CHAPTER IV

Accretive Operators and Nonlinear Cauchy Problems

IV.1. Accretive Operators in Hilbert Space

We briefly review various special results on real-valued, convex, lower-semi-continuous functionals on a Hilbert space. All of these have been obtained in a more general setting in II.7, but we collect them here for convenience and give usually substantially more simple proofs in Hilbert space to make the presentation independent. The subgradient motivates the extension of the notion of m-accretive operators as given in I.4, and in the remainder of this section we develop the properties of these multi-valued operators.

Let H be a Hilbert space. A sequence $\{x_n\}$ in H is *weakly convergent* to $x \in H$ if $\lim_{n\to\infty}(x_n, y)_H = (x, y)_H$ for every $y \in H$. In view of the isomorphism between H and its dual, H', this definition is consistent with weak convergence in a general Banach space, and we shall denote the weak convergence by "$x_n \rightharpoonup x$". The important property of weak sequential compactness for reflexive Banach spaces is particularly simple to verify for Hilbert space.

PROPOSITION 1.1. *If $\{x_n\}$ is a bounded sequence in the Hilbert space H, then it has a weakly convergent subsequence.*

PROOF. It suffices to consider the case in which H is *separable*, i.e., it has a countable dense subset $\{v_n\}$. (This is the case if H is the closed linear span of the sequence.) Then the general case is easily obtained from the projection Corollary I.2.2. Pick a subsequence $\{x_{1,n}\}$ of $\{x_n\}$ for which $\{(x_{1,n}, v_1)_H\}$ converges. Inductively, pick a subsequence $\{x_{j,n}\}$ of $\{x_{j-1,n}\}$ such that $\{(x_{j,n}, v_k)_H\}$ converges for $k = 1, 2, \cdots, j$. Thus, $\{x_{n,n}\}$ is a subsequence of $\{x_n\}$ for which $\lim_{n\to\infty}(x_{n,n}, v_j)_H$ exists for all $j \geq 1$. Now define $f(y) = \lim_{n\to\infty}(x_{n,n}, y)_H$ for all y in $\langle v_j \rangle$, the linear span of $\{v_j\}$. Then f is linear and continuous, so it has a unique extension to all of H, again denoted by $f \in H'$. From Corollary I.3 it follows there is an $x \in H$ such that $\lim_{n\to\infty}(x_{n,n}, y)_H = f(y) = (x, y)_H$ for $y \in \langle v_j \rangle$, and from the boundedness of $\{x_{n,n}\}$ it follows that $x_{n,n} \rightharpoonup x$ as desired. $\square$

Since strong convergence of a sequence implies weak convergence, every weakly-closed set is (strongly) closed. (Note that sequences are adequate to describe the (strong) closure of a set.) The following Proposition shows that the weak and strong closure are identical for convex sets.

PROPOSITION 1.2. *If K is closed and convex in H then K is weakly closed.*

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PROOF. Let x_0 be a point not in K; we contruct a weak-neighborhood separating them. Define $x = P_K(x_0)$, the indicated projection; we may assume $(x + x_0)/2 = 0$. Then $(x - x_0, y - x)_H \geq 0$ for $y \in K$, so we have $(x, y)_H \geq \|x\|^2$ for $y \in K$, and $(x, x_0)_H < 0$. That is, the linear functional $\mathcal{R}_H x \in H'$ separates K and x_0. $\qquad\square$

COROLLARY 1.1. *A convex set is weakly closed if and only if it contains all weak limits of sequences in K.*

Let the function $\varphi : H \to (-\infty, +\infty] \equiv \mathbb{R}_\infty$ be *convex*, i.e.,

$$\varphi(tx + (1 - t)y) \leq t\varphi(x) + (1 - t)\varphi(y) , \qquad x, y \in H , \ 0 \leq t \leq 1 .$$

Then its *domain* $\mathrm{dom}(\varphi) = \{x \in H : \varphi(x) < \infty\}$ is convex in H and each sublevel set $E_c = \{x \in H : \varphi(x) \leq c\}$ is convex. The function φ is *lower-semi-continuous* if each such E_c is closed, $c \in \mathbb{R}$. From Corollary 1.1 it follows that this is the same for weak and strongly closed and for weakly sequentially closed. Moreover we have the following criterion.

PROPOSITION 1.3. *The convex function $\varphi : H \to \mathbb{R}_\infty$ is lower-semi-continuous if and only if $\lim_{n\to\infty} x_n = x$ implies $\varphi(x) \leq \liminf_{n\to\infty} \varphi(x_n)$.*

PROOF. The limit condition implies each E_c is closed: $x_n \in E_c$ and $x_n \to x$ implies $\varphi(x) \leq c$, hence, $x \in E_c$. Conversely, suppose $\lim_{n\to\infty} x_n = x$ and that there is an $\varepsilon > 0$ for which

$$\varphi(x) - \varepsilon > b \equiv \liminf_{n\to\infty} \varphi(x_n) .$$

There is a subsequence $\{x_{n_k}\}$ such that $\varphi(x_{n_k}) \leq b + \varepsilon/2$ and $E_{b+\varepsilon/2}$ is closed so $\varphi(x) \leq b + \varepsilon/2$, a contradiction. $\qquad\square$

PROPOSITION 1.4. *Let K be closed convex and non-empty in H, and let $\varphi : H \to (-\infty, +\infty]$ be convex and lower-semi-continuous. If*

$$(1.1) \qquad\qquad \lim_{x\in K, \|x\|\to\infty} \varphi(x) = +\infty ,$$

then there exists a minimum point

$$x_0 \in K : \varphi(x_0) \leq \varphi(y) , \qquad y \in K .$$

PROOF. Let $\{x_n\}$ be a minimizing sequence in K. Then (1.1) shows either φ is $+\infty$ on K, so any $x_0 \in K$ is a solution, or that $\{x_n\}$ is bounded. Proposition 1.1 shows some subsequence is weakly convergent, Proposition 1.2 that the limit x_0 belongs to K, and Proposition 1.3 that x_0 is a minimum point. $\qquad\square$

We recall that the function φ is *proper* if $\mathrm{dom}(\varphi)$ is non-empty: $\varphi(x) < \infty$ for some $x \in H$. Assume that $\varphi : H \to \mathbb{R}_\infty$ is convex, lower-semi-continuous, and proper. This is equivalent to requiring that the *epigraph* $\mathrm{epi}(\varphi) \equiv \{[x, t] \in H \times \mathbb{R} : \varphi(x) \leq t\}$ be, respectively, convex, closed, and non-empty.

LEMMA 1.1. *The function φ is lower-bounded by a continuous affine function.*

PROOF. Take any point $[x_0, r_0] \in H \times \mathbb{R}$ for which $x_0 \in \text{dom}(\varphi)$ and $[x_0, r_0] \notin$ epi(φ) and define $[x_1, r_1]$ to be the projection of $[x_0, r_0]$ on epi(φ) in $H \times \mathbb{R}$. This is characterized by

$$[x_1, r_1] \in \text{epi}(\varphi) : ([x_1, r_1] - [x_0, r_0], [x, t] - [x_1, r_1])_{H \times \mathbb{R}} \geq 0 , \qquad [x, t] \in \text{epi}(\varphi) .$$

That is, we have $\varphi(x_1) \leq r_1$ and

$$(1.2) \qquad (x_1 - x_0, x - x_1)_H + (r_1 - r_0)(t - r_1) \geq 0 \quad \text{if} \quad \varphi(x) \leq t .$$

If we choose $x = x_1$ in this inequality, we find that $r_1 \geq r_0$. If $r_1 = r_0$ then we likewise obtain $(x_1 - x_0, x - x_1)_H \geq 0$ for all $x \in \text{dom}(\varphi)$, hence, $x_1 = x_0$, and this is a contradiction because $[x_0, r_0]$ does not belong to epi(φ). It follows that $r_1 > r_0$. Setting $x = x_1$ and $t = \varphi(x_1)$ in (1.2) shows that $\varphi(x_1) \geq r_1$, hence, $\varphi(x_1) = r_1$. Set $w = (x_0 - x_1)/(r_1 - r_0)$. Since we may take $t = \varphi(x)$ in (1.2) we obtain

$$(1.3) \qquad (w, x - x_1)_H \leq \varphi(x) - \varphi(x_1) , \qquad x \in H ,$$

so the continuous affine function ℓ defined by

$$\ell(x) = (w, x - x_1) + \varphi(x_1)$$

is a lower bound on φ. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Actually the function ℓ is *exact* at x_1, $\ell(x_1) = \varphi(x_1)$, so it is a maximal linear lower bound on the epigraph of φ. These maximal lower bounds correspond to the subgradients of φ.

DEFINITION. Let $\varphi : H \to \mathbb{R}_\infty$ be convex, proper, and lower-semi-continuous. Then $w \in H$ is a *subgradient* of φ at $x_1 \in H$ if

$$(w, x - x_1)_H \leq \varphi(x) - \varphi(x_1) , \qquad x \in H .$$

The set of all subgradients of φ at x_1 is denoted by $\partial_H \varphi(x_1)$.

REMARKS. If we identify H with H' by the Riesz isomorphism, $\mathcal{R}_H$, then the subgradients agree with the subdifferentials (II.7). In general, however, the subgradient and the Riesz isomorphism depend explicitly on the scalar product in H. In particular we have

$$w \in \partial_H \varphi(x_1) \Leftrightarrow \mathcal{R}_H w \in \partial \varphi(x_1) ,$$

and this is denoted by $\mathcal{R}_H \circ \partial_H \varphi = \partial \varphi$.

The subgradient $\partial_H \varphi$ is a multi-valued operator or *relation* on $H \times H$. It is accretive (see I.4) in the following sense: if $w_j \in \partial_H \varphi(x_j)$ for $j = 1, 2$, then $(w_1 - w_2, x_1 - x_2)_H \geq 0$. To see this, note that

$$(w_1, x_2 - x_1)_H \leq \varphi(x_2) - \varphi(x_1) ,$$
$$(w_2, x_1 - x_2)_H \leq \varphi(x_1) - \varphi(x_2) ,$$

and then add these inequalities. Moreover, we have the following.

