mathbb{R})$ *with* $\int \varphi_0 = 1$. *Then each* $\varphi \in C_0^\infty(\mathbb{R})$ *can be uniquely written as* $\varphi = \zeta + c\varphi_0$ *with* $\zeta \in \mathcal{H}$, *and this occurs when* $c = \int \varphi$.

Proposition 1.2.

(a) *For each distribution* S *there is a distribution* T *with* $\partial T = S$.

(b) *If* T_1, T_2 *are distributions with* $\partial T_1 = \partial T_2$, *then* $T_1 = T_2 +$ *constant.*

Proof.

(a) Define T on $\mathcal{H}$ by $T(\zeta) = -S(\psi)$, $\zeta \in H$, $\psi(x) = \int_{-\infty}^x \zeta$, and extend to all of $C_0^\infty(\mathbb{R})$ by $T(\varphi_0) = 0$.

(b) If $\partial T = 0$, then $T(\varphi) = T(\zeta + c\varphi_0) = T(\varphi_0) \int \varphi$, so $T = T(\varphi_0)\widetilde{1}$ is a constant. □

Corollary 1.1. *If* T *is a distribution on* $\mathbb{R}$ *with* $\partial T \in L^1_{\text{loc}}(\mathbb{R})$, *then* $T = \widetilde{f}$ *for some absolutely continuous* f, *and* $\partial T = \widetilde{Df}$.

Proof. Note first that if f is absolutely continuous then $Df(x)$ is defined for a.e. $x \in \mathbb{R}$ and $Df \in L^1_{\text{loc}}(\mathbb{R})$ with $\widetilde{Df} = \partial \widetilde{f}$ as before. In the converse situation of Corollary 1.1, let $g \in L^1_{\text{loc}}$ with $\widetilde{g} = \partial T$ and define $h(x) = \int_0^x g$, $x \in \mathbb{R}$. Then h is absolutely continuous, $\partial(T - \widetilde{h}) = 0$, so Proposition 1.2 shows $T = \widetilde{h} + \widetilde{c}$ for some $c \in \mathbb{R}$. Thus $T = \widetilde{f}$ with $f(x) = h(x) + c$, $x \in \mathbb{R}$. □

Corollary 1.2. *The weak formulations of the Dirichlet and Neumann problems are equivalent to the original formulations.*

Finally we describe the spaces that naturally arise in the consideration of such boundary-value problems. The Sobolev space $H^1(a,b)$ is given by

$$H^1(a,b) = \{u \in L^2(a,b) : \partial u \in L^2(a,b)\}$$

where we have identified $u \cong \widetilde{u}$. Thus each $u \in H^1(a,b)$ is absolutely continuous with

$$u(x) - u(y) = \int_y^x \partial u \ , \qquad a \le x \ , y \le b \ .$$

This gives the Hölder continuity estimate (see (2.1) below)

$$|u(x)-u(y)| \le |x-y|^{1/2}\|\partial u\|_{L^2(a,b)} \ , \quad u \in H^1(a,b), a \le x, y \le b \ .$$

If also we have $u(a)=0$ then there follow

$$|u(x)| \le (b-a)^{1/2}\|\partial u\|_{L^2(a,b)} \ , \qquad a \le x \le b \ , \tag{1.3}$$

$$\|u\|_{L^2(a,b)} \le \big((b-a)/\sqrt{2}\big)\|\partial u\|_{L^2(a,b)} \ , \tag{1.4}$$

and such estimates also hold for those $u \in H^1(a,b)$ with $u(b)=0$. Let $\lambda(x) = (x-a)(b-a)^{-1}$ and $u \in H^1(a,b)$. Then $\lambda u \in H^1(a,b)$ and $\partial(\lambda u) = \lambda\partial u + (b-a)^{-1}u$, so $\|\partial(\lambda u)\|_{L^2} \le \|\partial u\|_{L^2} + (b-a)^{-1}\|u\|_{L^2}$. The same holds for $\partial((1-\lambda)u)$ so by writing $u = \lambda u + (1-\lambda)u$ we obtain

$$\begin{aligned} \max\{|u(x)| : a \le x \le b\} &\le 2(b-a)^{1/2}\|\partial u\|_{L^2} \\ &\quad + 2(b-a)^{-1/2}\|u\|_{L^2} \ , \quad u \in H^1(a,b) \ . \end{aligned} \tag{1.5}$$

This simple estimate will be very useful throughout this introductory chapter.

More generally, we define for each integer $k \ge 1$ the *Sobolev space*

$$H^k(a,b) = \{u \in L^2(a,b) : \partial^j u \in L^2(a,b)\} \ , \qquad 1 \le j \le k \ .$$

Estimates analogous to those above can be easily obtained in appropriate subspaces.

I.2. Variational Method in Hilbert Space

Our objective is to review certain topics in the elementary theory of Hilbert space which lead directly to abstract variational or weak formulations of boundary value problems. Let V be a linear space over the reals $\mathbb{R}$ and the function $x, y \mapsto (x,y)$ from $V \times V$ to $\mathbb{R}$ be a *scalar product*. That is, $(x,x) > 0$ for non-zero $x \in V$, $(x,y) = (y,x)$ for $x, y \in V$, and for each $y \in V$ the function $x \mapsto (x,y)$ is linear from V to $\mathbb{R}$. For each pair $x, y \in V$ it follows that

$$\big|(x,y)\big|^2 \le (x,x)(y,y) \ . \tag{2.1}$$

To see this, we note that

$$0 \le (tx+y, tx+y) = t^2(x,x) + 2t(x,y) + (y,y) \ , \qquad t \in \mathbb{R} \ ,$$

and so the discriminant of the quadratic must be non-positive. From (2.1) it follows that $\|x\| \equiv (x,x)^{1/2}$, $x \in V$, defines a *norm* on V : $\|x\| \ge 0$, $\|tx\| = |t|\ \|x\|$, and $\|x+y\| \le \|x\| + \|y\|$ for $x, y \in V$ and $t \in \mathbb{R}$. Thus every scalar product induces a norm and corresponding *metric* $d(x,y) = \|x-y\|$. A sequence $\{x_n\}$ *converges to* x in V if $\lim_{n\to\infty}\|x_n - x\| = 0$. This is denoted by $\lim_{n\to\infty} x_n = x$. A convergent sequence is always *Cauchy*: $\lim_{m,n\to\infty}\|x_m - x_n\| = 0$. The space V with norm $\|\cdot\|$ is *complete* if each Cauchy sequence is convergent in V. A complete normed linear space is a *Banach space*, and a complete scalar product space is a *Hilbert space*.

Some familiar examples of Hilbert spaces include Euclidean space $\mathbb{R}^m = \{\vec{x} = (x_1, x_2, \dots, x_m) : x_j \in \mathbb{R}\}$ with $(\vec{x}, \vec{y}) = \sum_{j=1}^m x_j y_j$, the sequence space $\ell^2 = \{\vec{x} = \{x_1, x_2, x_3, \dots\} : \sum_{j=1}^\infty |x_j|^2 < \infty\}$ with $(\vec{x}, \vec{y}) = \sum_{j=1}^\infty x_j y_j$, and the Lebesgue space $L^2(\Omega) = \{$equivalence classes of measurable functions $f : \Omega \to \mathbb{R} : \int_\Omega |f|^2\, d\mu$

$< \infty\}$ with $(f,g) = \int_\Omega f(w)g(w)\,d\mu$, where (Ω,μ) is a measure space. Another example is the Sobolev space $H^1(a,b)$ with the scalar product

$$(u,v)_{H^1} = (u,v)_{L^2} + (\partial u, \partial v)_{L^2} \ , \qquad u,v \in H^1(a,b) \ .$$

To verify that this space is complete, let $\{u_n\}$ be a Cauchy sequence, so that both $\{u_n\}$ and $\{\partial u_n\}$ are Cauchy sequences in $L^2(a,b)$. Since $L^2(a,b)$ is complete there are $u,v \in L^2(a,b)$ for which $\lim u_n = u$ and $\lim \partial u_n = v$ in $L^2(a,b)$. For each $\varphi \in C_0^\infty(a,b)$ we have

$$-\int_a^b u_n \cdot D\varphi = \int_a^b \partial u_n \varphi \ , \qquad n \geq 1 \ ,$$

so letting $n \to \infty$ shows $v = \partial u$. Thus $u \in H^1(a,b)$ and $\lim u_n = u$ in $H^1(a,b)$.

Let V_1 and V_2 be normed linear spaces with corresponding norms $\|\cdot\|_1$, $\|\cdot\|_2$. A function $T : V_1 \to V_2$ is *continuous at* $x \in V_1$ if $\{T(x_n)\}$ converges to $T(x)$ in V_2 whenever $\{x_n\}$ converges to x in V_1. It is *continuous* if it is continuous at every x. For example, the norm is continuous from V_1 into $\mathbb{R}$. If T is linear, we shall also denote its value at x by Tx instead of $T(x)$.

PROPOSITION 2.1. *If $T : V_1 \to V_2$ is linear, the following are equivalent:*
(a) *T is continuous at 0,*
(b) *T is continuous at every $x \in V_1$,*
(c) *there is a constant $K \geq 0$ such that $\|Tx\|_2 \leq K\|x\|_1$ for all $x \in V_1$.*

PROOF. Clearly (c) implies (b) by linearity and (b) implies (a). If (c) were false there would be a sequence $\{x_n\}$ in V_1 with $\|Tx_n\|_2 > n\|x_n\|_1$, but then $y_n \equiv \|Tx_n\|_2^{-1} x_n$ is a sequence which contradicts (a). □

We shall denote by $\mathcal{L}(V_1,V_2)$ the set of all continuous linear functions from V_1 to V_2; these are called the *bounded linear functions* because of (c) above. Additional structure on this set is given as follows.

PROPOSITION 2.2. *For each $T \in \mathcal{L}(V_1,V_2)$ we have*

$$\begin{aligned}\|T\| &\equiv \sup\{\|Tx\|_2 : x \in V_1, \|x\|_1 \leq 1\} = \sup\{\|Tx\|_2 : \|x\|_1 = 1\} \\ &= \inf\{K > 0 : \|Tx\|_2 \leq K\|x\|_1 \ , \qquad x \in V_1\} \ ,\end{aligned}$$

and this gives a norm on $\mathcal{L}(V_1,V_2)$. If V_2 is complete, then $\mathcal{L}(V_1,V_2)$ is complete.

PROOF. Consider the two numbers

$$\lambda = \sup\{\|Tx\|_2 : \|x\|_1 \leq 1\} \ , \qquad \mu = \inf\{K > 0 : \|Tx\|_2 \leq K\|x\|_1 \ , \ x \in V_1\} \ .$$

If K is in the set defining μ, then for each $x \in V_1$ with $\|x\|_1 \leq 1$ we have $\|Tx\|_2 \leq K$, so $\lambda \leq K$. This holds for all such K so $\lambda \leq \mu$. If $x \in V_1$ with $\|x\|_1 > 0$ then $x/\|x\|_1$ is a unit vector and so $\|T(x/\|x\|_1)\|_2 \leq \lambda$. Thus $\|Tx\|_2 \leq \lambda\|x\|_1$ for all $x \neq 0$, and it clearly holds if $x = 0$, so we have $\mu \leq \lambda$. This establishes the equality of the three expressions for $\|T\|$, and it is easy to check that this defines a norm on $\mathcal{L}(V_1,V_2)$.

Suppose V_2 is complete and let $\{T_n\}$ be a Cauchy sequence in $\mathcal{L}(V_1,V_2)$. For each $x \in V_1$,

$$\|T_m x - T_n x\|_2 \leq \|T_m - T_n\| \ \|x\|_1$$

so $\{T_n x\}$ is Cauchy in V_2, hence, convergent to a unique Tx in V_2. This defines $T : V_1 \to V_2$ and it follows by continuity of addition and scalar multiplication that T is linear. Also

$$\|T_n x\|_2 \le \|T_n\| \; \|x\|_1 \le \sup\{\|T_m\|\}\|x\|_1$$

so letting $n \to \infty$ shows T is continuous with $\|T\| \le \sup\{\|T_m\|\}$. Finally, to show $\lim T_n = T$, let $\varepsilon > 0$ and choose N so large that $\|T_m - T_n\| < \varepsilon$ for $m, n \ge N$. Then for each $x \in V_1$ $\|T_m x - T_n x\|_2 \le \varepsilon \|x\|_1$, and letting $m \to \infty$ gives

$$\|Tx - T_n x\|_2 \le \varepsilon \|x\|_1 \ , \qquad x \in V_1 \ .$$

Thus, $\|T - T_n\| \le \varepsilon$ for $n \ge N$. □

As a consequence it follows that the *dual* $V' \equiv \mathcal{L}(V, \mathbb{R})$ of any normed linear space V is complete with the dual norm

$$\|f\|_{V'} \equiv \sup\{|f(x)| : x \in V, \|x\|_V \le 1\}$$

for $f \in V'$.