PROPOSITION 1.5. *If $\varphi : H \to \mathbb{R}_\infty$ is convex, proper, and lower-semi-continuous, then the range of $I + \partial_H \varphi$ is all of H.*

PROOF. Let $w_0 \in H$ and define $\psi : H \to \mathbb{R}_\infty$ by

$$\psi(x) = \varphi(x) + 1/2\|x\|^2 - (w_0, x)_H , \qquad x \in H .$$

Since φ has an affine lower bound, ψ satisfies $\lim_{\|x\| \to \infty} \psi(x) = +\infty$, so by Proposition 1.4 it attains a minimum value at some $x_0 \in H$: that is, $0 \in \partial_H \psi(x_0)$, or

$$(w_0, x - x_0)_H \le \varphi(x) - \varphi(x_0) + 1/2(\|x\|^2 - \|x_0\|^2) , \quad x \in H .$$

Replace x by $tx + (1-t)x_0$, use the convexity of φ, divide by $t > 0$, and let $t \to 0^+$ to obtain

$$(w_0, x - x_0)_H \le \varphi(x) - \varphi(x_0) + (x_0, x - x_0)_H , \qquad x \in H .$$

This shows $w_0 - x_0 \in \partial_H \varphi(x_0)$ as desired. $\qquad \square$

COROLLARY 1.2. *(cf. Proposition I.4.2). For every $\alpha > 0$, $Rg(I + \alpha \partial_H \varphi) = H$.*

The preceding results lead us to extend our notion of m-accretive operators to include possibly nonlinear or multi-valued relations. For any relation A on H, that is, a subset A of $H \times H$, we define the *domain* $D(A) = \{x : [x, y] \in A\}$, *range* $Rg(A) = \{y : [x, y] \in A\}$ and *inverse* $A^{-1} = \{[y, x] : [x, y] \in A\}$. This extends the notion of a function when we identify it with its graph. Conversely, we may regard A as a function into "subsets of H" with the notation $A(x) = \{y : [x, y] \in A\}$; then A is a function exactly when each $A(x)$ is a single point, and $D(A) = \{x : A(x)$ is non-empty$\}$. The linear operations lead to the definitions

$$\lambda A = \{[x, \lambda y] : [x, y] \in A\} \quad \text{for} \quad \lambda \in \mathbb{R} \quad \text{and}$$

$$A + B = \{[x, y + z] : [x, y] \in A \quad \text{and} \quad [x, z] \in B\} .$$

Then $D(\lambda A) = D(A)$ if $\lambda \ne 0$, and $D(A + B) = D(A) \cap D(B)$.

DEFINITION. The relation or operator A on H is *accretive* if $[x_j, w_j] \in A$ for $j = 1, 2$, (i.e., $w_j \in A(x_j)$) implies $(w_1 - w_2, x_1 - x_2)_H \ge 0$. It is *m-accretive* if, in addition, $Rg(I + A) = H$.

Such operators will play a central role in the following where we shall generalize and extend the results of I.5. We begin with some preliminary extensions of results from I.4 which characterize m-accretive operators.

LEMMA 1.2. *$(x, y)_H \ge 0$ if and only if*

$$\|x\| \le \|x + \alpha y\| , \qquad \alpha > 0 .$$

PROOF. Both of these conditions are equivalent to

$$0 \le 2\alpha(x, y)_H + \alpha^2 \|y\|^2 , \qquad \alpha > 0 . \qquad \square$$

COROLLARY 1.3. *The following are equivalent:*
(a) *A is accretive,*
(b) *$\|x_1 - x_2\| \le \|(x_1 + \alpha w_1) - (x_2 + \alpha w_2)\|$ for all $[x_j, w_j] \in A$, $j = 1, 2$, and $\alpha > 0$, and*
(c) *$(I + \alpha A)^{-1}$ is a contraction on $Rg(I + \alpha A)$ for all $\alpha > 0$.*

LEMMA 1.3. *The following are equivalent:*
(a) *A is accretive and $Rg(I + \alpha A) = H$ for some $\alpha > 0$,*
(b) *A is m-accretive, and*
(c) *A is accretive and $Rg(I + \alpha A) = H$ for all $\alpha > 0$.*

PROOF. It suffices to show (a) implies (c). Assume (a) and let $\beta > \alpha/2$. For a given $w \in H$ we define $T : H \to H$ by

$$T(x) = (I + \alpha A)^{-1} \left(\frac{\alpha}{\beta} w + \left(1 - \frac{\alpha}{\beta} \right) x \right) , \qquad x \in H .$$

Since $(I + \alpha A)^{-1}$ is a contraction and

$$\left| 1 - \frac{\alpha}{\beta} \right| = \frac{|\beta - \alpha|}{\beta} < 1 ,$$

it follows that T is a strict-contraction on H. Thus T has a unique fixed point, $x = T(x)$, which then satisfies $x + \beta A(x) \ni w$. This shows $Rg(I + \beta A) = H$ if $\beta > \alpha/2$, and (c) follows by an easy induction. □

Let A be m-accretive and define the corresponding *resolvents* of A by $J_\alpha \equiv (I + \alpha A)^{-1}$ for $\alpha > 0$. Each J_α is a contraction on H. From the equivalence of $y = J_\alpha(x)$ and $\frac{1}{\alpha}(x - y) \in A(y)$, we obtain

(1.4) $$\frac{1}{\alpha}\big(x - J_\alpha(x)\big) \in A\big(J_\alpha(x)\big) , \qquad x \in H , \ \alpha > 0 .$$

We can write (1.4) in the form

$$\frac{1}{\beta} \left(\frac{\beta}{\alpha} x + \left(1 - \frac{\beta}{\alpha} \right) J_\alpha(x) - J_\alpha(x) \right) \in A\big(J_\alpha(x)\big) , \qquad \beta > 0 ,$$

and this is equivalent to

$$\frac{\beta}{\alpha} x + \left(1 - \frac{\beta}{\alpha} \right) J_\alpha(x) \in (I + \beta A)\big(J_\alpha(x)\big) .$$

This proves the *resolvent identity*

(1.5) $$J_\alpha = J_\beta \circ \left(\frac{\beta}{\alpha} I + (1 - \frac{\beta}{\alpha}) J_\alpha \right) , \qquad \alpha, \beta > 0 .$$

PROPOSITION 1.6. *Let A be m-accretive.*
(a) *Then A is maximal accretive: if B is accretive and $A \subset B$, then $A = B$.*
Let $[x_n, y_n] \in A$ and $x_n \rightharpoonup x$, $y_n \rightharpoonup y$ in H.
(b) *If $\liminf_{n\to\infty}(x_n, y_n)_H \leq (x, y)_H$ then $[x, y] \in A$.*
(c) *If $\limsup_{n\to\infty}(x_n, y_n)_H \leq (x, y)_H$, then $\lim_{n\to\infty}(x_n, y_n)_H = (x, y)_H$.*

PROOF.
(a) If $w \in B(x)$, let $z = (I + A)^{-1}(w + x)$. Then $z \in D(A) \subset D(B)$ and $w + x \in (I + B)(z) \cap (I + B)(x)$, so $x = z \in D(A)$ and $w \in A(x)$.
(b) Taking the $\liminf$ of the inequality

$$(v - y_n, u - x_n)_H \geq 0 , \qquad [u, v] \in A , n \geq 1 ,$$

gives
$$(v - y, u - x)_H \geq 0 , \qquad [u, v] \in A .$$

Since A is maximal accretive, it equals $B = A \cup \{[x, y]\}$.

(c) From (b) we have $(y - y_n, x - x_n)_H \geq 0$, $n \geq 1$. Take the lim inf and obtain $\liminf_{n \to \infty}(x_n, y_n)_H \geq (x, y)_H$, so (c) follows immediately. $\qquad \square$

Actually, every maximal accretive operator in Hilbert space is m-accretive. Proposition 1.6.a is the easier half of this deep result of G. Minty on the equivalence of these two notions. The remaining parts of Proposition 1.6 give continuity properties of m-accretive operators. With the identity (1.4) these will lead us next to results on the approximations of I by J_α and of A by m-accretive Lipschitz operators.

PROPOSITION 1.7 (MINTY-ROCKAFELLAR). *If A is m-accretive then $\overline{D(A)}$ is convex   and $\lim_{\alpha \to 0} J_\alpha(x)$ $= \mathrm{Proj}_{\overline{D(A)}}(x)$ for each $x \in H$.*

PROOF. Let K be the closed convex hull of $D(A)$ and $x \in H$. Set $x_\alpha = J_\alpha(x)$ so $\frac{1}{\alpha}(x - x_\alpha) \in A(x_\alpha)$ by (1.4). Since A is accretive
$$\left(\frac{1}{\alpha}(x - x_\alpha) - v, x_\alpha - u \right)_H \geq 0 , \qquad [u, v] \in A ,$$

so we obtain
$$\|x_\alpha\|^2 \leq (x, x_\alpha - u)_H + (x_\alpha, u)_H - \alpha(v, x_\alpha - u)_H , \qquad \alpha > 0 .$$

Thus $\{x_\alpha\}$ is bounded, so some subsequence $\{x_{\alpha'}\}$ converges weakly to a vector $x_0 \in K$. Taking the lim inf in the above shows that
$$\|x_0\|^2 \leq (x, x_0 - u)_H + (x_0, u)_H , \qquad u \in D(A) ,$$

so we have
$$x_0 \in K : (x - x_0, u - x_0)_H \geq 0 , \qquad u \in K ,$$

and this shows $x_0 = \mathrm{Proj}_K(x)$. By the uniqueness of weak limits we have $x_\alpha \rightharpoonup x_0$. From the estimate on $\|x_\alpha\|$ we also obtain
$$\limsup \|x_\alpha\|^2 \leq (x, x_0 - u)_H + (x_0, u)_H , \qquad u \in K ,$$

and setting $u = x_0$ shows $\limsup \|x_\alpha\|^2 \leq \|x_0\|^2$. Hence $\lim_{\alpha \to 0} \|x_\alpha\| = \|x_0\|$ so we have the strong $\lim_{\alpha \to 0} x_\alpha = x_0$ in H. Finally, note that if $x \in K$ then $x_\alpha \in D(A)$ and $x_\alpha \to x$ so it follows $\overline{D(A)} = K$. $\qquad \square$

Consider now the following approximation of the m-accretive operator A by a Lipschitz function. For $\alpha > 0$ given, add αI to the inverse A^{-1} and then invert this sum: $A_\alpha = (A^{-1} + \alpha I)^{-1}$. For operators on $H = \mathbb{R}$, A_α is obtained from A by tilting the graph to obtain a maximum slope of $1/\alpha$. In general, we have the equivalence of
$$\begin{aligned} y &\in (A^{-1} + \alpha I)^{-1}x , & x &\in (\alpha I + A^{-1})y , \\ y &\in A(x - \alpha y) , & x &\in (I + \alpha A)(x - \alpha y) , \end{aligned}$$

and $y = \frac{1}{\alpha}\big(x - J_\alpha(x)\big)$. Compare this with (1.4).

DEFINITION. Let A be m-accretive on H. Then the *Yosida approximations* of A is the operator

$$(1.6) \qquad\qquad A_\alpha \equiv \frac{1}{\alpha}(I - J_\alpha) , \qquad \alpha > 0 .$$

Thus, $A_\alpha(x) \in A(J_\alpha(x))$, $x \in H$, and A_α is characterized by $y = A_\alpha(x)$ if and only if $y \in A(x - \alpha y)$. Since A is maximal accretive, we can show that each $A(x)$ is closed and convex in H. This allows us to define the *minimal section* operator A^0 by

$$A^0 x = \mathrm{Proj}_{A(x)}(0) = \{y : y \in A(x) , y \text{ of minimal norm }\}.$$

THEOREM 1.1. *Let A be m-accretive.*
(a) *Each A_α is m-accretive and Lipschitz with constant $\frac{1}{\alpha}$, $\quad \alpha > 0$.*
(b) *$(A_\alpha)_\beta = A_{\alpha+\beta}$, $\quad \alpha, \beta > 0$.*
(c) *For each $x \in D(A)$, $\|A_\alpha x\|$ converges upward to $\|A^0 x\|$, $\lim_{\alpha \to 0} A_\alpha(x) = A^0 x$, and*

$$\|A_\alpha x - A^0 x\|^2 \le \|A^0 x\|^2 - \|A_\alpha x\|^2 , \qquad \alpha > 0 .$$

(d) *For each $x \notin D(A)$, $\|A_\alpha x\|$ is increasing and unbounded as $\alpha \to 0$.*

PROOF.
(a) From the definition of A_α and (1.4) we obtain

$$\begin{aligned}
(A_\alpha x_1 &- A_\alpha x_2, x_1 - x_1)_H \\
&= \left(A_\alpha x_1 - A_\alpha x_2, \alpha(A_\alpha x_1 - A_\alpha x_2) + (J_\alpha x_1 - J_\alpha x_2)\right)_H \\
&\ge \alpha\|A_\alpha x_1 - A_\alpha x_2\|^2
\end{aligned}$$

since A is accretive. Therefore A_α is accretive and satisfies

$$\|A_\alpha x_1 - A_\alpha x_2\| \le \frac{1}{\alpha}\|x_1 - x_2\| , \qquad x_1, x_2 \in H .$$

From the definition of A_α it follows that x satisfies the resolvent equation $w = (I + \beta A_\alpha)(x)$ if x is a fixed-point of the function $T(x) = w - \beta A_{alpha}(x)$. Such a fixed-point exists if $\beta < \alpha$, for then T is a strict contraction.
(b) This follows from the characterization of $y = A_\alpha x$ by $y \in A(x - \alpha y)$.
(c) If $x \in D(A)$ then

$$0 \le (A^0 x - A_\alpha x, x - J_\alpha x)_H = \alpha(A^0 x - A_\alpha x, A_\alpha x)_H$$

so we have $\|A_\alpha x\|^2 \le (A^0 x, A_\alpha x)_H$ and $\|A_\alpha x\| \le \|A^0 x\|$. Applied to A_β with (b) we obtain $\|A_{\alpha+\beta} x\|^2 \le (A_\beta x, A_{\alpha+\beta} x)_H$ and $\|A_{\alpha+\beta} x\| \le \|A_\beta x\|$, so it follows that $\|A_\alpha(x)\|$ increases as α decreases and satisfies

$$\|A_\alpha x - A_{\alpha+\beta} x\|^2 \le \|A_\alpha x\|^2 - \|A_{\alpha+\beta} x\|^2 , \qquad \alpha, \beta > 0 ,$$

for each $x \in H$. If $\{\|A_\alpha x\|\}$ is bounded then this inequality shows that there is a $y \in H$ for which $\lim_{\alpha \to 0} A_\alpha(x) = y$. Also (1.6) shows $\lim_{\alpha \to 0} J_\alpha(x) = x$, and so from Proposition 6 and (1.4) we obtain $y \in A(x)$. Also $\|y\| = \lim_{\alpha \to 0} \|A_\alpha x\| \le \|A^0 x\|$, so $y = A^0 x$. This also proves (d). $\qquad\square$

Finally we return to the special case of a subgradient, $A = \partial\varphi$, and show the approximations A_α are derivatives likewise.

DEFINITION. Let $f : H \to \mathbb{R}$ be a function and $x \in H$. Then a vector $f'(x) \in H$ is the Frechet *gradient* of f at x if

$$\lim_{y \to x}\left[\left(f(y) - f(x) - \left(f'(x), y - x\right)_H\right)\Big/\|y - x\|\right] = 0$$

If such a $f'(x)$ exists we say f is Frechet *differentiable* at x.

PROPOSITION 1.8 (MOREAU). *Let $\varphi : H \to \mathbb{R}_\infty$ be proper, convex and lower-semi-contin-uous; set $A = \partial\varphi$. Define for each $\alpha > 0$*

$$\varphi_\alpha(x) \equiv \min\left\{\frac{1}{2\alpha}\|y - x\|^2 + \varphi(y) : y \in H\right\} , \qquad x \in H .$$

Then $\varphi_\alpha(x) = \frac{\alpha}{2}\|A_\alpha x\|^2 + \varphi(J_\alpha(x))$, and φ_α is convex and Frechet differentiable with gradient $\varphi_\alpha' = A_\alpha$. Also $\varphi_\alpha(x)$ converges upward to $\varphi(x)$ as $\alpha \downarrow 0$ for each $x \in H$.

PROOF. From the proof of Proposition 1.5 it follows that the minimum in the definition of φ_α is attained at y if and only if $0 \in \frac{1}{\alpha}(y - x) + \partial\varphi(y)$, and this is equivalent to $y = J_\alpha(x)$. This shows $\varphi_\alpha(x) = \frac{1}{2\alpha}\|x - J_\alpha x\|^2 + \varphi(J_\alpha(x))$ as desired.

Let $x, y \in H$; from $A_\alpha(x) \in \partial\varphi(J_\alpha x)$ it follows that $(A_\alpha x, J_\alpha y - J_\alpha x)_H \leq \varphi(J_\alpha y) - \varphi(J_\alpha x)$, hence,

$$\varphi_\alpha(y) - \varphi_\alpha(x) \geq \frac{\alpha}{2}\left\{\|A_\alpha y\|^2 - \|A_\alpha x\|^2 + \frac{2}{\alpha}(A_\alpha x, J_\alpha y - J_\alpha x)_H\right\} .$$

Writing $J_\alpha = I - \alpha A_\alpha$ gives

$$\varphi_\alpha(y) - \varphi_\alpha(x)$$
$$\geq \frac{\alpha}{2}\left\{\|A_\alpha y\|^2 - \|A_\alpha x\|^2 + 2(A_\alpha x, A_\alpha x - A_\alpha y)_H + \frac{2}{\alpha}(A_\alpha x, y - x)_H\right\}$$
$$= \frac{\alpha}{2}\|A_\alpha y - A_\alpha x\|^2 + (A_\alpha x, y - x)_H$$
$$\geq (A_\alpha x, y - x)_H .$$

Interchanging x, y gives

$$\varphi_\alpha(x) - \varphi_\alpha(y) \geq (A_\alpha y, x - y)_H = (A_\alpha x, x - y)_H + (A_\alpha y - A_\alpha x, x - y)_H$$

so we obtain

$$0 \leq \varphi_\alpha(y) - \varphi_\alpha(x) - (A_\alpha x, y - x)_H \leq \frac{1}{\alpha}\|y - x\|^2 .$$

This shows $\varphi_\alpha'(x) = A_\alpha(x)$. Since A_α is accretive, it follows by Proposition II.7.4 that φ_α is convex.

Now from the definition of φ_α we obtain

$$\varphi_\beta(x) \leq \varphi_\alpha(x) \leq \varphi(x) , \qquad 0 < \alpha < \beta$$

and we have seen above that

$$\varphi(J_\beta x) \leq \varphi_\beta(x) .$$

By Proposition 1.7 it follows that for each $x \in \overline{D(A)}$, $\lim J_\alpha(x) = x$ and so

$$\varphi(x) \le \liminf \varphi(J_\alpha x) \le \liminf \varphi_\alpha(x) \le \limsup \varphi_\alpha(x) \le \varphi(x) \ .$$

That is, $\lim \varphi_\alpha(x) = \varphi(x)$. But for each $x \notin \overline{D(A)}$, $\|x - J_\alpha x\| \to \|\operatorname{Proj}_{\overline{D(A)}}(x) - x\| > 0$ and $\alpha \|A_\alpha x\|^2 = \|A_\alpha x\| \, \|x - J_\alpha x\| \to +\infty$, so

$$\varphi_\alpha(x) \to +\infty = \varphi(x) \ . \qquad \qquad \Box$$

COROLLARY 1.4. $\operatorname{dom}(\varphi) \subset \overline{D(A)}$, hence, $D(A) \subset \operatorname{dom}(\varphi) \subset \overline{\operatorname{dom}(\varphi)} \subset \overline{D(A)}$.