Hereafter we let V denote a Hilbert space with norm $\|\cdot\|$, scalar product $(\cdot,\cdot)$, and dual space V'. A subset K of V is called *closed* if each $x_n \in K$ and $\lim x_n = x$ imply $x \in K$. The subset K is *convex* if $x, y \in K$ and $0 \le t \le 1$ imply $tx + (1-t)y \in K$. The following *minimization principle* is fundamental.

THEOREM 2.1. *Let K be a closed, convex, non-empty subset of the Hilbert space V, and let $f \in V'$. Define $\varphi(x) \equiv (1/2)\|x\|^2 - f(x)$, $x \in V$. Then there exists a unique*

$$x \in K : \varphi(x) \le \varphi(y) \ , \qquad y \in K \ . \tag{2.2}$$

PROOF. Set $d \equiv \inf\{\varphi(y) : y \in K\}$ and choose $x_n \in K$ such that $\lim_{n\to\infty} \varphi(x_n) = d$. Then we obtain successively

$$\begin{aligned} d \le \varphi\big(1/2(x_m + x_n)\big) &= (1/2)\big(\varphi(x_m) + \varphi(x_n)\big) - (1/8)\|x_n - x_m\|^2 \ , \\ (1/4)\|x_n - x_m\|^2 &\le \varphi(x_m) + \varphi(x_n) - 2d \ , \end{aligned}$$

and this last expression converges to zero. Thus $\{x_n\}$ is Cauchy, it converges to some $x \in V$ by completeness, and $x \in K$ since it is closed. Since φ is continuous, $\varphi(x) = d$ and x is a solution of (2.2). If x_1 and x_2 are both solutions of (2.2), the last inequality shows $(1/4)\|x_1 - x_2\| \le d + d - 2d = 0$, so $x_1 = x_2$. □

The solution of the minimization problem (2.2) can be characterized by a *variational inequality*. For $x, y \in V$ and $t > 0$ we have $(1/t)\Big(\varphi\big(x + t(y-x)\big) - \varphi(x)\Big) = (x, y-x) - f(y-x) + (1/2)t\|y-x\|^2$, so the *derivative* of φ at x in the direction $y - x$ is given by

$$\begin{aligned} \varphi'(x)(y-x) &= \lim_{t\to 0}(1/t)\Big(\varphi\big(x + t(y-x)\big) - \varphi(x)\Big) \\ &= (x, y-x) - f(y-x) \ . \end{aligned} \tag{2.3}$$

An easy calculation shows the above equals $\varphi(y) - \varphi(x) + (x, y) - (1/2)\|x\|^2 - (1/2)\|y\|^2$, so (2.1) gives

$$\varphi'(x)(y - x) \le \varphi(y) - \varphi(x) \ , \qquad x, y \in V \ . \tag{2.4}$$

Suppose x is a solution of (2.2). Since for each $y \in K$ we have $x + t(y - x) \in K$ for small $t > 0$, it follows from (2.3) that

$$x \in K : \varphi'(x)(y - x) \ge 0 \ , \qquad y \in K \ .$$

Conversely, for any such x it follows from (2.4) that it satisfies (2.2). Thus, we have shown that (2.2) is equivalent to

$$x \in K : (x, y - x) \ge f(y - x) \ , \qquad y \in K \ . \tag{2.5}$$

The equivalence of (2.2) and (2.5) is merely the fact that the point where a quadratic function takes its minimum is characterized by having a non-negative derivative in each direction into the set.

As an example, let $x_0 \in V$ and define $f \in V'$ by $f(y) = (x_0, y)$ for $y \in V$. Then $\varphi(x) = (1/2)(\|x - x_0\|^2 - \|x_0\|^2)$ so (2.2) means that x is that point of K which is closest to x_0. Recalling that the angle θ between $x - x_0$ and $y - x$ is determined by

$$(x - x_0, y - x) = \cos(\theta)\|x - x_0\| \ \|y - x\| \ ,$$

we see (2.5) means x is that point of K for which $-\pi/2 \le \theta \le \pi/2$ for every $y \in K$. We define x to be the *projection* of x_0 on K and denote it by $P_K(x_0)$.

Corollary 2.1. *For each closed convex non-empty subset K of V there is a projection operator $P_K : V \to K$ for which $P_K(x_0)$ is that point of K closest to $x_0 \in V$; it is characterized by*

$$P_K(x_0) \in K : \big(P_K(x_0) - x_0 \ , y - P_K(x_0)\big) \ge 0 \ , \qquad y \in K \ .$$

It follows from this characterization that the function P_K satisfies

$$\|P_K(x_0) - P_K(y_0)\|^2 \le (P_K(x_0) - P_K(y_0), x_0 - y_0) \ , \qquad x_0, y_0 \in V \ .$$

From this we see that P_K is a *contraction*, i.e.,

$$\|P_K(x_0) - P_K(y_0)\| \le \|x_0 - y_0\| \ , \qquad x_0, y_0 \in V \ ,$$

and that P_K satisfies the *angle condition*

$$(P_K(x_0) - P_K(y_0), x_0 - y_0) \ge 0 \ , \qquad x_0, y_0 \in V \ .$$

That is, the operator P_K is *monotone* (cf., Section II.2).

COROLLARY 2.2. *For each closed subspace K of V and each $x_0 \in V$ there is a unique*

$$x \in K : (x - x_0, y) = 0 \ , \qquad y \in K \ .$$

Two vectors $x, y \in V$ are called *orthogonal* if $(x, y) = 0$, and the *orthogonal complement* of the set S is $S^\perp \equiv \{x \in V : (x, y) = 0 \text{ for } y \in S\}$. Corollary 2.2 says each $x_0 \in V$ can be uniquely written in the form $x_0 = x_1 + x_2$ with $x_1 \in K$ and $x_2 \in K^\perp$ whenever K is a closed subspace. We denote this orthogonal decomposition by $V = K \oplus K^\perp$.

The *Riesz map* $\underset{\sim}{\mathcal{R}}$ of V into V' is defined by $\underset{\sim}{\mathcal{R}}x(y) = (x, y)$ for $x, y \in V$. It is clear that $\|\underset{\sim}{\mathcal{R}}x\|_{V'} = \|x\|_V$; Theorem 1 with $K = V$ shows by way of (2.5) that $\underset{\sim}{\mathcal{R}}$ is onto V', so $\underset{\sim}{\mathcal{R}}$ is an isometric isomorphism of the Hilbert space V onto its dual V'. Specifically, for each $f \in V'$ there is a unique $x = \underset{\sim}{\mathcal{R}}^{-1}(f) \in V$.

COROLLARY 2.3. *For each $f \in V'$ there is a unique*

$$x \in V : (x, y) = f(y) \ , \qquad y \in V \ . \tag{2.6}$$

We recognize (2.6) as the weak formulation of certain boundary value problems. Specifically, when $V = H_0^1$ or H^1, (2.6) is the Dirichlet or Neumann problem, respectively, with $c = 1$. An easy but useful generalization is obtained as follows. Let $a : V \times V \to \mathbb{R}$ be bilinear (linear in each variable separately), continuous (cf. (1.1)), symmetric ($a(x, y) = a(y, x), x, y \in V$) and V-*elliptic*: there is a $c_0 > 0$ such that

$$a(x, x) \geq c_0 \|x\|^2 \ , \qquad x \in V \ . \tag{2.7}$$

Thus, $a(\cdot, \cdot)$ determines an *equivalent scalar product* on V: a sequence converges in V with $\|\cdot\|$ if and only if it converges with $a(\cdot, \cdot)^{1/2}$. Thus we may replace $(\cdot, \cdot)$ by $a(\cdot, \cdot)$ above.

THEOREM 2.1A. *Let $a(\cdot, \cdot)$ be a bilinear, symmetric, continuous and V-elliptic form on the Hilbert space V, let K be a closed, convex and non-empty subset of V, and let $f \in V'$. Set $\varphi(x) = (1/2)a(x, x) - f(x)$, $x \in V$. Then there is a unique*

$$x \in K : \varphi(x) \leq \varphi(y) \ , \qquad y \in K \ . \tag{2.8}$$

The solution of (2.8) *is characterized by*

$$x \in K : a(x, y - x) \geq f(y - x) \ , \qquad y \in K \ . \tag{2.9}$$

If, in addition, K is a subspace of V, then (2.9) *is equivalent to*

$$x \in K : a(x, y) = f(y) \ , \qquad y \in K \ . \tag{2.10}$$

Now (2.10) is precisely our weak formulation, and we see it is the special case of a *variational inequality* (2.9) which is the characterization of the solution of the minimization problem (2.8). When $a(\cdot, \cdot)$ is not symmetric we can still solve the linear problem (2.10), although it no longer is related to a minimization problem.

THEOREM 2.2 (LAX-MILGRAM). *Let $a(\cdot,\cdot)$ be bilinear, continuous and V-coercive (see 1.2), and let $f \in V'$. Then there is a unique*

$$x \in V : a(x,y) = f(y) \ , \qquad y \in V \ . \tag{2.11}$$

PROOF. For each $x \in V$ the function "$y \mapsto a(x,y)$" belongs to V', so by Corollary 3 there is a unique $\alpha(x) \in V : (\alpha(x), y) = a(x,y)$, $y \in V$. This defines $\alpha \in \mathcal{L}(V,V)$, and we similarly construct $\beta \in \mathcal{L}(V,V)$ with $(x, \beta(y)) = a(x,y)$ for $x, y \in V$. Since (2.11) is equivalent to $\alpha(x) = \underset{\sim}{\mathcal{R}}^{-1}(f)$, it suffices to show α is invertible. First, α is one-to-one, since

$$c_0\|x\|^2 \le |a(x,x)| = \big|\big(\alpha(x), x\big)\big| \le \|\alpha(x)\| \ \|x\| \ ,$$

and so $\alpha(x) = 0$ implies $x = 0$. Also, $c_0\|x\| \le \|\alpha(x)\|$ for all $x \in V$. Second, we show the range of $\alpha, Rg(\alpha)$, is closed. If $\lim_{n\to\infty} z_n = z$ and $z_n = \alpha(x_n)$, then $c_0\|x_n - x_m\| \le \|z_n - z_m\|$ so $\{x_n\}$ is Cauchy, hence, convergent to some $x \in V$. But α is continuous, so $\alpha(x) = z \in Rg(\alpha)$. Finally, since $K \equiv Rg(\alpha)$ is a closed subspace, hence $V = Rg(\alpha) \oplus Rg(\alpha)^\perp$, we need only show $Rg(\alpha)^\perp = \{0\}$. But if $y \in Rg(\alpha)^\perp$ then for every $x \in V$, $0 = (\alpha(x), y) = (x, \beta(y))$, so $\beta(y) = 0$. As above, β is one-to-one, so $y = 0$. Thus $Rg(\alpha) = V$. □

Let's consider briefly the dependence of the solution of (2.11) on the data $a(\cdot,\cdot)$ and f in the problem. We denote by $\mathcal{A}$ the operator in $\mathcal{L}(V,V')$ which is equivalent to the bilinear form $a(\cdot,\cdot)$ and given by

$$\mathcal{A}x(y) = a(x,y) \ , \qquad x, y \in V \ .$$

Thus (2.11) is equivalent to $\mathcal{A}x = f$, and we note that $\mathcal{A} = \mathcal{R} \circ \alpha$, where α arose in the proof of Theorem 2. Suppose we are given two such bilinear forms and linear functionals. Denote the corresponding operators by $\mathcal{A}_1$, $\mathcal{A}_2$, and consider the solutions of $\mathcal{A}_1(x_1) = f_1$, $\mathcal{A}_2(x_2) = f_2$. Then we have

$$\mathcal{A}_1(x_1 - x_2)(x_1 - x_2) = (f_1 - f_2)(x_1 - x_2) + (\mathcal{A}_2 - \mathcal{A}_1)x_2(x_1 - x_2) \ ,$$

and from here we obtain

$$c_0\|x_1 - x_2\|^2 \le (\|f_1 - f_2\|_{V'} + \|(\mathcal{A}_2 - \mathcal{A}_1)(x_2)\|_{V'})\|x_1 - x_2\| \ ,$$

where c_0 is the constant in (2.7) for $\mathcal{A}_1$. This yields in turn the estimate

$$c_0\|x_1 - x_2\| \le \|f_1 - f_2\|_{V'} + \|\mathcal{A}_2 - \mathcal{A}_1\|_{\mathcal{L}(V,V')}\|f_2\|_{V'}/c_0 \ .$$

COROLLARY 2.4. *Assume, in addition, that $a(\cdot,\cdot)$ is V-elliptic. Then the solution of* (2.11) *in V depends continuously on f in V' and on $\mathcal{A}$ in $\mathcal{L}(V,V')$.*

Finally we show that the nonlinear problem (2.9) can be resolved for non-symmetric forms. The proof will make use of the following elementary fixed-point theorem.