IV.2. Construction of m-accretive Operators

We first recall some examples of m-accretive operators which are subgradients of convex functions as discussed in Section II.8. This class includes a variety of elliptic boundary-value problems. Then we introduce the realization of an m-accretive operator in the Hilbert space H that is distributed over $L^2(\Omega, H)$ and realizations of first-order derivatives of either real or vector-valued functions of several variables corresponding to initial-value problems. By adding such operators we shall obtain evolution equations and (as special cases) a variety of parabolic initial-boundary-value problems.

EXAMPLE 2.A: CONVEX FUNCTIONS ON $\mathbb{R}$.

Let $F : \mathbb{R} \to \overline{\overline{\mathbb{R}}}$ be a monotone function and denote its left and right limits at $x \in \mathbb{R}$ by $F^-(x)$ and $F^+(x)$, respectively. Then $F^-(x) = F^+(x)$ at all but at most a countable set of points. The subgradient of the convex function

$$\varphi(x) \equiv \int_{x_0}^x F^-(s) \, ds = \int_{x_0}^x F^+(s) \, ds$$

for a given $x_0 \in \operatorname{dom}(F)$ is given by

$$\partial \varphi(x) = [F^-(x), F^+(x)] \ , \qquad x \in \mathbb{R} \ .$$

EXAMPLE 2.B: CONVEX INTEGRANDS.

Let Ω be a measureable subset of $\mathbb{R}^n$ and $\varphi : \mathbb{R} \to \mathbb{R}_\infty$ be proper, convex, and lower-semi-continuous. We shall also suppose that either $0 = \varphi(0) = \min(\varphi)$. (Note that if Ω has finite measure, then we may use an affine exact lower bound for φ to change variable and thereby reduce to this case. That is, if $\varphi(x) \ge \ell(x)$ for the affine continuous function, $\ell(\cdot)$, then we replace $\varphi(\cdot)$ by $\varphi(\cdot) - \ell(\cdot)$.) Then

$$\Phi(u) \equiv \int_\Omega \varphi(u(x)) \, dx$$

defines a proper, convex, lower-semi-continuous function $\Phi : L^2(\Omega) \to \mathbb{R}_\infty^+$ with domain $\operatorname{dom}(\Phi) = \{u \in L^2(\Omega) : \varphi \circ u \in L^1(\Omega)\}$, and the subgradient is computed pointwise, i.e., $f \in \partial \Phi(u)$ if and only if

$$u, f \in L^2(\Omega) \text{ and } f(x) \in \partial \varphi(u(x)) \ , \quad \text{a.e. } x \in \Omega \ .$$

(See Proposition II.8.1.)

EXAMPLE 2.C: L^2-REALIZATIONS.

Let A be m-accretive on the Hilbert space H. Set $\mathcal{H} = L^2(\Omega, H)$, the square-summable H-valued functions on Ω, and define $\mathcal{A}$ on $\mathcal{H}$ by $v \in \mathcal{A}(u)$ if and only if

$$u, v \in \mathcal{H} \quad \text{and} \quad v(x) \in A(u(x)) , \quad \text{a.e.} \ x \in \Omega .$$

If $A(0) \ni 0$ or if Ω has finite measure, it follows easily that $\mathcal{A}$ is m-accretive and $(I + \alpha\mathcal{A})^{-1} f(x) = (I + \alpha A)^{-1}(f(x))$, a.e. $x \in \Omega$. Specifically, given $f \in \mathcal{H}$ we define $u(x) = (I + \alpha A)^{-1}(f(x))$, $x \in \Omega$, and check that $u \in \mathcal{H}$.

Suppose additionally that A is a subgradient, i.e., $A = \partial\varphi$ for an appropriate $\varphi : H \to \mathbb{R}_\infty$. Define $\Phi : \mathcal{H} \to \mathbb{R}_\infty$ as above and note that $\mathcal{A} \subset \partial\Phi$. But $\mathcal{A}$ is maximal, so we see $\mathcal{A} = \partial\Phi$. Moreover, for $\alpha > 0$ we have $\mathcal{A}_\alpha u(x) = A_\alpha(u(x))$ and this leads to

$$\begin{aligned}
\Phi_\alpha(u) &= \frac{\alpha}{2}\|\mathcal{A}_\alpha u\|^2 + \Phi\big((I + \alpha\mathcal{A})^{-1} u\big) \\
&= \int_\Omega \left\{ \frac{\alpha}{2}\|A_\alpha(u(x))\|^2 + \varphi\big(J_\alpha(u(x))\big) \right\} dx \\
&= \int_\Omega \varphi_\alpha(u(x)) \, dx .
\end{aligned}$$

Thus the approximations of Φ and φ correspond.

EXAMPLE 2.D: INITIAL-VALUE PROBLEMS.

Let H be the Hilbert space $L^2(0,1)$ and $u_0 \in \mathbb{R}$. Define $A = \frac{d}{dt}$ on the domain $D(A) = \{u \in H^1(0,1) : u(0) = u_0\}$. Then A is m-accretive and the resolvent is given by

$$(2.1) \qquad (I + \alpha A)^{-1} f(t) = e^{-t/\alpha} u_0 + \frac{1}{\alpha}\int_0^t e^{-\frac{(t-s)}{\alpha}} f(s) \, ds .$$

Note that this is of the form $e^{-t/\alpha} u_0 + (1 - e^{-t/\alpha})w(t)$, where $w(t)$ is a convex combination of the values of f and thus $(I + \alpha A)^{-1} f(t)$ is a convex combination of u_0 and $w(t)$. This observation is very useful when considering data in a given closed convex set, for then the resolvent remains in that set.

The above extends directly to the case of vector-valued functions, i.e., on $\mathcal{H} = L^2((0,1), H)$. Moreover, it can be extended to $\mathcal{H} = L^2(\Omega, H)$ where Ω is a smoothly bounded domain in $\mathbb{R}^n$, $A = \frac{\partial}{\partial x_k}$, $1 \le k \le n$, and $D(A) = \{u \in \mathcal{H} : Au \in \mathcal{H}$ and $u|_{\Gamma_k} = u_0\}$ where $\Gamma_k = \{y \in \partial\Omega : \nu_k(y) < 0\}$. Here ν_k is the k-th component of the unit outward normal vector $\vec{\nu}$ on $\partial\Omega$.

EXAMPLE 2.E: BOUNDARY-VALUE PROBLEMS.

Let Ω be a bounded domain in $\mathbb{R}^n$ whose boundary $\Gamma = \partial\Omega$ is a C^1-manifold of dimension $n - 1$ with Ω locally on one side. Thus we have a *trace* functional $\gamma : H^1(\Omega) \to L^2(\Gamma)$ which gives generalized boundary values. (See Section II.4.) Let $\psi : \mathbb{R} \to \mathbb{R}$ be convex and continuous with

$$(2.2) \qquad\qquad 0 \le \psi(s) \le C(s^2 + 1) , \qquad s \in \mathbb{R} ,$$

and define $\Psi : H = L^2(\Omega) \to \mathbb{R}_\infty$ by

$$\Psi(u) \equiv \frac{1}{2}\int_\Omega |\vec{\nabla} u|^2 \, dx + \int_\Gamma \psi(\gamma u) \, ds \ \text{ if } \ u \in H^1(\Omega) , \ +\infty \ \text{ otherwise } .$$

Note that $\mathrm{dom}(\Psi) = H^1(\Omega)$ and that Ψ is proper and convex. To see Ψ is lower-semi-continuous, let $u_n \to u$ in $L^2(\Omega)$ with $\Psi(u_n) \le c$; after passing to a subsequence we have $u_n \rightharpoonup u$ in $H^1(\Omega)$ and $\gamma u_n \rightharpoonup \gamma u$ in $L^2(\Gamma)$, so we have

$$\frac{1}{2}\int_\Omega |\vec{\nabla} u|^2\, dx \le \liminf_{n\to\infty}\frac{1}{2}\int_\Omega |\vec{\nabla} u_n|^2\, dx \ , \ \int_\Gamma \psi(\gamma u)\, ds \le \liminf_{n\to\infty}\int_\Gamma \psi(\gamma u_n)\, ds$$

by weak-lower-semicontinuity of the respective terms. The result follows by the super-additivity of $\liminf$.

To compute the subgradient, let $F \in \partial\Psi(u)$, that is, $F \in L^2(\Omega)$, $u \in H^1(\Omega)$ and

$$\int_\Omega F(v - u)\, dx \le \Psi(v) - \Psi(u) \ , \qquad v \in H^1(\Omega) \ .$$

Let $w \in H^1(\Omega)$, $0 \le t \le 1$, and set $v = tw + (1-t)u$ above, use convexity of ψ and then let $t \to 0^+$ to obtain

$$(2.3) \quad \int_\Omega F(w - u)\, dx \le \int_\Omega \vec{\nabla} u \cdot \vec{\nabla}(w - u)\, dx + \int_\Gamma \big(\psi(\gamma w) - \psi(\gamma u)\big)\, dx, w \in H^1(\Omega) \ .$$

Choosing $w = u \pm \varphi$ with $\varphi \in C_0^\infty(\Omega)$ in (2.3) yields the partial differential equation

$$(2.4.\mathrm{a}) \qquad\qquad -\Delta u = F \ \text{ in } \ L^2(\Omega)$$

in the sense of distributions on Ω. Let $B = \gamma[H^1(\Omega)]$ be the range of γ with the norm induced by the quotient map from $H^1(\Omega)/H_0^1(\Omega)$. Then according to Proposition II.5.3 we have the *abstract Green's theorem*

$$\int_\Omega \vec{\nabla} u \cdot \vec{\nabla} v\, dx = \int_\Omega (-\Delta u)v\, dx + \partial u(\gamma v) \ , \qquad v \in H^1(\Omega)$$

with a $\partial u \in B'$ which extends the notion of normal derivative $\frac{\partial}{\partial \nu} = \vec{\nu} \cdot \vec{\nabla}$ on the boundary Γ, so we obtain

$$\partial u(\zeta - \gamma u) \le \int_\Gamma (\psi(\zeta) - \psi(\gamma u))\, ds \ , \qquad \zeta \in B \ .$$

Using the growth-estimate (2.2) on ψ gives a $c_1 > 0$ such that

$$|\partial u(\zeta)| \le c_1(\|\zeta\|_{L^2(\Gamma)}^2 + 1) \ , \qquad \zeta \in B \ ,$$

and B is dense in $L^2(\Gamma)$, so we obtain $\partial u \in L^2(\Gamma)$ and the generalized *boundary condition*

$$(2.4.\mathrm{b}) \qquad\qquad \partial u + \partial\psi(\gamma u) \ni 0 \ \text{ in } \ L^2(\Gamma) \ .$$

Thus $F \in \partial\Psi(u)$ implies (2.4), and the converse is immediate, so this characterizes $\partial\Psi$.

It is very useful to be able to *add* operators. The problem is that the domain of the sum is the intersection of the domains, and this can be too small unless the operators are compatible. For example, the two operators $-\Delta$ on $D_1 = H_0^1 \cap H^2$ (Dirichlet) and $-\Delta$ on $D_2 = \{u \in H^2 : \frac{\partial u}{\partial \nu} = 0\}$ (Neuman) are both m-accretive, but $D_1 \cap D_2$ is too small.

LEMMA 2.1. *If A is m-accretive and B is accretive and Lipschitz, then $A + B$ is m-accretive.*

PROOF. First note that $A + B$ is clearly accretive. Given $f \in H$ we want $u \in \mathrm{dom}(A) : f \in u + Bu + A(u)$. Since A is m-accretive if and only if αA is m-accretive $\forall \alpha > 0$, we may assume B is a strict contraction. Thus, the desired u is characterized by $u = (I + A)^{-1}(f - Bu)$, the fixed point of a *strict contraction* on H. $\qquad\square$

Consider a pair A, B of m-accretive operators, and the equation for their sum, $f \in u + Au + Bu$. A "penalty method" of approximation is the system

$$u_\alpha + Au_\alpha + \frac{1}{\alpha}(u_\alpha - v_\alpha) \ni f \; , \quad Bv_\alpha + \frac{1}{\alpha}(v_\alpha - u_\alpha) \ni 0$$

in which $\frac{1}{\alpha}(u_\alpha - v_\alpha)$ is the "penalty": as $\alpha \to 0$ one hopes $u_0 \cong v_0$ in the limit. This system is equivalent to the single equation

(2.5) $$u_\alpha + A(u_\alpha) + B_\alpha(u_\alpha) \ni f$$

which has a solution by Lemma 2.1.

PROPOSITION 2.1 (BREZIS-CRANDALL-PAZY). *Let A and B be m-accretive. If the sequence $\{B_\alpha u_\alpha\}$ is bounded, then there exists $u : f \in u + Au + Bu$ and $u_\alpha \to u$.*

PROOF. Set $w_\alpha = f - u_\alpha - B_\alpha u_\alpha \in A(u_\alpha)$. Then

$$\|u_\alpha - u_\beta\|^2 + (w_\alpha - w_\beta, u_\alpha - u_\beta) + (B_\alpha u_\alpha - B_\beta u_\beta, u_\alpha - u_\beta) = 0$$

and $u_\alpha - u_\beta = (\alpha B_\alpha u_\alpha - \beta B_\beta u_\beta) + ((I + \alpha B)^{-1} u_\alpha - (I + \beta B)^{-1} u_\beta)$ imply that

$$\|u_\alpha - u_\beta\|^2 \leq \|B_\alpha u_\alpha - B_\beta u_\beta\| \, \|\alpha B_\alpha u_\alpha - \beta B_\beta u_\beta\|$$

so $\{u_\alpha\}$ is Cauchy and $u_\alpha \to u \in H$. Since $\{B_\alpha u_\alpha\}$ and $\{w_\alpha\}$ are bounded, for some subsequence we have $B_\alpha u_\alpha \rightharpoonup v$, $w_\alpha \rightharpoonup w$. But then $w \in A(u)$; since $(I + \alpha B)^{-1} u_\alpha = -\alpha B_\alpha u_\alpha + u_\alpha \to u$ and $B((I + \alpha B)^{-1} u_\alpha) \ni f - u_\alpha - w_\alpha$, we have $v \in B(u)$. Thus

$$u + w + v = f \; , \qquad w \in A(u) \, , v \in B(u) \; . \qquad\square$$

In order to use this result, one needs an *a-priori estimate* on $\{B_\alpha u_\alpha\}$. One can show also that $\{B_\alpha u_\alpha\}$ is bounded *only if* there exists a solution. Here is a useful observation.

LEMMA 2.2. *If $D(A) \cap D(B) \neq \varphi$, then $\{u_\alpha\}$ is bounded.*

PROOF. Let $u_0 \in D(A) \cap D(B)$ and pick $f_\alpha \in u_0 + A(u_0) + B_\alpha(u_0)$. Since A, B_α are accretive, $|u_\alpha - u_0|^2 \leq (f - f_\alpha, u_\alpha - u_0)$, hence $|u_\alpha - u_0| \leq |f - f_\alpha|$; note $|B_\alpha u_0| \leq |B^0 u_0|$. $\qquad\square$

Thus, if also A *or* B is *bounded*, Proposition 2.1 applies.

PROPOSITION 2.2 (BREZIS). *Let B be m-accretive and $\varphi : H \to \mathbb{R}_\infty$ be proper, convex and lower semi-continuous. Suppose there is a $C > 0$ such that*

(2.6) $$\varphi((I + \alpha B)^{-1}u) \le \varphi(u) + C\alpha, u \in H, \alpha > 0 .$$

Then $\partial\varphi + B$ is m-accretive.

PROOF. Let $f \in H$ and u_α the solution for $\alpha > 0$ of (2.5),

$$u_\alpha + \partial\varphi(u_\alpha) + B_\alpha(u_\alpha) \ni f .$$

Thus we have

(2.7) $$(f - B_\alpha(u_\alpha) - u_\alpha, v - u_\alpha) \le \varphi(v) - \varphi(u_\alpha) , \qquad v \in H .$$

Set $v = (I + \alpha B)^{-1}(u_\alpha)$ to get

$$(f - B_\alpha(u_\alpha) - u_\alpha, -\alpha B_\alpha(u_\alpha)) \le C\alpha , \qquad \alpha > 0 ,$$

from which there follows

$$\|B_\alpha(u_\alpha)\|^2 \le \|f - u_\alpha\| \, \|B_\alpha(u_\alpha)\| + C .$$

If $D(\partial\varphi) \cap D(B) \ne \varphi$, then by Lemma 2.2 we are done. In general, it suffices to set $v = v_0 \in \mathrm{dom}(\varphi) \cap D(B)$ in (2.7) to get

$$(f - B_\alpha(u_\alpha) - u_\alpha, v_0 - u_\alpha) \le \varphi(v_0) - \varphi(u_\alpha) .$$

Since B_α is accretive we have

$$(f - B_\alpha(v_0) - u_\alpha, v_0 - u_\alpha) \le \varphi(v_0) - \varphi(u_\alpha) ;$$

since φ has an affine lower bound, this shows $\{u_\alpha\}$ is bounded, and we are done.□

Note that if φ is the indicator function of the closed, convex, non-empty set K, $\varphi = I_K$ where $I_K(u) = \begin{cases} 0 , & u \in K \\ +\infty , & u \notin K , \end{cases}$ then (2.6) is equivalent to $(I + \alpha B)^{-1}[K] \subset K$. For additional characterizations of (2.6), see Proposition 5.4.

EXAMPLE 2.F: DOUBLY-NONLINEAR BOUNDARY-VALUE-PROBLEM.
Here we add Examples 2.B and 2.E:

$$\Phi(u) \cong \int_\Omega \varphi(u(x)) \, dx , \quad \Psi(u) = \frac{1}{2} \int_\Omega |\vec{\nabla} u|^2 \, dx + \int_\Gamma \psi(\gamma u) \, ds$$

It will follow that $\partial\Phi + \partial\Psi$ is m-accretive and equal to $\partial(\Phi + \psi)$ if we can show

$$\Psi((I + \alpha\partial\Phi)^{-1}f) \le \Psi(f) , \qquad f \in H .$$

But this is immediate since $(I + \alpha\partial\varphi)^{-1}$ is a monotone contraction on $\mathbb{R}$. In particular, it follows that for every $F \in L^2(\Omega)$ there is a unique

$$u \in H^1(\Omega) : -\Delta u + \partial\varphi(u) + u \ni F \text{ in } L^2(\Omega)$$

(2.8) $$\partial u + \partial\psi(\gamma u) \ni 0 \text{ in } L^2(\Gamma)$$

when ψ is quadratically-bounded. (The general case is similar when the boundary condition is interpreted in B'.) This is a *doubly-nonlinear* elliptic boundary-value problem.

EXAMPLE 2.G: CAUCHY PROBLEM.

This time we add Examples 2.C and 2.D: we let H be a Hilbert space and set

$$\Phi(u) = \int_0^1 \varphi(u(t))\, dt , \quad A = \frac{d}{dt} \text{ with } u(0) = u_0 \text{ on } \mathcal{H} = L^2(0,1;H)$$

where the function $\varphi : H \to \mathbb{R}_\infty^+$ is convex and LSC as in Example 2.C. Recall from (2.1) that

$$(I + \alpha A)^{-1} f(t) = u_0 e^{-t/\alpha} + (1 - e^{-t/\alpha}) \frac{1}{\alpha(1 - e^{-t/\alpha})} \int_0^t e^{-(t-s)/\alpha} f(s)\, ds .$$

From here by convexity of φ we get

$$\varphi((I+\alpha A)^{-1} f(t)) \le e^{-t/\alpha} \varphi(u_0) + (1 - e^{-t/\alpha}) \frac{1}{\alpha(1 - e^{-t/\alpha})} \int_0^t e^{-(t-s)/\alpha} \varphi(f(s))\, ds ,$$

and integrating this over (0,1) gives

$$(2.9) \qquad \Phi((I + \alpha A)^{-1} f) \le \alpha(1 - e^{-1/\alpha}) \varphi(u_0) + \Phi(f) .$$

Thus (2.6) is satisfied with $C = \varphi(u_0)$ if $u_0 \in \text{dom}(\varphi)$.