PROPOSITION 2.3. *Let K be a closed non-empty subset of a Banach space V, and let $T : K \to K$ be a strict contraction:*

$$\|T(x) - T(y)\| \le \lambda \|x - y\| \ , \qquad x, y \in V \ ,$$

where $0 \le \lambda < 1$. Then T has a unique fixed point, an $x \in K : T(x) = x$.

PROOF. Let $x_0 \in K$ and define $\{x_n\}$ by $T(x_n) = x_{n+1}$, $n \ge 0$. For n, $k \ge 1$ we have

$$\begin{aligned} \|x_{n+k+1} - x_n\| &\le \sum_{j=n}^{n+k} \|x_{j+1} - x_j\| \\ &\le \sum_{j=n}^{n+k} \lambda^j \|x_1 - x_0\| \\ &\le \lambda^n (1-\lambda)^{-1} \|x_1 - x_0\| \ , \end{aligned}$$

so $\{x_n\}$ is a Cauchy sequence in K. The limit of this sequence is a fixed point in K. Finally, if x_1 and x_2 are fixed points, $\|x_1 - x_2\| = \|T(x_1) - T(x_2)\| \le \lambda \|x_1 - x_2\|$, so $(1-\lambda)\|x_1 - x_2\| = 0$. Thus, $x_1 = x_2$, so there is exactly one fixed point of T.□

COROLLARY 2.5. *Let K and V be as above and assume T^n is a strict-contraction for some integer $n \ge 1$. Then T has a unique fixed point in K.*

THEOREM 2.3 (LIONS-STAMPACCHIA). *Let $a(\cdot,\cdot)$ be a bilinear, continuous and V-elliptic form on V, and let K be a closed, convex and nonempty subset of V. Then for each $f \in V'$ there exists a unique*

$$x \in K : a(x, y - x) \ge f(y - x) \ , \qquad y \in K \ , \tag{2.12}$$

and the mapping $f \mapsto x : V' \mapsto K$ is continuous.

PROOF. Let x_1 and x_2 be solutions corresponding to f_1 and f_2. Then $a(x_1, x_2 - x_1) \ge f(x_2 - x_1)$, $a(x_2, x_1 - x_2) \ge f(x_1 - x_2)$, and we add these to get $a(x_1 - x_2, x_1 - x_2) \le (f_1 - f_2)(x_1 - x_2)$. This gives $\|x_1 - x_2\| \le (1/c_0)\|f_1 - f_2\|_{V'}$ from which follows the uniqueness and continuous dependence.

To prove existence, let $r > 0$ and define $F(x) \in V'$ for each $x \in V$ by

$$F(x)(y) = (x, y) - r a(x, y) + r f(y) \ , \qquad y \in V \ .$$

Then note that x is a solution (2.12) if and only if

$$x \in K : (x, y - x) \ge F(x)(y - x) \ , \qquad y \in K \ .$$

But this is equivalent to $x = P_K(\underset{\sim}{\mathcal{R}}^{-1} F(x))$; so x is characterized as the fixed point of the function $P_K \underset{\sim}{\mathcal{R}}^{-1} F$. Now P_K is a contraction, and $\mathcal{R}$ is an isometric isomorphism, so it suffices to show F is a strict contraction. But we have

$$\big|\big(F(x_1) - F(x_2)\big)(y)\big| = \big|(x_1 - x_2, y) - r\big(\alpha(x_1 - x_2), y\big)\big|$$

where $\alpha : V \to V$ was constructed in Theorem 2.2, and

$$\|x - r\alpha(x)\|^2 = \|x\|^2 - 2r a(x, x) + r^2 \|\alpha(x)\|^2 \le (1 - 2rc_0 + r^2 C^2)\|x\|^2 \ .$$

Choose $r < 2c_0/C^2$ so $\lambda \equiv (1 - 2rc_0 + r^2C^2)^{1/2} < 1$. Then we have $\|F(x_1) - F(x_2)\|_{V'} \le \lambda\|x_1 - x_2\|$, so it follows that $P_K \underset{\sim}{\mathcal{R}}^{-1} F$ has a unique fixed point. $\square$

In the following section we shall illustrate how the Theorems of this section can be applied to the Sobolev spaces constructed in Section 1 to show that the boundary-value problems of Dirichlet, Neumann and various other types are all well-posed. Extensions of these Theorems to considerably more general situations and related nonlinear boundary-value problems in $\mathbb{R}^n$ will be developed in Chapter II. Also see Theorem III.2.1 for an extension of Theorem 2.2 which is applicable to linear evolution equations.

I.3. Applications to Stretched String Problems

We consider the simplest problem of elasticity, the vertical displacement $u(x)$ within a fixed plane of a string of length $\ell > 0$ whose initial position ($u = 0$) is the interval $0 \le x \le \ell$. The string is stretched by a tension $T > 0$ and it is flexible and elastic, so this tension acts in the direction of the tangent and the string has no resistance to bending. A vertical load or force $F(x)$ per unit length is applied and this results in the displacement $u(x)$ at each point x. For each segment $[x_1, x_2]$ the vertical components of force must balance, and this gives

$$-T \sin\theta_{x_2} + T \sin\theta_{x_1} = \int_{x_1}^{x_2} F(x)\, dx$$

where $\sin\theta_x = u'(x)/\sqrt{1 + (u'(x))^2}$ is the vertical component of the unit tangent at $(x, u(x))$. We assume displacements are small, so $1 + (u')^2 \cong 1$ and we obtain

$$-T\big(u'(x_2) - u'(x_1)\big) = \int_{x_1}^{x_2} F(x)\, dx\ , \qquad 0 \le x_1 < x_1 \le \ell\ .$$

If F is locally integrable on $(0, \ell)$ it follows that $u' = \partial u$ is locally absolutely continuous and the equation

$$-T\partial^2 u = F \qquad \text{in } L^1_{\text{loc}}(0, \ell) \tag{3.1}$$

describes the displacement of the string in the interior of the interval $(0, \ell)$. At the end-points we need to separately prescribe the behavior. For example, at $x = 0$ we could specify either the position, $u(0) = c$, the vertical force, $T\partial u(0) = f_0$, or some combination of these such as an elastic restoring force of the form $T\partial u(0) = h(u(0) - c)$. Such conditions will be prescribed at each of the two boundary points of the interval.

To calculate the energy that is added to the string to move it to the position u, we take the product of the forces and displacements. These tangential and vertical changes are given by

$$T \int_0^\ell \big(\sqrt{1 + (u')^2} - 1\big)\, dx - \int_0^\ell F(x)u(x)\, dx$$

where the first term depends on the change in length of the string, and for small displacements this gives us the approximate *potential energy* functional

$$\varphi(u) = \int_0^\ell \big((T/2)(\partial u)^2 - Fu\big)\, dx \ . \tag{3.2}$$

Here we have used the expansion for small values of r

$$\sqrt{1+r^2} = 1 + \frac{1}{2}r^2 + \cdots \ .$$

We shall see the displacement u corresponding to the external load F can be obtained by minimizing (3.2) over the appropriate set of admissable displacements. Moreover, this applies to more general loads. For example, a "point load" of magnitude $F_0 > 0$ applied at the point c, $0 < c < \ell$, leads to the energy functional

$$\varphi(u) = T/2 \int_0^\ell (\partial u)^2\, dx - F_0 u(c)$$

in which the second term is just a Dirac functional concentrated at c.

Next we give a set of boundary-value problems on the interval (a, b). Each is guaranteed to have a unique solution by Theorem 2.1A. Each example will be related to a stretched-string problem, and for certain special cases we shall compute the displacement u to see if it appears to be consistent with the physical problem. In all of these examples, we rescale the load so that we may assume $T = 1$. Thus, the load becomes the ratio of the actual load to the tension.

Example 3.A. The displacement of a string fixed at both ends is given by the solution of

$$u \in H_0^1(0,\ell) : -\partial^2 u = f$$

where $f \in H_0^1(0,\ell)'$. This problem is well-posed by Theorem 2.1A; note that the corresponding bilinear form is H_0^1-elliptic by the estimate (1.4). If we apply a load $F(x) = \operatorname{sgn}(x - \ell/2)$, where the *sign function* is given by $\operatorname{sgn}(x) = x/|x|$, $x \neq 0$, the resulting displacement is

$$u(x) = \begin{cases} -\frac{1}{2}x(\ell/2 - x)\ , & 0 < x < \ell/2 \\ \frac{1}{2}(x - \ell/2)(\ell - x)\ , & \ell/2 < x < \ell \end{cases}$$

If we apply a point load, δ_c concentrated at $x = c$, the displacement is

$$u(x) = 1/2(c - |x - c|) + (1/2 - c/\ell)x$$

with maximum value $u(c) = c(1 - c/\ell)$. Both of these solutions can be computed directly from the ordinary differential equation by using Proposition 1.2.

Example 3.b. Non-homogeneous boundary conditions arise when the displacements at the end-points are fixed at non-zero levels. For example, the solution to

$$u \in H^1(0,\ell) : u(0) = f_1 \ , \quad u(\ell) = f_2 \ , \quad -\partial^2 u = F$$

is obtained by minimizing (3.2) over the set of admissable displacements

$$K = \{v \in H^1(0,\ell) : v(0) = f_1, v(\ell) = f_2\} \ .$$

This minimum u satisfies (2.9) where

$$a(u,v) = \int_0^\ell \partial u \partial v \, dx \ , \qquad f(v) = \int_0^\ell F v \, dx \qquad u, v \in H^1(0,\ell) \ .$$

Since the set K is the translate of the subspace $H_0^1(0,\ell)$ by the function

$$u_0(x) = (\ell - x) f_1/\ell + x f_2/\ell \ ,$$

this variational inequality is equivalent to

$$u \in K : a(u,\varphi) = f(\varphi) \ , \qquad \varphi \in H_0^1(0,\ell) \ .$$

Moreover, this problem is actually a "linear" problem for the unkown $w \equiv u - u_0$ in the form

$$w \in H_0^1(0,\ell) : a(w,\varphi) = f(\varphi) - a(u_0,\varphi) \ , \qquad \varphi \in H_0^1(0,\ell) \ ,$$

and thus it is well-posed by any one of the Theorems of Section 2.