An alternative derivation of (2.9) is as follows. Set $u_\alpha = (I + \alpha A)^{-1} f$ so that

$$\alpha \frac{du_\alpha}{dt} + u_\alpha = f , \qquad u_\alpha(0) = u_0 ,$$

and let φ_β, $\beta > 0$, be given by Proposition 1.8. Then we have

$$\varphi_\beta(f(t)) - \varphi_\beta(u_\alpha(t)) \ge \left(\varphi_\beta'(u_\alpha(t)) , f(t) - u_\alpha(t) \right)$$

$$= \left(\varphi_\beta'(u_\alpha(t)), \alpha \frac{du_\alpha}{dt} \right)$$

$$= \alpha \frac{d}{dt} \varphi_\beta(u_\alpha(t))$$

and an integration gives (by Example 2.C above)

$$\Phi_\beta(u_\alpha) \le \Phi_\beta(f) + \alpha \varphi_\beta(u_0) .$$

Taking the limit as $\beta \to 0$ yields (2.6). Thus it follows that if $u_0 \in \text{dom}(\varphi)$ then $A + \partial\Phi$ is m-accretive on $L^2((0,1), H)$. This corresponds to an *abstract Cauchy problem* in the Hilbert space H. That is, $F \in (A + \partial\Phi)(u)$ is characterized by

$$\frac{du(t)}{dt} + \partial\varphi(u(t)) \ni F(t) , \quad \text{a.e. } t \in (0,1) ,$$

$$u(0) = u_0 .$$

In particular, if we apply this to Example 2.F by choosing $\varphi = \Phi + \Psi$ on $H = L^2(0,1)$ from Example 2.B and Example 2.E, with the obvious duplicated use of φ, it follows that $\alpha I + A + \partial\Phi + \partial\Psi$ is surjective for each $\alpha > 0$. Thus, for every

$$F \in L^2((0,1); L^2(\Omega)) \cong L^2((0,1) \times \Omega)$$

and $u_0 \in \mathrm{dom}(\Phi + \Psi)$ there is a unique $u \in L^2((0,1), H^1(\Omega))$ such that

$$(2.10.a) \qquad \alpha u + \frac{\partial u}{\partial t} - \Delta u + \partial\varphi(u) \ni F \text{ in } L^2((0,1) \times \Omega) \ ,$$

$$(2.10.b) \qquad \partial u + \partial\psi(\gamma u) \ni 0 \text{ in } L^2((0,1) \times \Gamma) \ ,$$

$$(2.10.c) \qquad u(0,\cdot) = u_0 \text{ in } L^2(\Omega) \ .$$

This is a *doubly-nonlinear* parablic initial-boundary-value problem.

REMARK. If $\partial\varphi$ is *strongly-accretive*, i.e., if $\partial\varphi - \alpha I$ is accretive for some $\alpha > 0$, then the preceding results hold also for $\alpha = 0$ by a change of notation. More generally, consider the Cauchy problem in the Hilbert space H

$$\frac{du}{dt} + \alpha u + A(u(t)) \ni f(t) \ , \qquad 0 < t < 1 \ , \ u(0) = u_0$$

with an m-accretive A in H. Let $v(t) = e^{\alpha t} u(t)$ to get an equivalent problem

$$\frac{dv}{dt} + e^{\alpha t} A(e^{-\alpha t} v(t)) \ni e^{\alpha t} f(t) \ , \qquad 0 < t < 1 \ , \ v(0) = u_0 \ .$$

Take scalar product with $w \in L^2((0,1), H)$ to get at a.e. $t \in (0,1)$

$$\left(\frac{dv(t)}{dt}, w(t) \right)_H + \left(A(e^{-\alpha t} v(t)), e^{\alpha t} w(t) \right)_H = \left(e^{\alpha t} f(t), w(t) \right)_H \ .$$

But in order to exploit the accretive estimates on A we need to multiply by $e^{-2\alpha t}$ before integrating, and then we have

$$\int_0^1 (v'(t), w(t))_H e^{-2\alpha t} \, dt + \int_0^1 \left(A(e^{-\alpha t} w(t)), e^{-\alpha t} w(t) \right) dt$$

$$= \int_0^1 \left(e^{\alpha t} f(t), w(t) \right) e^{-2\alpha t} \, dt \ .$$

Thus in the space $\mathcal{H} = L^2((0,1), H; e^{-2\alpha t} \, dt)$ with the indicated *density* or *weight* $e^{-2\alpha t}$ the operator $\mathcal{A}(v)(t) = e^{\alpha t} A(e^{-\alpha t} v(t))$ is accretive. The results of Example 3 carry over here and thereby one can replace A by $A + \alpha I$ for *any* $\alpha \in \mathbb{R}$.

It is not difficult to give sufficient additional conditions on an m-accretive operator A which imply that it is surjective. For example, if A is *strongly-accretive*, i.e., there is a $c > 0$ for which

$$(y_1 - y_2, x_1 - x_2) \geq c\|x_1 - x_2\|^2 \ , \qquad [x_j, y_j] \in A \ , \ j = 1, 2 \ ,$$

then $A - cI$ is m-accretive, hence, $A = (A - cI) + cI$ is surjective. For another example, consider $A = \partial\varphi$. If $f \in H$ and

$$\lim_{\|v\| \to \infty} (\varphi(v) - (f, v)) = \infty \ ,$$

then the convex function $\varphi(\cdot) - (f, \cdot)$ attains a minimum at some $u \in H$,

$$\varphi(u) - (f, u) \leq \varphi(v) - (f, v) \ , \qquad v \in H \ ,$$

hence, $f \in \partial\varphi(u) = A(u)$. This holds for every $f \in H$ if

$$\lim_{\|u\|\to\infty} \frac{\varphi(u)}{\|u\|} = \infty \ .$$

This is a *coercivity* condition on φ, a convenient sufficient condition for $\partial\varphi$ to be surjective. More generally we have the following.

PROPOSITION 2.3. *Let A be m-accretive and let A^{-1} be bounded, that is,*

$$\lim_{\|v\|\to\infty} \|A^0(v)\| = +\infty \ .$$

Then A is surjective: $Rg(A) = H$.

PROOF. Note that $Rg(A)$ is closed, for if $y_n \to y$ in H with $y_n \in Rg(A)$, then there is a sequence $x_n \in A^{-1}(y_n)$ which by assumption is bounded. By passing to a subsequence, which we denote again by $\{x_n\}$, we have $x_n \rightharpoonup x$ and $[x, y] \in A$ by Proposition 1.6.

Let $[x_0, y_0] \in A$ and $y \in H$. For each $\alpha > 0$ set $x_\alpha \equiv (\alpha I + A)^{-1}(y + \alpha x_0)$ and $y_\alpha \equiv y + \alpha(x_0 - x_\alpha) \in A(x_\alpha)$. Since A is accretive we obtain $(y_\alpha - y_0, x_\alpha - x_0) \geq 0$ and this shows $(y_\alpha - y_0, y - y_\alpha) \geq 0$. Consequently follow $\|y_\alpha - y_0\|^2 \leq (y_\alpha - y_0, y - y_0)$ and $\|y_\alpha - y_0\| \leq \|y - y_0\|$. Thus $\{y_\alpha\}$ is bounded and by assumption $\{x_\alpha\}$ is bounded. This shows $y_\alpha \to y \in \overline{Rg(A)} = Rg(A)$.   □

Proposition 2.3 asserts that for an m-accretive operator A to be surjective, it suffices to know *a-priori estimates* on solutions, i.e., if $A(u) \ni f$, then $\|u\| \leq C(\|f\|)$ where $C : \mathbb{R}^+ \to R^+$ is a *bounded* function. Such estimates often occur as a *coercive* property of A:

$$\lim_{\|u\|\to\infty} \frac{(A^0u, u)}{\|u\|} = \infty \ .$$

Note that it is not true that A^{-1} being bounded implies that A is coercive. For example, the operator $A[x_1, x_2] = [-x_2, x_1]$ on $\mathbb{R}^2$ is not coercive.

It is not the case that coercivity is preserved by addition. For example, set $A = \frac{d}{dt}$ on $D(A) = \{u \in H^1(0, 1) : u(0) = 0\}$ and $B = -\frac{d}{dt}$ on $D(B) = \{u \in H^1(0, 1) : u(1) = 0\}$. They are both m-accretive and coercive on $L^2(0, 1)$, but $A + B = 0$. From the calculation $\|(A + B)u\|^2 = \|Au\|^2 + \|Bu\|^2 + 2(Au, Bu)_H$, it is clear that one needs some control of the "angle" between Au and Bu. This does occur in the situation of Proposition 2.2.

COROLLARY 2.1. *In the situation of Proposition 2.2, we obtain the estimate*

$$\|B^0u\| \leq \|(\partial\varphi + B)^0u\| + \sqrt{C}$$

Thus $(\partial\varphi + B)^{-1}$ is bounded if B^{-1} or $(\partial\varphi)^{-1}$ is bounded.

PROOF. For $w \in \partial\varphi(u)$ we have

$$\varphi((I + \alpha B)^{-1}u) - \varphi(u) \geq (w, (I + \alpha B)^{-1}u - u)$$

and by (2.6) we have $\alpha C \geq (w, -\alpha B_\alpha u)$, $\alpha > 0$. Thus for $u \in D(B) \cap D(\partial\varphi)$ there follows

$$(B^0(u), w) \geq -C \ , \qquad w \in \partial\varphi(u) \ .$$

Let $f \in (\partial\varphi + B)(u)$, that is,

$$f = v + w \ , \ \text{where } v \in B(u) \ , w \in \partial\varphi(u) \ .$$

Then

$$(B^0(u), f) = (B^0(u), v) + (B^0(u), w) \geq \|B^0(u)\|^2 - C \ ,$$

and the desired estimate follows from this. □

EXAMPLE 2.G (CONTINUED). Since A is coercive it follows that $A + \partial\Phi + \partial\Psi$ is coercive on $\mathcal{H} = L^2(0,1;H)$ and, hence, (again) we find for every $F \in L^2((0,1), L^2(\Omega))$ and $u_0 \in \text{dom}(\Phi + \Psi)$ there is a unique solution of (2.10) with $\alpha = 0$.

We close with some equivalent notions of coercivity for subgradients.

PROPOSITION 2.4 (BREZIS). *Let $\varphi : H \to \mathbb{R}_\infty^+$ be proper, convex and lower semi-contin-uous, and set $A = \partial\varphi$. The following are equivalent:*

(a) $\displaystyle\lim_{\|u\|\to\infty} \left(\frac{\varphi(u)}{\|u\|}\right) = \infty.$

(b) *for every $u_0 \in \text{dom}(\varphi)$,* $\displaystyle\lim_{\substack{\|u\|\to\infty \\ [u,f]\in A}} \frac{(f, u - u_0)}{\|u\|} = \infty.$

(c) *there exists $u_0 \in H$ such that* $\displaystyle\lim_{\substack{\|u\|\to\infty \\ u\in\text{dom}(A)}} \frac{(A^0u, u - u_0)}{\|u\|} = \infty.$

(d) $\displaystyle\lim_{\substack{\|u\|\to\infty \\ u\in\text{dom}(A)}} \|A^0u\| = \infty.$

(e) A^{-1} *is bounded, hence, $Rg(A) = H$.*

PROOF. From the inequality $\varphi(u) \leq \varphi(u_0) + (f, u - u_0)$ it follows that (a) $\Rightarrow$ (b). Clearly (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e). To show (e) $\Rightarrow$ (a), let $R > 0$ and note that by (e), for every $z \in H$ with $\|z\| \leq R$ there is a $v : A(v) \ni z$ and $\|v\| \leq M$. Thus from

$$\varphi(u) - \varphi(v) \geq (z, u - v) \ , \qquad u \in \text{dom}(\varphi) \ ,$$

we obtain

$$(z, u) \leq \varphi(u) + MR \quad \forall \quad z \in H \ , \ |z| \leq R \ .$$

Hence $R\|u\| \leq \varphi(u) + MR$ and

$$\frac{\varphi(u)}{\|u\|} \geq R - \frac{MR}{\|u\|} \ ,$$

and so we obtain

$$\liminf_{\|u\|\to\infty} \frac{\varphi(u)}{\|u\|} \geq R$$

for every $R > 0$. □

IV.3. The Cauchy Problem in Hilbert space

Let A be a m-accretive operator in the Hilbert space H. As before we denote by $J_\alpha = (I + \alpha A)^{-1}$ the resolvent, by $A_\alpha = \frac{1}{\alpha}(I - J_\alpha)$ the Yosida approximation, and by A^0 the minimal section of A. We shall study the evolution equation

$$(3.1) \qquad \frac{du}{dt}(t) + A\big(u(t)\big) \ni 0 , \qquad 0 < t .$$

A *solution* of (3.1) is a continuous function $u : [0, \infty) \to H$ which is absolutely continuous on each $[a, b]$, $0 < a < b$, hence, u is differentiable a.e. with $\frac{du}{dt} \in L^1(a, b; H)$, and for a.e. $t > 0$, $u(t) \in D(A)$ and (3.1) holds. The *Cauchy problem* is to find a solution of (3.1) with $u(0) = u_0$ given in H. The uniqueness of a solution of the Cauchy problem is immediate when A is accretive or Lipschitz. Existence is similarly easy when A is Lipschitz; existence of a solution for the m-accretive case is the goal of this section. This will be done by replacing A by the Lipschitz A_α, $\alpha > 0$, to obtain a sequence of solutions u_α of this *approximate* or *regularized* problem and then showing $\lim_{\alpha \to 0} u_\alpha$ exists and is a solution of the Cauchy problem for (3.1).

The Yosida approximation of A in (3.1) and a re-scaled time variable suggest that we consider the differential equation

$$(3.2) \qquad \frac{du}{dt}(t) + u(t) - J\big(u(t)\big) = 0$$

in which J is Lipschitz.

LEMMA 3.1. *Assume K is a closed, convex non-empty set in H and $J : K \to K$ is Lipschitz: $\|J(x) - J(y)\| \le L\|x - y\|$, $x, y \in K$. Then for each $u_0 \in K$ there exists a unique absolutely continuous $u : [0, \infty) \to H$ which is a solution of (3.2) with $u(t) \in K$, $t > 0$, and with $u(0) = u_0$.*

REMARK. We shall need below only the case $K = H$ in Lemma 3.1. However, that the equation (3.2) would leave the set K invariant is suggested by the observation that for each u on the boundary of K the vector $J(u) - u$ is directed into K, hence, the equation (3.2) implies that the direction of the solution is back into K.

PROOF. If u is such a solution then

$$\frac{d}{dt}\big(e^t u(t)\big) = e^t J\big(u(t)\big) , \qquad t > 0 ,$$

so we find by integrating that

$$(3.3) \qquad u(t) = e^{-t} u_0 + \int_0^t e^{s-t} J\big(u(s)\big) \, ds , \qquad t \ge 0 .$$

Let's solve (3.3) in the Banach space $C([0, T], H)$ by a fixed-point theorem. Consider the closed, convex $\mathcal{K} = \{u \in C([0, T], H) : u(t) \in K \text{ for all } t \in [0, T]\}$, and define

$$\mathbb{T}(u)(t) = e^{-t} u_0 + \int_0^t e^{s-t} J u(s) \, ds , \qquad 0 \le t \le T ,$$

for $u \in \mathcal{K}$. Then $\mathbb{T}(u) \in C([0,T], H)$ and the indicated convex combination satisfies

$$v_1(t) = \int_0^t e^{s-t} J(u(s)) \, ds \Big/ \int_0^t e^{s-t} \, ds \in K \, , \qquad t > 0 \, .$$

Note that $\int_0^t e^{s-t} \, ds = 1 - e^{-t}$ so

$$\mathbb{T}(u)(t) = e^{-t} u_0 + (1 - e^{-t}) v_1(t) \in K \, , \qquad t > 0 \, .$$

Thus $\mathbb{T}$ maps $\mathcal{K}$ into $\mathcal{K}$. For any pair u_1, $u_2 \in \mathcal{K}$ we have

$$\|\mathbb{T}(u_1)(t) - \mathbb{T}(u_2)(t)\| \le L \int_0^t e^{s-t} \|u_1(s) - u_2(s)\| \, ds$$

and this shows, successively,

$$\|\mathbb{T}u_1 - \mathbb{T}u_2\|_{C([0,t],H)} \le Lt \|u_1 - u_2\|_{C([0,T],H)} \, ,$$

$$\|\mathbb{T}^2(u_1(t)) - \mathbb{T}^2(u_2(t))\| \le L \int_0^t e^{s-t} Ls \, ds \|u_1 - u_2\|_{C([0,T],H)}$$

$$\le \frac{L^2 t^2}{2} \|u_1 - u_2\|_{C([0,T],H)}$$

and by induction we obtain for $k \ge 1$

$$\|\mathbb{T}^k u_1 - \mathbb{T}^k u_2\|_{C([0,t],H)} \le \frac{(Lt)^k}{k!} \|u_1 - u_2\|_{C([0,T],H)} \, , \qquad 0 \le t \le T \, .$$

Thus, $\mathbb{T}^k$ is a strict-contraction for k sufficiently large, so (3.3) has a unique solution in $\mathcal{K}$. $\qquad\qquad\square$

COROLLARY 3.1. *If $u_0 \in K$ and $\alpha > 0$ there exists a unique solution (as above) of*

(3.4)
$$\frac{du}{dt} + \frac{1}{\alpha} \Big(u(t) - J(u(t)) \Big) = 0$$

with $u(t) \in \mathcal{K}$, and it is characterized by

$$u(t) = e^{-t/\alpha} u_0 + \frac{1}{\alpha} \int_0^t e^{(s-t)/\alpha} J(u(s)) \, ds \, .$$

PROOF. The change-of-variable $v(t) = u(\alpha t)$ shows (3.4) is equivalent to (3.2). $\square$

We shall show the Cauchy problem for (3.1) has a solution if $u(0) = u_0 \in D(A)$. For each $\alpha > 0$ let u_α be the solution of

(3.5)
$$\frac{du_\alpha}{dt}(t) + A_\alpha(u_\alpha(t)) = 0 \, , \qquad t \ge 0 \, ,$$

with $u_\alpha(0) = u_0$ as is given in $C^1([0,\infty), H)$ by Corollary 3.1. If $h > 0$ then $u_\alpha(t+h)$ is a solution of (3.5) and since A_α is accretive we obtain

$$\frac{d}{dt} \|u_\alpha(t+h) - u_\alpha(t)\|^2 = -2 \Big(A_\alpha(u_\alpha(t+h)) - A_\alpha(u_\alpha(t)), u_\alpha(t+h) - u_\alpha(t) \Big)_H \le 0$$

so $\|u_\alpha(t+h) - u_\alpha(t)\| \le \|u_\alpha(h) - u_\alpha(0)\|$. Letting $h \to 0$ shows $\|u'_\alpha(t)\| \le \|u'_\alpha(0)\|$, hence

(3.6) $\|A_\alpha\big(u_\alpha(t)\big)\| \le \|A_\alpha(u_0)\| \le \|A^0 u_0\|$, $t \ge 0$, $\alpha > 0$.