Example 3.c. Unilateral boundary constraints arise when the admissable displacements are given by

$$K = \{v \in H^1(0,\ell) : v(0) = 0 \ , \quad v(\ell) \geq 0\} \ .$$

Since K is a *cone*, the displacement u that minimizes (3.2) over K is characterized by

$$\begin{aligned} u \in K : a(u,\varphi) \geq f(\varphi) \ , \quad &\text{for} \quad \varphi \in H^1(0,\ell) \ , \ \varphi(0) = 0 \ , \ \varphi(\ell) \geq 0 \ , \\ &\text{and} \quad a(u,u) = f(u) \ , \end{aligned}$$

and this is equivalent to

$$\begin{aligned} &u \in H^1(0,\ell) : -\partial^2 u = F \quad \text{in } L^2(0,\ell) \ , \\ &u(0) = 0 \ , \qquad \text{and} \qquad u(\ell) \geq 0 \ , \partial u(\ell) \geq 0 \ , \ \partial u(\ell) u(\ell) = 0 \ . \end{aligned}$$

The conditions at ℓ state that the displacement is non-negative, the vertical force of the constraint is non-negative, and one or the other is equal to zero. For the problem in which a force is loaded at the point c, $0 < c < \ell$, with magnitude F_0, the functional is

$$f(v) = F_0 v(c) = F_0 \delta_c(v) \ , \qquad v \in H^1(0,\ell) \ ,$$

and the corresponding solutions can be computed as follows. The general solution of

$$-\partial^2 u = F_0 \delta_c \ , \qquad u(0) = 0$$

is given by

$$u(x) = c_1 x - F_0(x-c)H(x-c) \ , \qquad 0 \le x \le \ell \ ,$$

and we need additionally to have

$$\begin{aligned} u(\ell) &= c_1 \ell - F_0(\ell - c) \ge 0 \ , \\ \partial u(\ell) &= c_1 - F_0 \ge 0 \end{aligned}$$

with equality in at least one. Thus we have either

$$c_1 = F_0 \ge 0 : u(x) = F_0\big(x - (x-c)H(x-c)\big) \ ,$$

or else

$$c_1 = (1 - c/\ell)F_0 \le 0 : u(x) = F_0\big((1-c/\ell)x - (x-c)H(x-c)\big)$$

depending on the sign of F_0. Thus, if $F_0 \ge 0$

the string does not touch the constraint at $x = \ell$ and has zero vertical force acting on that end, while if $F_0 < 0$ the string is supported by the constraint with a force given by $\partial u(\ell) = -(c/\ell)F_0 > 0$.

Example 3.d. Unilateral internal constraints on the string occur, for example, if the entire string is constrained as for the admissable displacements

$$K = \{v \in H_0^1(0,\ell) : v(x) \ge 0 \ , 0 \le x \le \ell\} \ .$$

With a distributed load $F(x)$ as above the solution is characterized by

$$u \in H_0^1(0,\ell) : u \ge 0 \ , \quad \partial^2 u + F \le 0 \ , \quad (\partial^2 u + F)(u) = 0 \ . \tag{3.3}$$

For the specific load given by $F(x) = \operatorname{sgn}(x - \ell/2)$ we can directly compute a strong solution of this problem, and it is given by

$$u(x) = \begin{cases} \frac{1}{2}(\frac{\ell}{2})^2 + (\sqrt{2}-1)(\frac{\ell}{2})(x-\ell) - \frac{1}{2}(x-\ell/2)^2 \operatorname{sgn}(x-\ell/2) \ , \\ \qquad\qquad\qquad\qquad (1-\sqrt{2}/2)\ell \le x \le \ell \ , \\ \qquad 0 \ , \qquad\qquad 0 \le x \le (1-\sqrt{2}/2)\ell \ . \end{cases}$$

Thus we find the interval is divided into two distinct regions. In the first, the displacement is up against the obstacle or constraint, $u = 0$. In the other, the displacement is free of the constraint and it satisfies the differential equation. These two alternatives are implied by the equation in (3.3). The first inequality is the constraint and the second states that the effect of the constraint is that of a non-negative force, $-(\partial^2 u + F)$, distributed along the string. Note that the dependence of the displacement u on the force F is non-linear. The location of the first region

of constraint is the major difficulty in this problem, for if it were known then one could obtain the solution by resolving a Dirichlet problem on the complementary region. This is an example of a *free-boundary problem* wherein the crux is to locate the unknown boundary between the contact region and its complement.

EXAMPLE 3.E. Our next problem arises from a stretched string which has a specified vertical force at the left end and an elastic restoring force at the right end. Thus, define $V = H^1(0,\ell)$ and

$$a(u,v) = \int_0^\ell \partial u \partial v \, dx + hu(\ell)v(\ell) \ , \qquad u,v \in V \ ,$$
$$f(v) = \int_0^\ell Fv \, dx - f_0 v(0) + f_\ell v(\ell) \ , \qquad v \in V \ .$$

From (1.5) we find these are continuous functionals on V, and by similar estimates it follows that $a(\cdot,\cdot)$ is $H^1(0,\ell)$-elliptic if $h > 0$. With this assumption, it follows from Theorem 2.1A that there is a unique weak solution of the boundary-value problem

$$u \in H^1(0,\ell) : -\partial^2 u = F \quad \text{in } L^2(0,\ell) \ ,$$
$$\partial u(0) = f_0 \ , \qquad \partial u(\ell) + hu(\ell) = f_\ell \ .$$

Note that the equation in $L^2(0,\ell)$ shows that ∂u is absolutely continuous on $[0,\ell]$, so the boundary conditions are meaningful. Finally, we see that $a(\cdot,\cdot)$ fails to be H^1-elliptic if $h = 0$. In that case there is non-uniqueness of solutions, since any constant added to a solution gives another solution, and there exists a solution only if

$$\int_0^\ell F(x) \, dx - f_0 + f_\ell = 0 \ ,$$

i.e., the total force on the string is zero. In the absence of a restoring force ($h = 0$) this constraint on the forces is necessary for existence of a physically consistent displacement of the stretched string.

Orientation.

We saw in Section 1 that various boundary-value problems have a weak formulation in Sobolev spaces which are actually equivalent to the respective strong or direct formulations. Then in Section 2 we found that this weak formulation is equivalent to a minimization principle (2.2), and that it corresponds to a variational problem, (2.5) or (2.11). Section 3 indicates that these can be related to "minimum energy" statements in a classical setting.

Linear boundary-value-problems can be characterized by operators on Hilbert space in two rather natural but different ways. The first is motivated by the abstract variational problem (2.11). Each bilinear continuous form $a(\cdot,\cdot)$ on the Hilbert space V is equivalent to a continuous linear operator $\mathcal{A} : V \to V'$, and these are related by

$$a(x,y) = \mathcal{A}x(y) \ , \qquad x,y \in V \ .$$

The problem (2.11) is equivalent to the operator equation

$$x \in V : \mathcal{A}x = f \ \text{ in } \ V' \ .$$

This operator $\mathcal{A}$ from the space to its dual is one-to-one or onto exactly when the boundary-value problem has uniqueness or is always solvable, respectively, and $\mathcal{A}^{-1}$ being continuous (e.g., $a(\cdot,\cdot)$ is V-coercive) corresponds to the continuous dependence of the solution x on the data f in (2.11).

The second approach is to characterize the boundary-value problem as an *unbounded* operator A on a single space H: the domain $D(A)$ of the operator A is the set of functions in H which satisfy all the boundary conditions, and the value of A is just the action of the differential equation on any such function in $D(A)$. Since the domain and range both are in the same space, various polynomials (or more general functions) of A or its *resolvent*, $(\lambda I+A)^{-1}$, can be constructed in H. However, such an operator is almost never continuous, and the domain is usually a proper subset of the Hilbert space, so the analysis is rather delicate. We shall briefly study this second way of characterizing boundary-value problems and its connection with the corresponding variational operators $\mathcal{A}$. Then we study in Section 5 the exponential function of such an unbounded operator A which characterizes the corresponding Cauchy problem.

I.4. Unbounded Operators

Let H be a Hilbert space, D a subspace (algebraic) of H and let $A : D \to H$ be linear. Such a map we call an *unbounded operator* on H with *domain* D. The *graph* of A is the subspace

$$G(A) = \{[x, Ax] : x \in D\}$$

of the product $H \times H$. Note that $H \times H$ is also a Hilbert space with componentwise addition and scalar multiplication, and its scalar product is

$$\big([x_1, x_2], [y_1, y_2]\big)_{H\times H} = (x_1, y_1)_H + (x_2, y_2)_H \ .$$

The operator A is called *closed* if $G(A)$ is a closed subspace of $H \times H$. That is, A is closed if whenever $x_n \in D$, $x_n \to x$ and $Ax_n \to y$ in H imply that $x \in D$ and $Ax = y$. This is a much weaker condition than continuity of A, since convergence of $\{Ax_n\}$ is an assumption, not a conclusion.

Suppose $A : D \to H$ is an unbounded operator on H and that D is *dense* in H. The *adjoint* of A is defined as follows. Let D^* be the subspace of those $y \in H$ for which the linear map $x \mapsto (Ax, y) : D \to \mathbb{R}$ is continuous. Such a map has a unique extension by uniform continuity to all of H and thus there is a unique vector $A^*y \in H$ such that

$$(Ax, y)_H = (x, A^*y)_H \ , \qquad x \in D \ , y \in D^* \ . \tag{4.1}$$

It follows easily that $A^* : D^* \to H$ is an unbounded operator (i.e., linear) on H, and it is called the *adjoint* of A.

Since A^* is defined by (4.1) with D^* being "maximal", it follows directly that A^* is closed. Note that we can define A^* only if D is dense.

Lemma 4.1. *If A is closed and D is dense, then D^* is dense.*

PROOF. Let $P : H \times H \to G(A)^\perp$ be the projection onto the orthogonal complement of $G(A)$ in $H \times H$. If $y \in H$, $y \neq 0$, then $[0, y] \notin G(A)$ so $P[0, y] = [u, v]$ satisfies $([u, v], [0, y])_{H\times H} = (v, y)_H \neq 0$. But

$$(u, x)_H + (v, Ax)_H = 0 \ , \qquad x \in D$$

so $v \in D^*$. Thus, we see that for each $y \neq 0$ there is a $v \in D^*$ with $(v, y)_H \neq 0$. This shows $(D^*)^\perp = \{0\}$, so D^* is dense. □

DEFINITION. An (unbounded) operator $A : D \to H$ is *accretive* if

$$(Ax, x)_H \geq 0 \ , \qquad x \in D \ ,$$

and it is *m-accretive* if, in addition, $A + I$ maps D onto H, i.e., $Rg(A + I) = H$.

Such operators will occur frequently and play an important role in our work below. We give some examples in $H = L^2(a, b)$.

EXAMPLE 4.A. Set $D = H_0^1(a, b)$ and $A = \partial$. Then A is closed, D is dense, and the adjoint is given by $A^* = -\partial$ on $D^* = H^1(a, b)$. Note that A is accretive, A^* is not accretive, and that $Rg(A+I)$ is the orthogonal complement of $\{\exp(x)\}$. Thus A is not m-accretive. Finally, the second adjoint A^{**} is equal to A; in particular, it has the same domain.

EXAMPLE 4.B. Set $D = \{v \in H^1(a, b) : v(a) = cv(b)\}$ and $A = \partial$. Then $A^* = -\partial$ on $D^* = \{v \in H^1(a, b) : v(b) = cv(a)\}$ and A is closed, as follows directly or by comparison with A^*. Furthermore, A and A^* are accretive only if $|c| \leq 1$, and then both are m-accretive with

$$\begin{aligned} (I + A)u = f \qquad & \text{if and only if} \\ u(x) = \int_a^b G(x, s)f(s)\, ds \ , \qquad & a \leq x \leq b \ , \end{aligned}$$

where

$$G(x, s) = \frac{e^{-(x-s)}}{e^{b-a} - c} \begin{cases} e^{b-a} \ , & a \leq s < x \ , \\ c \ , & x < s \leq b \ . \end{cases}$$

The integrand $G(\cdot, \cdot)$ is the Green's function for $A + I$, and it is characterized for each $s \in (a, b)$ as the solution of

$$G(\cdot, s) \in D \ , \qquad (I + A)G(\cdot, s) = \delta_s \ ,$$

where δ_s is the Dirac functional at s.

EXAMPLE 4.C. Set $D = \{v \in H_0^1(a, b) : \partial^2 v \in L^2(a, b)\}$ and $A = -\partial^2$. Then one can show directly that A is closed, or that $A = A^*$, from which it follows that A is closed. It is easy to check that A is accretive and from Section 2 that A is m-accretive. This operator corresponds to the *Dirichlet problem*.

LEMMA 4.2. *If A is m-accretive then $(I + A)^{-1}$ is a contraction on H, hence, A is closed.*

PROOF. For $u \in D$ we have

$$\|u\|_H^2 \le \big((I+A)u, u\big)_H \le \|(I+A)u\|_H \|u\|_H$$

so $(I+A)^{-1}$ is a contraction. Since an operator is closed if and only if its inverse is closed, and it is closed if and only if its sum with a multiple of the identity is closed, the result follows. □

Actually Example 4.C illustrates a general situation that occurs frequently. Let V be a Hilbert space which is dense in another Hilbert space H, and assume the identity $V \to H$ is continuous. Let $a(\cdot,\cdot)$ be a continuous bilinear form on V. Then we define D to be the set of all $u \in V$ such that the function $v \mapsto a(u,v)$ is continuous on V *with the H-norm*. For each such $u \in D$ there is then a unique $Au \in H$ such that

$$a(u,v) = (Au, v)_H \ , \qquad u \in D \ , v \in V \ ,$$

and this defines a linear operator $A : D \to H$. This construction is similar to that of the adjoint above, and the special case of

$$H = L^2(a,b) \ , \qquad V = H_0^1(a,b) \ , \qquad a(u,v) = (\partial u, \partial v)_H$$

gives our last example.

Consider the *adjoint form* on V given by $b(u,v) = a(v,u)$, $u, v \in V$. This leads likewise to an operator $B : D(B) \to H$ given by

$$a(u,v) = (u, Bv)_H \ , \qquad u \in V, v \in D(B) \ .$$

Then we obtain the following.

PROPOSITION 4.1. *Assume there is a $\lambda \in \mathbb{R}$ and a $c > 0$ such that*

$$a(v,v) + \lambda\|v\|_H^2 \ge c\|v\|_V^2 \ , \qquad v \in V \ .$$

Then D is dense in H, the operator $A + \lambda I : D(A) \to H$ is one-to-one and onto and its inverse is continuous, A is closed, and $A^* = B$.

PROOF. If $F \in H$ then $v \mapsto (F,v)_H$ is continuous and linear on V, so by Theorem 2.2 there is a unique

$$u \in V : a(u,v) + \lambda(u,v)_H = (F,v)_H \ , \qquad v \in V \ .$$

Thus $u \in D$ and $(A+\lambda)u = F$, so $A + \lambda$ maps D one-to-one onto H. Similarly, $B + \lambda$ maps $D(B)$ one-to-one onto H. If $w \in D^\perp$ in H then there is a $v \in D(B)$ with $(B+\lambda)v = w$, and so

$$0 = (u,w)_H = \big(u, (B+\lambda)v\big)_H = \big((A+\lambda)u, v\big)_H \ , \qquad u \in D \ .$$

Since $A + \lambda I$ is onto, $v = 0$ and $w = 0$, so $D^\perp = \{0\}$. This shows D is dense. We can deduce that A is closed from the fact that $(A+\lambda)^{-1}$ is continuous or from $A = B^*$, which follows by symmetry from the next part of the proof.

Suppose $v \in D(B)$. Then for every $u \in D$ we have $(Au, v)_H = a(u,v) = (u, Bv)$, and this shows that $v \in D(A^*)$ with $A^*v = Bv$. That is, A^* is an extension

of B. Next let $u \in D^*$ and choose $u_0 \in D(B)$ so that $(B+\lambda)u_0 = (A^*+\lambda)u$. For each $v \in D$ we have

$$\big((A+\lambda)v, u\big)_H = \big(v, (A^*+\lambda)u\big)_H = \big(v, (B+\lambda)u_0\big)_H = \big((A+\lambda)v, u_0\big)_H \ .$$

Since $A+\lambda$ is onto H this implies $u = u_0 \in D(B)$, hence, $D^* = D(B)$. □

COROLLARY 4.1. *Assume that $a(\cdot,\cdot)$ is non-negative, i.e.,*

$$a(v,v) \geq 0 \ , \qquad v \in V \ ,$$

and that there is a $c > 0$ for which

$$a(v,v) + \|v\|_H^2 \geq c\|v\|_V^2 \ , \qquad v \in V \ .$$

Then A is m-accretive.

Here is another example to illustrate the situation, the *Neumann problem*.

EXAMPLE 4.D. Set $H = L^2(a,b)$, $V = H^1(a,b)$ and define

$$a(u,v) = \int_a^b \partial u \partial v \ , \qquad u, v \in V \ .$$

As above this determines an unbounded operator A on $L^2(a,b) : Au = F \in L^2(a,b)$ is equivalent to

$$u \in V : a(u,v) = (F,v)_{L^2} \ , \qquad v \in V \ ,$$

and this weak Neumann problem is equivalent to

$$u \in V \ , \qquad -\partial^2 u = F \ , \qquad \partial u(a) = \partial u(b) = 0 \ .$$

Thus, we find

$$D = \big\{u \in V : \partial^2 u \in L^2(a,b) \ , \partial u(a) = \partial u(b) = 0\big\}$$

and $A = -\partial^2 : D \to L^2(a,b)$. From Proposition 1 it follows that A is m-accretive, $A = A^*$, and that $A + \lambda I$ is a bijection of D onto $L^2(a,b)$ for every $\lambda > 0$. The situation is different at $\lambda = 0$. Specifically, it is clear that $Au = F$ is possible only if $\int_a^b F = 0$, i.e., F is orthogonal in L^2 to the constant functions. Conversely, one can show that this condition on F is sufficient for the existence of a solution. (For example, solve the mixed problem

$$-\partial^2 u = F \ , \qquad u(a) = \partial u(b) = 0 \ ,$$

and then note that $\int_a^b F = \partial u(a)$.) In summary, we find the range of A and kernel of A are given by

$$\begin{aligned} &\mathrm{Ker}(A) = \{ \text{ constant functions } \} \ , \\ &\mathrm{Rg}\ (A) = \mathrm{Ker}(A)^\perp \ . \end{aligned}$$

We shall characterize the m-accretive operators among those which are accretive. But first, note that if $B \in \mathcal{L}(H)$ is a strict contraction, $\|B\|_{\mathcal{L}(H)} < 1$, then $I - B$ is a bijection of H onto itself. This follows directly, either from the power-series representation $(I-B)^{-1} = \sum_{n=0}^\infty B^n$ in $\mathcal{L}(H)$, from Proposition 2.3, or from Proposition 4.1 with $V = H$ and $a(u,v) = \big((I-B)u, v\big)_H$.

PROPOSITION 4.2. *The following are equivalent:*
(a) $A : D \to H$ *is accretive and there exists a* $\mu > 0$ *such that* $Rg(\mu I + A) = H$,
(b) A *is* m*-accretive, and*
(c) A *is accretive,* D *is dense in* H*, and* $Rg(\lambda I + A) = H$ *for every* $\lambda > 0$.

PROOF. Clearly, (c) implies (b) and (b) implies (a). Suppose (a) holds. Then $(\mu + A)^{-1} \in \mathcal{L}(H)$ and $\|(\mu + A)^{-1}\| \le 1/\mu$. From our preceding remark it follows that if $|\lambda - \mu|/\mu < 1$, then $\big(I + (\lambda - \mu)(\mu + A)^{-1}\big)^{-1} \in \mathcal{L}(H)$. In this case we have

$$(\lambda + A)(\mu + A)^{-1} = \big((\lambda - \mu) + (\mu + A)\big)(\mu + A)^{-1} = \big((\lambda - \mu)(\mu + A)^{-1} + I\big) ,$$

so $\mathrm{Rg}(\lambda + A) = H$ whenever $0 < \lambda < 2\mu$. By induction it follows that $R(\lambda + A) = H$ for every $\lambda > 0$. To show D is dense, let $z \in D^{\perp}$, set $z = (\mu + A)x$ for some $x \in D$, and note that $0 = (z, x)_H \ge \mu\|x\|^2$, so $x = 0$, hence, $z = 0$. This shows $D^{\perp} = \{0\}$. The preceding shows (a) implies (c), and so the equivalences are now obvious. □

In the next section we shall show that the m-accretive operators are further characterized as those for which the *initial-value problem*

$$\frac{du(t)}{dt} + Au(t) = 0 , \qquad 0 < t , \quad u(0) = u_0$$

is a well-posed problem. Formally, we write $u(t) \cong \exp(-tA)u_0$; the m-accretive operators are precisely those for which the indicated "exponential operators" can be constructed as contractions on H.

It is easy to see how the unbounded operator A with domain D in H constructed as above from the continuous bilinear form $a(\cdot, \cdot)$ on V is related to the continuous $\mathcal{A} \in \mathcal{L}(V, V')$ which is equivalent to $a(\cdot, \cdot)$. In fact, the graph of A is the restriction of the graph of $\mathcal{A}$ to $V \times H$. That is, note that $H' \hookrightarrow V'$ by "restriction to V" of functionals on H, so $D = \{u \in V : \mathcal{A}u \in H'\}$ and then $Au \in H$ is just that $\mathcal{A}u \in H'$ which corresponds through the identification of H with H' by its Riesz map. Thus, with this identification $H = H'$ in the proof of Proposition 4.1, it is clear that $\mathcal{A} + \lambda I$ is an isomorphism of V onto V' and $A + \lambda I$ is just its (necessarily onto) restriction to $H \subset V'$. (More generally, if $\mathcal{R}$ is the Riesz map of H onto H', then $A = \mathcal{R}^{-1}\mathcal{A}$.) Finally, note that A is accretive on H exactly when the linear operator $\mathcal{A}$ satisfies

$$\mathcal{A}v(v) \ge 0 , \qquad v \in V .$$

This property of $\mathcal{A}$ is called *monotone* and will predominate in Chapters II and III. Not every m-accretive A corresponds to a monotone $\mathcal{A}$ as above; those which do are a special class.

DEFINITIONS. Let V, H be Hilbert spaces with $H \cong H'$ and let $\mathcal{A} \in \mathcal{L}(V, V')$ be *monotone*:

$$\mathcal{A}v(v) \ge 0 , \qquad v \in V .$$

The corresponding unbounded operator on H, $A = \mathcal{A}|_{V \times H}$, is then accretive and we shall call it *regular accretive* when it is so determined by a triple $\{\mathcal{A}, V, H\}$. Assume further that for every $\varepsilon > 0$, $\mathcal{A} + \varepsilon I$ is V-elliptic. (This implies that $\mathcal{A}$ is monotone.) Then $\mathrm{Rg}(\mathcal{A} + \varepsilon I) = V'$ for each $\varepsilon > 0$ and so $\mathrm{Rg}(A + \varepsilon I) = H$, hence, A is m-accretive, and we shall call it *regular* m*-accretive*.

One can show that every accretive self-adjoint operator on H is regular m-accretive. For this, let V be the closure of $D(A)$ with the scalar product $(u,v)_V \equiv (u,v)_H + (Au,v)_H$.

I.5. The Cauchy Problem

Let H be a Hilbert space, D a subspace, and $A : D \to H$ an unbounded linear operator. The *Cauchy Problem* for the evolution equation

$$u'(t) + Au(t) = 0 \ , \qquad t > 0 \ , \tag{5.1}$$

is to find a *solution* $u \in C([0,\infty), H) \cap C^1((0,\infty), H)$ such that $u(t) \in D(A)$ for $t > 0$ and $u(0) = u_0$, where $u_0 \in H$ is prescribed. The continuity or differentiability of the vector-valued function $u : [0,\infty) \to H$ is defined exactly as in the real-valued case $H = \mathbb{R}$, but with absolute-value replaced by the H norm.

Suppose that for each $u_0 \in D$ there is a unique solution u of the Cauchy Problem; then define $S(t)u_0 = u(t)$ for $t \geq 0$, $u_0 \in D$. Since A is linear it follows each $S(t) : D \to D$ is linear for $t \geq 0$. Furthermore, since the translate $u(t+\tau)$ is a solution of (5.1) for each $\tau \geq 0$, we find from the uniqueness that

$$S(t+\tau)u_0 = S(t)S(\tau)u_0 \ , \qquad t,\tau \geq 0 \ ; S(0) = I \ .$$

If $u(t) = S(t)u_0$ then

$$1/2 \frac{d}{dt}\|u(t)\|^2 = -\big(Au(t), u(t)\big)_H \ ,$$

so if A is accretive then $\|u(t)\|$ is decreasing for $t \geq 0$, hence, $\|S(t)u_0\| \leq \|u_0\|$ and thus each $S(t)$ is a contraction on D. If D is dense in H each $S(t)$ has a unique extension to a contraction on H and we obtain the following.

Definition. $\{S(t) : t \geq 0\}$ is a *linear contraction semigroup* (or LCS) if $S(t) : H \to H$ is a linear contraction for each $t \geq 0$,

$$S(t+\tau) = S(t)S(\tau) \ \text{ for } \ t,\tau \geq 0 \ , \ S(0) = I \ ,$$

and

$$S(\cdot)x \in C\big([0,\infty), H\big) \ \text{ for each } \ x \in H \ .$$

For example, suppose $A \in \mathcal{L}(H)$ and A is accretive. Then one can define by power series in $\mathcal{L}(H)$ the operator

$$\exp(A) = \sum_{n=0}^{\infty} A^n/n!$$

and thereby the family of operators $S(t) = \exp(-tA)$, $t \geq 0$. It follows that

$$\frac{d}{dt}S(t) = -A \cdot S(t) \ \text{ in } \ \mathcal{L}(H)$$

and hence $u(t) \equiv S(t)u_0$ is the solution of the Cauchy Problem for (5.1) with $u(0) = u_0$. Note that the operator $-A$ can be recovered from the semigroup by computing the right-derivative at $t = 0$,

$$-Au_0 = \lim_{t \to 0^+} \{\big(u(t) - u(0)\big)/t\} \equiv D^+u(0) \ .$$

DEFINITION. The *generator* of the LCS $\{S(t) : t \geq 0\}$ is the operator B defined by $Bx \equiv D^+(S(0)x)$ for each x belonging to

$$D(B) = \{x \in H : \lim_{h \to 0^+} h^{-1}\big(S(h)x - x\big) \equiv D^+\big(S(0)x\big) \ \text{ exists}\} \ .$$

Our preceding remarks verify most of the following.

PROPOSITION 5.1. *Let $A \in L(D, H)$ be closed and accretive, D dense in H, and assume for every $u_0 \in D$ there exists a solution $u \in C^1([0, \infty), H)$ of (5.1) on $t \geq 0$ with $u(0) = u_0$. Construct $\{S(t) : t \geq 0\}$ as above, so $u(t) = S(t)u_0$, $t \geq 0$. Then $\{S(t) : t \geq 0\}$ is a LCS on H whose generator is an extension of $-A$.*

PROOF. Since A is accretive, each Cauchy problem has at most one solution, so the construction of $\{S(t) : t \geq 0\}$ is done as above. For each $u_0 \in D$ we have

$$S(t)u_0 - u_0 = \int_0^t u' = -\int_0^t Au(s)\, ds \ , \qquad t > 0 \ ,$$

and the integrand is continuous on $[0, \infty)$, so $D^+(S(0)u_0) = -Au_0$. □

Our objective is to find sufficient conditions on an operator A in order that the Cauchy problem for (5.1) will have a solution. Thus we shall characterize those operators which are generators of linear contraction semigroups. To begin, suppose that $B : D(B) \to H$ is the generator of $\{S(t) : t \geq 0\}$ as above. Then for each $x \in D(B)$

$$S(t+h)x - S(t)x = \big(S(h) - I\big)S(t)x = S(t)\big(S(h) - I\big)x \ , \qquad h > 0 \ , \ t \geq 0 \ ,$$

so dividing by h and letting $h \to 0$ shows

$$D^+S(t)x = BS(t)x = S(t)\, Bx \ , \qquad x \in D(B) \ , \ t \geq 0 \ .$$

In particular, $S(t) : D(B) \to D(B)$. Also, for $0 < h < t$

$$S(t)x - S(t-h)x = S(t-h)\big(S(h) - I\big)x$$

and the $S(t-h)$ are uniformly bounded so we may divide this by h and take the limit to obtain

$$D^-S(t)x = S(t)\, Bx \ , \qquad x \in D(B) \ , \ t > 0 \ ,$$

where D^- denotes the left-derivative.

LEMMA 5.1. *For each $x \in D(B)$ the function $S(\cdot)x$ belongs to $C^1([0, \infty), H)$, and for each $t \geq 0$, $S(t)x \in D(B)$ and*

$$S(t)x - x = \int_0^t BS(s)x\, ds = \int_0^t S(s)\, Bx\, ds \ .$$

COROLLARY 5.1. *B is closed.*

PROOF. If $x_n \in D(B)$, $x_n \to x$ and $Bx_n \to y$ in H, then

$$h^{-1}\big(S(h) - I\big)x_n = h^{-1}\int_0^h S(s)\,Bx_n\,ds\ , \qquad h > 0\ ,\ n \geq 1\ ,$$

so we let $n \to \infty$ and then $h \to 0$ to obtain $D^+S(0)x = y$. □

COROLLARY 5.2. *If $u_0 \in D(B)$ and $u(t) \equiv S(t)u_0$, $t \geq 0$, then $u \in C^1([0,\infty), H)$ satisfies $u'(t) = Bu(t)$, $t \geq 0$, and $u(0) = u_0$.*

LEMMA 5.2. *$D(B)$ is dense in H, $\int_0^t S(s)x\,ds \in D(B)$ and*

$$S(t)x - x = B\int_0^t S(s)x\,ds\ , \qquad t \geq 0\ ,\ x \in H\ .$$

PROOF. For each $t \geq 0$ we set $x_t = \int_0^t S(s)x\,ds$. Then if $h > 0$ we have

$$\begin{aligned} S(h)x_t - x_t &= \int_0^t \big(S(h+s) - S(s)\big)x\,ds \\ &= \int_h^{t+h} S(s)x\,ds - \int_0^t S(s)x\,ds \pm \int_t^h S(s)x\,ds \\ &= \int_t^{t+h} S(s)x\,ds - \int_0^h S(s)x\,ds\ . \end{aligned}$$

Dividing by $h > 0$ and taking the limit shows $x_t \in D(B)$ and $Bx_t = S(t)x - x$ as desired. Finally, note that $x_t/t \in D(B)$ and $x_t/t \to x$ as $t \to 0$, so $D(B)$ is dense.□

Fix $\lambda > 0$ and note that $\{e^{-\lambda t}S(t) : t \geq 0\}$ is a linear contraction semigroup whose generator is $B - \lambda$. We have let λ denote the operator λI. From Lemma 5.1 and Lemma 5.2 we obtain, respectively,

$$\begin{aligned} e^{-\lambda t}S(t)x - x &= \int_0^t e^{-\lambda s}S(s)(B-\lambda)x\,ds\ , \qquad x \in D(B)\ ,\ t \geq 0\ , \\ e^{-\lambda t}S(t)y - y &= (B-\lambda)\int_0^t e^{-\lambda s}S(s)y\,ds\ , \qquad y \in H\ ,\ t \geq 0\ . \end{aligned}$$

Take the limit as $t \to \infty$ in these to get

$$\begin{aligned} x &= \int_0^\infty e^{-\lambda s}S(s)(\lambda - B)x\,ds\ , \qquad x \in D(B) \\ y &= (\lambda - B)\int_0^\infty e^{-\lambda s}S(s)y\,ds\ , \qquad y \in H\ . \end{aligned}$$

The first shows $\lambda - B$ is one-to-one and the second that it is onto H, so it follows that

$$\|(\lambda - B)^{-1}y\| \leq \int_0^\infty e^{-\lambda s}\,ds\|y\| = \lambda^{-1}\|y\|\ .$$

This completes the proof of the first half of our main result.

THEOREM 5.1 (HILLE-YOSIDA). *A necessary and sufficient condition for B to be the generator of a linear contraction semigroup is that $D(B)$ is dense and $\lambda(\lambda - B)^{-1}$ is a contraction on H for every $\lambda > 0$.*

PROOF (CONTINUED). The sufficiency will be established in three steps. First we approximate B by $B_\lambda \in \mathcal{L}(H)$, then construct the semigroup $\{\exp(tB_\lambda) : t \geq 0\}$ for $\lambda > 0$, and finally show the limit $\lim_{\lambda\to\infty} \exp(tB_\lambda)$ is the desired linear contraction semigroup.

LEMMA 5.3. *Define $B_\lambda = \lambda B(\lambda - B)^{-1}$, $\lambda > 0$. Then $B_\lambda = -\lambda + \lambda^2(\lambda - B)^{-1} \in \mathcal{L}(H)$ and for each $x \in D(B)$, $\|B_\lambda x\| \leq \|Bx\|$, $\lim_{\lambda\to\infty} B_\lambda x = Bx$.*

PROOF. For $x \in D(B)$

$$(B_\lambda + \lambda)(\lambda - B)x = \lambda^2 x \quad \text{and } \lambda - B \text{ is onto } H \ , \quad \text{so}$$
$$B_\lambda = -\lambda + \lambda^2(\lambda - B)^{-1} \ .$$

From this follows $(\lambda - B)B_\lambda = \lambda B$ and $B_\lambda = \lambda(\lambda - B)^{-1}B$. Since $\lambda(\lambda - B)^{-1}$ is a contraction, the desired estimate follows. Also, for $x \in D(B)$, $\|\lambda(\lambda - B)^{-1}x - x\| = \frac{1}{\lambda}\|B_\lambda x\| \to 0$, so $\lambda(\lambda - B)^{-1}x \to x$ for each $x \in D(B)$, hence, for each $x \in H$. Thus $B_\lambda \to Bx$ for each $x \in D(B)$. □

LEMMA 5.4. *Define $S_\lambda(t) = \exp(tB_\lambda) \in \mathcal{L}(H)$, $t \geq 0$, $\lambda > 0$. Then $\{S_\lambda(t) : t \geq 0\}$ is a linear contraction semigroup with generator B_λ for each $\lambda > 0$. For $x \in H$, $S_\lambda(t)x$ converges in H as $\lambda \to \infty$, uniformly on each interval $[0, T]$, $T > 0$.*

PROOF. From $S_\lambda(t) = \exp(-\lambda t)\exp(\lambda^2(\lambda - B)^{-1}t)$ we obtain

$$\|S_\lambda(t)\| \leq \exp(-\lambda t)\exp(\lambda t) = 1$$

for $t \geq 0, \lambda > 0$. The remaining semigroup properties are obtained from calculus of power series in $\mathcal{L}(H)$. Similarly we obtain

$$S_\lambda(t) - S_\mu(t) = \int_0^t \frac{d}{ds}\big(S_\mu(t-s)S_\lambda(s)\big)\,ds = \int_0^t S_\mu(t-s)S_\lambda(s)(B_\lambda - B_\mu)\,ds$$

by calculus in $\mathcal{L}(H)$, and thus the estimate

$$\|S_\lambda(t)x - S_\mu(t)x\| \leq t\|B_\lambda(x) - B_\mu(x)\| \ , \qquad x \in D(B) \ .$$

With Lemma 5.3 this shows that $\{S_\lambda(t)x\}$ is uniformly Cauchy in H for $0 \leq t \leq T$. Each $S_\lambda(t)$ is a contraction so the limit exists for each $x \in H$, uniformly on bounded intervals. □

To finish the proof of Theorem 5.1, we define $S(t)x = \lim_{\lambda\to\infty} S_\lambda(t)x$. It follows that $\{S(t) : t \geq 0\}$ is a linear contraction semigroup. For $x \in D(B)$, $S_\lambda(\cdot)B_\lambda x \to S(\cdot)Bx$ uniformly on bounded intervals, so from

$$S_\lambda(t)x - x = \int_0^t S_\lambda(s)B_\lambda x\,ds$$

we obtain in the limit

$$S(t)x - x = \int_0^t S(s)Bx\,ds \ , \qquad x \in D(B) \ , \ t \geq 0 \ .$$

This shows $D^+S(0)x = Bx$, $x \in D(B)$, so B is a (possible) restriction of the generator of $\{S(t) : t \geq 0\}$. But $I - B$ is onto H, so B equals the generator. $\square$

From Proposition 4.2 we obtain the following.

COROLLARY 5.3. *A necessary and sufficient condition for $-A : D(A) \to H$ to be the generator of a linear contraction semigroup is that A be m-accretive.*

We consider two fundamental examples.

EXAMPLE 5.A. Let $H = L^2(0,1)$ and $A = \partial$ on $D(A) = \{u \in H^1(0,1) : u(0) = cu(1)\}$ with $|c| \leq 1$. We showed in Section 4 that A is m-accretive, so by Theorem 5.1 we see the initial-boundary-value problem

$$\partial_t u(x,t) + \partial_x u(x,t) = 0 \ , \qquad 0 < x < 1 \ , \ t \geq 0 \ , \tag{5.2.a}$$

$$u(0,t) = cu(1,t) \tag{5.2.b}$$

$$u(x,0) = u_0(x) \tag{5.2.c}$$

has a unique solution for each $u_0 \in D(A)$. An explicit representation for this solution can be easily found. Since any solution of (5.2.a) is of the form $u(x,t) = F(x-t)$, it follows that

$$u(x,t) = u_0(x-t) \ , \qquad 0 \leq t \leq x \leq 1 \ ,$$

and then (5.2.b) implies

$$u(x,t) = cu_0(1+x-t) \ , \qquad x \leq t \leq x+1 \ .$$

By an easy induction we obtain

$$u(x,t) = c^n u_0(n+x-t) \ , \qquad n-1+x \leq t \leq n+x \ , \ n \geq 1 \ .$$

This representation of the solution gives some additional information. First, the Cauchy problem can be solved only if $u_0 \in D(A)$, because $u(\cdot,t) \in D(A)$ implies $u(\cdot,t)$ is (absolutely) continuous and this is possible only if u_0 satisfies the boundary condition (5.2.b). Second, the solution satisfies $u(\cdot,t) \in H^1(0,1)$ for every $t \geq 0$ but will *not* belong to $H^2(0,1)$ unless $\partial u_0 \in D(A)$. That is, we do not in general have $u(\cdot,t) \in H^2(0,1)$, no matter how smooth the initial function u_0 may be. Finally, the representation above defines a solution of (5.2) on $-\infty < t < \infty$ by allowing n to be *any* integer. Thus, the problem can be solved backwards in time as well as forward. This is related to the fact that $-A$ generates a *group* of operators.

EXAMPLE 5.B. For our second example we take $H = L^2(\mathbb{R})$, $V = H^1(\mathbb{R})$ and define $a(u,v) = (\partial u, \partial v)_H$ for $u, v \in V$. Corollary 4.1 shows that the operator

$$A = -\partial^2 \ , D(A) = \{u \in V : \partial^2 u \in L^2(\mathbb{R})\}$$

is m-accretive. Thus by Theorem 5.1 we obtain existence and uniqueness for the initial-value problem

$$\partial_t u(x,t) = \partial_x^2 u(x,t) \ , \qquad -\infty < x < \infty \ , \ t > 0 \ , \tag{5.3.a}$$

$$u(x,0) = u_0(x) \ , \tag{5.3.b}$$

with $u_0 \in D(A)$. We shall obtain a useful representation for the solution. First note the only bounded solutions of the form $\exp(\alpha x + \beta t)$ are $\exp(\pm i\mu x - \mu^2 t)$, $\mu \in \mathbb{R}$. Taking real parts and then integrating all these together lead to

$$\widetilde{u}(x,t) = \int_0^\infty e^{-\mu^2 t} \cos(\mu x)\, d\mu = \frac{1}{\sqrt{t}} \int_0^\infty e^{-\lambda^2} \cos\left(\frac{x\lambda}{\sqrt{t}}\right) d\lambda$$

and this is a solution for $t > 0$. To evaluate this integral, set

$$K(s) = \int_0^\infty e^{-\lambda^2} \cos(s\lambda)\, d\lambda$$

and check that $K'(s) = -(s/2)K(s)$, hence, $K(s) = M\exp(-s^2/4)$. Finally, $K(0) = M = \sqrt{\pi}/2$, so we have shown

$$\widetilde{u}(x,t) = (\pi/4t)^{1/2} \exp(-x^2/4t) \ .$$

By a re-scaling suggested by the following result, we are led to the special function

$$K(x,t) = (4\pi t)^{-1/2} \exp(-x^2/4t) \ , \qquad t > 0 \ .$$

Lemma 5.5.

(i) *K is $C^\infty(\mathbb{R} \times \mathbb{R}_+)$ and each $\partial_x^m \partial_t^n K$ converges to zero exponentially as $|x| \to \infty$.*

(ii) *$\partial_t K = \partial_x^2 K$.*

(iii) *$K > 0$, $\int_{\mathbb{R}} K(x,t)\, dx = 1$, $t > 0$.*

(iv) *For $\delta > 0$, $\lim_{t\to 0} K(x,t) = 0$ uniformly for $|x| \geq \delta$.*

(v) *For $\delta > 0$, $\lim_{t\to 0} \int_{|x|\geq\delta} K(x,t)\, dx = 0$.*

Proof.

(i) This follows from l'Hopital's rule.

(ii) Differentiate and check.

(iii) By a change of variable this integral is

$$(4\pi t)^{-1/2} \int_{\mathbb{R}} \exp(-u^2)(4t)^{1/2}\, du$$

so the result follows.

(iv) For $|x| \geq \delta$ we have $K(x,t) \leq (4\pi t)^{-1/2} \exp(-\delta^2/4t)$.

(v) $\displaystyle\int_{|x|\geq\delta} K(x,t)\, dx = \pi^{-1/2} \int_{|u|\geq\delta/(4t)} e^{-u^2}\, du \to 0 \quad \text{as } t \to 0 \ .$ □

Proposition 5.2. *Let u_0 be bounded and continuous on $\mathbb{R}$. Define*

$$u(x,t) = \begin{cases} \int_{\mathbb{R}} K(x-\xi,t) u_0(\xi)\, d\xi \ , & t > 0 \ , \\ u_0(x) \ , & t = 0 \ . \end{cases}$$

Then u is bounded and continuous on $\{(x,t) : t \geq 0\}$, it is infinitely differentiable on $\{(x,t) : t > 0\}$, and it satisfies (5.3).

PROOF. The function u is well-defined and C^∞ by (i), a solution of the heat equation by (ii), and from (iii) we obtain $\sup_{x,t}(u) \le \sup_x u_0$, so u is bounded. It remains only to check the continuity on $\mathbb{R} \times [0, \infty)$.

Let $x_0 \in \mathbb{R}$ and $\varepsilon > 0$; choose $A > |x_0|$. Since u_0 is uniformly continuous on $[-A, A]$ there is a $\delta > 0$ such that $(x_0 - 2\delta, x_0 + 2\delta) \subset [-A, A]$ and $x, y \in [-A, A]$, $|x-y| < 2\delta$ imply $|u_0(x) - u_0(y)| < \varepsilon/2$. By a change of variable and (iii) we obtain

$$\begin{aligned} u(x,t) - u_0(x_0) = &\int_{|s|\ge\delta} K(s,t)\big(u_0(x+s) - u_0(x_0)\big)\, ds \\ &+ \int_{|s|\le\delta} K(s,t)\big(u_0(x+s) - u_0(x_0)\big)\, ds \ . \end{aligned}$$

If $|x - x_0| < \delta$ then for $|s| \le \delta$ we have

$$x + s \in [-A, A] \ , \qquad |x + s - x_0| < 2\delta$$

so $|u_0(x+s) - u_0(x_0)| < \varepsilon/2$. Thus the second term above is bounded by $\varepsilon/2$ by (iii). Thus we have

$$|u(x,t) - u_0(x_0)| \le \int_{|s|\ge\delta} K(s,t)\, ds \cdot 2 \sup_x |u_0| + \varepsilon/2$$

and the result follows from (v). □

Example 5.B illustrates the *regularizing effects* that occur with evolutions governed by regular m-accretive operators. Consider the case of such an operator A which arises from a triple $\{a(\cdot,\cdot), V, H\}$ for which $a(\cdot,\cdot)$ is symmetric. Thus A is *self-adjoint*: $A = A^*$. Let $\{S(t) : t \ge 0\}$ be the semigroup generated by $-A$, $u_0 \in D(A)$, and $u(t) = S(t)u_0$. Thus, (5.1) holds and $u(0) = u_0$. We seek estimates which imply "regularity" of the solution $u(t)$. First, from $(Au(t), u(t))_H = -\frac{1}{2}\frac{d}{dt}\|(u(t)\|_H^2$ we obtain

$$\int_0^T a(u(t), u(t))\, dt = \frac{1}{2}(\|u_0\|_H^2 - \|u(T)\|_H^2) \ . \tag{5.4}$$

For each $h > 0$ and $t > \tau > 0$, $u(t+h) - u(t) = S(t-\tau)(u(\tau+h) - u(\tau))$, so $S(t-\tau)$ being a contraction shows $\|u'(t)\|_H \le \|u'(\tau)\|$, hence, $\|u'(\cdot)\|_H$ is non-increasing. We have

$$t\|u'(t)\|_H^2 = -\frac{t}{2}\frac{d}{dt}a(u(t), u(t)) = -\frac{d}{dt}\left(\frac{t}{2}a(u(t), u(t))\right) + \frac{1}{2}a(u(t), u(t))$$

since $a(\cdot,\cdot)$ is symmetric, and this yields

$$\int_0^T t\|u'(t)\|_H^2\, dt + \frac{T}{2}a(u(T), u(T)) = \frac{1}{2}\int_0^T a(u(t), u(t))\, dt \ . \tag{5.5}$$

Using the non-increase of $\|u'(\cdot)\|_H$ and (5.4), we obtain

$$\|u'(T)\|_H^2 \frac{T^2}{2} \le \frac{1}{4}\|u_0\|_H^2 \ ,$$

and this leads to the following *parabolic regularizing property* of the LCS.

THEOREM 5.2. *If A is regular m-accretive and self-adjoint, the generated LCS satisfies the following:*

$$S(t) \text{ maps } H \text{ into } D(A) \ , \text{ and } \|tAS(t)\|_{\mathcal{L}(H)} \leq \frac{1}{\sqrt{2}} \ , \qquad t > 0 \ .$$

PROOF. Let $w \in H$ and $w_n \in D(A)$ for $n \geq 1$ with $w_n \to w$. We have $S(t)w_n \to S(t)w$ and $\|AS(t)(w_m - w_n)\|_H \leq \|w_m - w_n\|_H/\sqrt{2}t$, so $\{AS(t)w_n\}$ converges in H. But A is closed, and so the desired result follows. □

In the proof of Lemma 5.1 we found that A commutes with $S(t)$ on $D(A)$, $AS(t)x = S(t)Ax \in D(A)$ for $x \in D(A)$, so it follows that $S(t)$ maps $D(A)$ into $D(A^2)$ for $t > 0$. Thus $S(t) = S(t/2)S(t/2)$ maps H into $D(A^2)$ with

$$\|A^2S(t)\|_{\mathcal{L}(H)} = \|AS(t/2)AS(t/2)\|_{\mathcal{L}(H)} \leq \left(\frac{2}{t\sqrt{2}}\right)^2 \ .$$

By induction we obtain the following.

COROLLARY 5.4. *For every $t > 0$ and integer $p \geq 1$*

$$S(t) \text{ maps } H \text{ into } D(A^p) \ , \text{ and } \|A^pS(t)\|_{\mathcal{L}(H)} \leq \left(\frac{p}{t\sqrt{2}}\right)^p \ .$$

This gives the *spatial regularity* of the solution $u(t)$ of the Cauchy problem.

COROLLARY 5.5. *For every $u_0 \in H$ there is a unique solution $u \in C([0,\infty), H) \cap C^\infty((0,\infty), H)$ of (5.1) with $u(0) = u_0$, and it satisfies $u(t) \in D(A^p)$ for every $t > 0$ and $p \geq 1$.*

For any m-accretive operator, $A + I$ is a bijection of $D(A)$ onto H, and if we define a norm on $D(A)$ by

$$\|x\|_D \equiv (\|x\|_H^2 + \|Ax\|_H^2)^{1/2} \ , \qquad x \in D(A) \ ,$$

it follows that $D(A)$ is a Hilbert space isomorphic to H. Similarly, $D(A^p)$ is a Hilbert space with scalar-product

$$(x, y)_{D^p} \equiv (x, y)_H + (A^px, A^py)_H \ , \qquad x, y \in D(A^p) \ ,$$

and $(A + I)^p$ is an isomorphism of $D(A^p)$ onto H. For the special case of a self-adjoint regular accretive operator as above we can deduce from the identity

$$A^p\frac{u(t+h) - u(t)}{h} = \frac{S(t-\varepsilon+h) - S(t-\varepsilon)}{h}A^pS(\varepsilon)u_0 \ , \qquad 0 < \varepsilon < t \ , \ h > 0 \ ,$$

that $\lim_{h\to 0^+}\left(\frac{u(t+h)-u(t)}{h}\right) = u'(t)$ in the space $D(A^p)$. When A is a differential operator this shows $u'(t)$ agrees with the partial derivative with respect to time and a corresponding *temporal regularity* of the solution of (5.1).

I.6. Wave Equations

Consider again the vertical displacement within a fixed plane of a string of length $\ell > 0$. As before, let $T > 0$ be the tension applied to stretch the string. Here we shall study the time-dependent displacement $u(x,t)$ for $t > 0$, $0 < x < \ell$, and describe the vibrations of the string resulting from an initial displacement $u(x,0) = u_0(x)$ and an initial velocity $u_t(x,0) = u_1(x)$ as well as specific boundary conditions at the endpoints $x = 0, \ell$.

To derive the equation for these transverse vibrations of the string, recall from Section 3 that for a section of the string, $x_1 < x < x_2$, during the time interval, $t_1 < t < t_2$, the total impulse is given by

$$\int_{t_1}^{t_2} T\big(u_x(x_2,s) - u_x(x_1,s)\big)\, ds \ .$$

If the density of the string at x is given by $\rho > 0$, the change in momentum of this section is just

$$\int_{x_1}^{x_2} \rho\big(u_t(x,t_2) - u_t(x,t_1)\big)\, dx$$

and so this equals the impulse by Newton's second law. For a sufficiently smooth displacement $u(x,t)$, the equality of these integrals for each such $x_1 < x_2$ and $t_1 < t_2$ shows that the displacement satisfies the *wave equation*

$$\frac{\partial^2}{\partial t^2}\big(\rho u(x,t)\big) - T\frac{\partial^2}{\partial x^2}u(x,t) = 0 \ , \qquad 0 < x < \ell,\ t > 0 \ . \tag{6.1}$$

Notice that the total energy of the string at each time t is given by the sum

$$\frac{1}{2}\int_0^\ell \rho u_t^2(x,t)\, dx + \frac{1}{2}\int_0^\ell T u_x^2(x,t)\, dx$$

of kinetic and potential energy which are determined, respectively, by the velocity u_t and vertical force Tu_x. We shall show that various mixed initial-boundary-value problems for (6.1) are well-posed by reducing this second-order evolution equation to an equivalent first-order system to which Theorem 5.1 directly applies. This abstract development will be followed by a specific example for which we can display the solution explicitly and compare its properties with those guaranteed by the theory.

We shall regard (6.1) as a special case of the evolution equation

$$u''(t) + Bu'(t) + Au(t) = 0 \ , \qquad 0 < t \ , \tag{6.2}$$

where A is an unbounded operator with domain D in a Hilbert space H and B is another operator. To obtain (6.1) from (6.2) we shall choose A to correspond to an appropriate Dirichlet or Neumann problem as in Corollary 4.1; thus, A will be regular m-accretive. The operator B will be continuous and linear from V to H and accretive on H. Such an operator arises naturally from models of friction in the wave equation and it also is convenient to include it in our discussion for the following reason. The function u is a solution of (6.2) if and only if the function w given by $w(t) = e^{-\lambda t}u(t)$, $t > 0$, satisfies

$$w''(t) + (2\lambda I + B)w'(t) + (\lambda^2 I + \lambda B + A)w(t) = 0 \ , \qquad t > 0 \ . \tag{6.3}$$

Thus, (6.2) and (6.3) are equivalent, and this observation permits us to relax the hypothesis on A and B. Specifically, it suffices to require that the operators $\lambda_0^2 I + \lambda_0 B + A$ and $2\lambda_0 I + B$ satisfy certain conditions for some λ_0.

As in Section 4, let V be a Hilbert space which is dense and continuously imbedded in another Hilbert space H. Let $a(\cdot,\cdot)$ be a continuous, symmetric, bilinear and elliptic form on V, and let $A : D \to H$ be the unbounded operator on H constructed in Proposition 4.1; that is, A is the regular m-accretive operator constructed from $a(\cdot,\cdot)$, V and H with

$$a(u,v) = (Au,v)_H \ , \qquad u \in D \ , \ v \in V \ . \tag{6.4}$$

Note that $a(\cdot,\cdot)$ is a scalar-product on V whose norm is equivalent to the V-norm; hereafter we take $a(\cdot,\cdot)$ as the scalar-product on V. Let $B : V \to H$ be continuous and linear, and suppose B is accretive in H:

$$(Bu,u)_H \geq 0 \ , \qquad u \in V \ .$$

Consider the product space $\mathcal{H} = V \times H$ with the scalar-product

$$(\vec{u},\vec{v})_{\mathcal{H}} = a(u_1,v_1) + (u_2,v_2)_H \ , \qquad \vec{u},\vec{v} \in \mathcal{H} \ .$$

This is a Hilbert space, and we define on it the operator $\mathbb{A} : \mathcal{D} \to \mathcal{H}$ given by

$$\mathbb{A}\vec{u} = [-u_2, Au_1 + Bu_2] \ , \qquad \vec{u} \in \mathcal{D} \ ,$$

with domain $\mathcal{D} = D \times V$. This is the operator that arises when we write (6.2) as a first-order system

$$\begin{aligned} u'(t) \qquad\qquad - \quad v(t) &= 0 \\ v'(t) + Au(t) + Bv(t) &= 0 \ , \end{aligned}$$

for then the function $\vec{u}(t) = [u(t), v(t)]$, $t > 0$, is a solution of the equation

$$\vec{u}'(t) + \mathbb{A}\vec{u}(t) = \vec{0} \ , \qquad t > 0 \ , \tag{6.5}$$

in the space $\mathcal{H}$. Conversely, the first component of a solution of (6.5) satisfies the second-order (6.2).

Our plan is to show the Cauchy problem for (6.5) is well-posed; by Theorem 5.1 it is sufficient to show that $\mathbb{A}$ is m-accretive on $\mathcal{H}$. First note that for $\vec{u} \in \mathcal{D}$

$$\begin{aligned} (\mathbb{A}\vec{u},\vec{u})_{\mathcal{H}} &= -a(u_2,u_1) + (Au_1 + Bu_2, u_2)_H \\ &= (Bu_2,u_2)_H \ , \end{aligned}$$

so $\mathbb{A}$ is accretive. Note the critical dependence of this calculation on the choice of the scalar-product on V and $\mathcal{H}$, and that $\mathbb{A}$ is accretive exactly when B is accretive. To show that the range of $\lambda I + \mathbb{A}$ is $\mathcal{H}$, we let $\vec{f} = [f_1, f_2] \in \mathcal{H}$ be given and seek a $\vec{u} = [u_1, u_2] \in \mathcal{D}$ such that $\lambda\vec{u} + \mathbb{A}\vec{u} = \vec{f}$, i.e.,

$$u_1 \in D \ , \quad u_2 \in V : \lambda u_1 - u_2 = f_1 \ , \quad \lambda u_2 + Au_1 + Bu_2 = f_2 \ .$$

This system is equivalent to

$$\begin{aligned} u_1 \in D : (\lambda^2 I + \lambda B + A)u_1 &= (\lambda + B)f_1 + f_2 \ , \\ u_2 &= \lambda u_1 - f_1 \ , \end{aligned}$$

so it suffices to show that the range of $\lambda^2 I + \lambda B + A$ is all of H. But the bilinear form $\widetilde{a}(\cdot,\cdot)$ defined on V by

$$\widetilde{a}(u,v) = \lambda^2 (u,v)_H + \lambda (Bu,v)_H + a(u,v) \ , \qquad u,v \in V$$

is continuous and coercive, so the desired result follows from Theorem 2.2. That is, for each $h \in H$ there is a unique

$$u \in V : \widetilde{a}(u,v) = (h,v)_H \ , \qquad v \in V \ ,$$

and this is equivalent to

$$u \in D : (\lambda^2 I + B + A)u = h \ .$$

Thus, $\mathbb{A}$ is m-accretive on $\mathcal{H}$ and this leads to the following result.

PROPOSITION 6.1. *Let V and H be Hilbert spaces with V dense and continuously imbedded in H. Let $a(\cdot,\cdot)$ be a continuous, bilinear and symmetric form on V and let $B \in \mathcal{L}(V,H)$ for which either B is symmetric and there are numbers λ_0, $c > 0$ such that*

$$\begin{aligned} a(u,u) + \lambda_0 (Bv,v)_H + \lambda_0^2 \|v\|_H^2 \geq c\|v\|_V^2 \ , &\qquad v \in V \ , \\ (Bv,v)_H + 2\lambda_0 \|v\|_H^2 \geq 0 \ , &\qquad v \in V \ , \end{aligned}$$

or else the above hold with $\lambda_0 = 0$. Let $A : D \to H$ be the operator constructed from $a(\cdot,\cdot)$, V and H according to (6.4). *Then for each pair $u_0 \in D$, $u_1 \in V$ there is a unique solution $u \in C^1\big([0,\infty),V\big) \cap C^2\big([0,\infty),H\big)$ of*

$$u''(t) + Bu'(t) + Au(t) = 0 \ , \qquad t \geq 0 \tag{6.6.a}$$

$$u(0) = u_0 \ , \qquad u'(0) = u_1 \ . \tag{6.6.b}$$

PROOF (CONTINUED). By our remarks leading to (6.3), if B is symmetric it suffices to let $\lambda_0 = 0$ in our hypotheses. Since $\mathcal{A}$ is m-accretive the Cauchy problem for (6.5) with $\vec{u}(0) = [u_0, u_1]$ has a unique solution $\vec{u} \in C^1\big([0,\infty),\mathcal{H}\big)$. Then the first component is the corresponding solution of (6.6). □

Example. We shall apply Proposition 6.1 to resolve the initial-boundary-value problem for the wave equation with null displacement at the endpoints. After rescaling the coefficients and interval, this problem has the form

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) = 0 \ , \qquad 0 < x < 1 \ , \ t > 0 \ , \tag{6.7.a}$$

$$u(0,t) = u(1,t) \ , \qquad t > 0 \ , \tag{6.7.b}$$

$$u(x,0) = u_0(x) \ , \quad \partial_t u(x,0) = u_1(x) \ , \qquad 0 < x < 1 \ . \tag{6.7.c}$$

Set $H = L^2(0,1)$, $V = H_0^1(0,1)$, and define $a(u,u) = \int_0^1 \partial u(x) \partial v(x)\, dx$ on V, so the corresponding regular m-accretive operator is given by $A = -\partial^2$ on $D(A) = H_0^1(0,1) \cap H^2(0,1)$. According to Proposition 6.1, there is a unique solution $u \in C^1([0,\infty), H_0^1(0,1)) \cap C^2([0,\infty), L^2(0,1))$ if the initial data satisfies $u_0 \in D(A)$, $u_1 \in V$.

For the special case of (6.7) we can find an explicit representation for the solution. Extend the functions u_0 and u_1 from the interval $(0,1)$ to all of $\mathbb{R}$ so they

are *odd* with respect to both of the endpoints, $x = 0$, $x = 1$. These extensions are 2-periodic, and the above requirements on the initial data in Proposition 6.1 are sufficient to make the extensions of u_0, ∂u_0 and u_1 all continuous on $\mathbb{R}$. Using these extensions, we define

$$u(x,t) = \frac{1}{2}(u_0(x+t) + u_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} u_1(s)\,ds\ .$$

Due to the indicated continuity conditions, among others, it follows that u is the solution to (6.7) given by Proposition 6.1. This representation shows that the conditions imposed on the initial data are appropriate. Since the problem is *reversible*, that is, it is well-posed for all $t \in \mathbb{R}$, it follows that these conditions are actually *necessary* for a solution as given above.