We shall show u_α is Cauchy in $C([0,T], H)$. For α, $\beta > 0$

$$\frac{d}{dt} \|u_\alpha(t) - u_\beta(t)\|^2 = -2\big(A_\alpha u_\alpha(t) - A_\beta u_\beta(t), u_\alpha(t) - u_\beta(t)\big)_H .$$

Write $u_\alpha = \alpha A_\alpha u_\alpha + J_\alpha u_\alpha$ and likewise for u_β to get

$$(A_\alpha u_\alpha - A_\beta u_\beta, u_\alpha - u_\beta)_H = (A_\alpha u_\alpha - A_\beta u_\beta, \alpha A_\alpha u_\alpha - \beta A_\beta u_\beta)_H$$
$$+ (A_\alpha u_\alpha - A_\beta u_\beta, J_\alpha u_\alpha - J_\beta u_\beta)_H .$$

The latter term is non-negative, since A is accretive, so the right side is bounded below by

$$\alpha\|A_\alpha u_\alpha\|^2 + \beta\|A_\beta u_\beta\|^2 - (\alpha + \beta)(A_\alpha u_\alpha, A_\beta u_\beta)_H$$
$$\ge \alpha\|A_\alpha u_\alpha\|^2 + \beta\|A_\beta u_\beta\|^2 - \alpha\Big(\|A_\alpha u_\alpha\|^2 + \frac{1}{4}\|A_\beta u_\beta\|^2\Big)$$
$$- \beta\Big(\|A_\beta u_\beta\|^2 + \frac{1}{4}\|A_\alpha u_\alpha\|^2\Big)$$
$$\ge -\frac{\alpha + \beta}{4}\|A^0 u_0\|^2 .$$

This shows

$$\frac{d}{dt}\|u_\alpha(t) - u_\beta(t)\|^2 \le \frac{\alpha + \beta}{2}\|A^0 u_0\|^2$$

and, hence,

$$\|u_\alpha(t) - u_\beta(t)\| \le \|A^0 u_0\|\Big(\frac{\alpha + \beta}{2}t\Big)^{1/2} ,   t \ge 0 .$$

On each interval, $[0,T]$, the sequence $\{u_\alpha\}$ is uniformly Cauchy, hence, uniformly convergent to some $u \in C([0,T], H)$ with

$$\|u_\alpha(t) - u(t)\| \le (\alpha T/2)^{1/2}\|A^0 u_0\| ,   0 \le t \le T , \alpha > 0 .$$

From the estimate

$$\|J_\alpha u_\alpha(t) - u_\alpha(t)\| = \alpha\|A_\alpha(u_\alpha(t))\| \le \alpha\|A^0 u_0\| ,   \alpha > 0$$

it follows that $J_\alpha u_\alpha$ converges in $C([0,T], H)$ to u. The estimate (3.6) implies $\{u'_\alpha\}$ is bounded in the Hilbert space $L^2(0,T; H)$ so there is a subsequence which converges weakly. We take limits in

$$u_\alpha(t) = u_0 + \int_0^t u'_\alpha(s)\, ds$$

to find the limit is u', and by uniqueness of weak limits the original sequence $\{u'_\alpha\}$ converges weakly to u'. Note that the realization $\mathcal{A}$ of A in $L^2(0,T; H)$ is m-accretive, i.e., $\mathcal{A}(u)(t) = A(u(t))$. Since $-u'_\alpha \in \mathcal{A}(J_\alpha u_\alpha)$ it follows from Proposition 1.6 that $-u' \in \mathcal{A}(u)$. This proves the first part of the following.

PROPOSITION 3.1 (KŌMURA-KATO). *Let A be m-accretive in the Hilbert space H. For each $u_0 \in D(A)$ there is a unique absolutely continuous $u : [0, \infty) \to H$ such that $u(0) = u_0$ and (3.1) holds at a.e. $t > 0$. Also, u is Lipschitz with $\|u'\|_{L^\infty(0,\infty;H)} \le \|A^0 u_0\|$; at every $t \ge 0$, $u(t) \in D(A)$ and the right-derivative satisfies*

$$(3.7) \qquad D^+ u(t) + A^0\big(u(t)\big) = 0 \; ;$$

and the function $t \mapsto A^0(u(t))$ is right-continuous with $\|A^0(u(t))\|$ decreasing.

PROOF (CONTINUED). Let $t \ge 0$. Since $J_\alpha(u_\alpha(t)) \to u(t)$ and (3.5) holds, it follows from Proposition 1.6 that $u(t) \in D(A)$. Moreover, some subsequence $A_{\alpha_n}(u_{\alpha_n}(t))$ converges weakly to a $w \in A(u(t))$ and by weak lower-semi-continuity of the norm

$$\|w\| \le \liminf_{n \to \infty} \|A_{\alpha_n}(u_{\alpha_n}(t))\| \le \|A^0(u_0)\| \; .$$

Thus, we obtain

$$\|A^0(u(t))\| \le \|A^0(u_0)\| \; , \qquad t \ge 0 \; .$$

Let $t_n \downarrow 0$ with $A^0(u(t_n)) \rightharpoonup \xi$. Then Proposition 1.6 shows $\xi \in A(u_0)$ and $\|\xi\| \le \liminf_{n \to \infty} \|A^0(u(t_n))\| \le \|A^0(u_0)\|$, so $\xi = A^0(u_0)$ and we have shown $w - \lim_{n \to \infty} A^0(u(t_n)) = A^0(u_0)$. Furthermore,

$$\|A^0(u_0)\| \le \liminf \|A^0(u(t_n))\| \le \limsup \|A^0(u(t_n))\| \le \|A^0(u_0)\|$$

so $\lim_{n \to \infty} \|A^0 u(t_n)\| = \|A^0(u_0)\|$. This shows the strong limit exists and satisfies $\lim_{n \to \infty} A^0 u(t_n) = A^0(u_0)$, so $A^0 u(t)$ is continuous at $t = 0$. But for each $t_0 \ge 0$, the translate $u(t + t_0)$ is the solution of (3.1) with initial data $u(t_0)$, so $\lim_{n \to \infty} A^0(u(t_n + t_0)) = A^0(u(t_0))$. That is, $A^0(u(t))$ is right-continuous at every $t_0 \ge 0$; also $\|A^0(u(t + t_0))\| \le \|A^0(u(t_0))\|$, so $\|A^0(u(t))\|$ is decreasing.

For all $t \ge 0$ and $h > 0$ we have similarly

$$\|u(t + h) - u(t)\| \le h\|A^0(u(t))\|$$

from (3.6), so dividing by $h > 0$ and taking the limit show

$$\|u'(t)\| \le \|A^0(u(t))\| \; , \qquad \text{a.e. } t \ge 0 \; .$$

With (3.1) this shows that

$$u'(t) + A^0(u(t)) = 0 \; , \qquad \text{a.e. } t \ge 0 \; .$$

Finally, from the representation

$$u(t) = u_0 - \int_0^t A^0(u(s)) \, ds \; , \qquad t \ge 0$$

and the right-continuity of the integrand we obtain (3.7) at every $t \ge 0$. □

We shall next show that the Cauchy problem for (3.1) can be solved with initial data u_0 in $\overline{D(A)}$ for those operators A which are subgradients. Such operators are quite special, even within the linear examples, because the solution of (3.1) will satisfy $u(t) \in D(A)$ for $t > 0$ even though one starts with $u(0) \in \overline{D(A)}$. In examples with partial differential equations this means $u(t)$ is strictly smoother in

its spatial variables than is $u(0)$, so this is generally called the *regularizing property* of (3.1).

In order to discuss this more easily we define for each $t \geq 0$ the function $S(t) : D(A) \rightarrow D(A)$ by $S(t)u_0 = u(t)$ where u is the solution of the Cauchy problem for (3.1) with $u(0) = u_0$ in the situation of Proposition 1. By taking the (uniformly) continuous extension of each $S(t)$ we obtain a family of functions $S(t) : \overline{D(A)} \rightarrow \overline{D(A)}$ which satisfies

(3.8.a) $\|S(t)u_1 - S(t)u_2\| \leq \|u_1 - u_2\|$, $t \geq 0$, $u_1, u_2 \in \overline{D(A)}$,

(3.8.b) $S(t_1 + t_2) = S(t_1) \circ S(t_2)$, $t_1, t_2 \geq 0$,

(3.8.c) $S(\cdot)u_0$ is continuous on $\mathbb{R}_+$ for each $u_0 \in \overline{D(A)}$.

This gives the (strongly) continuous *semigroup of contractions* generated by $-A$. The *regularizing property* is that $S(t) : \overline{D(A)} \rightarrow D(A)$ for $t > 0$.

PROPOSITION 3.2 (BREZIS). *Let $\varphi : H \rightarrow \mathbb{R}_\infty$ be proper, convex and lower-semi-continu-ous; set $A = \partial\varphi$ and let $\{S(t) : t \geq 0\}$ be the continuous semigroup of contractions generated by $-A$. Then for each $u_0 \in \overline{D(A)}$ and $t > 0$ it follows that $S(t)u_0 \in D(A)$ and*

$$(3.9) \qquad \|A^0(S(t)u_0)\| \leq \|A^0 v\| + \frac{1}{t}\|u_0 - v\| , \qquad v \in D(A) .$$

PROOF. From Proposition 1.8 we have $A_\alpha = \varphi'_\alpha$. Let u_α denote as before the solution of (3.5) with $u_\alpha(0) = u_0 \in \overline{D(A)}$. Fix $v \in H$ and $\alpha > 0$; define

$$\psi(w) = \varphi_\alpha(w) - \varphi_\alpha(v) - (A_\alpha v, w - v)_H , \qquad w \in H$$

and note $\psi'(w) = \varphi'_\alpha(w) - A_\alpha v$, $w \in H$, $\min \psi = \psi(v) = 0$, and

$$u'_\alpha(t) + \psi'(u_\alpha(t)) = -A_\alpha v , \qquad t \geq 0 .$$

Since $(\psi'(u_\alpha), v - u_\alpha) \leq \psi(v) - \psi(u_\alpha) = -\psi(u_\alpha)$ it follows that

$$\psi(u_\alpha(t)) \leq \left(u'_\alpha(t) + A_\alpha v, v - u_\alpha(t)\right)_H$$
$$= -\frac{1}{2}\frac{d}{dt}\|v - u_\alpha(t)\|^2 + \left(A_\alpha v, v - u_\alpha(t)\right)_H ,$$

so we obtain by an integration

$$(3.10) \quad \int_0^T \psi\left(u_\alpha(t)\right) dt \leq \frac{1}{2}\left(\|v - u_0\|^2 - \|v - u_\alpha(T)\|^2\right) + \int_0^T \left(A_\alpha v, v - u_\alpha(t)\right)_H dt .$$

This is the *energy estimate*. In order to estimate the derivative, u'_α, we take its scalar-product with its equation above, multiply by $t \geq 0$ and integrate to obtain,

successively,

$$t\|u_\alpha'(t)\|^2 + t\frac{d}{dt}\psi\big(u_\alpha(t)\big) = t\frac{d}{dt}\big(A_\alpha v, v - u_\alpha(t)\big)_H \ ,$$

$$\int_0^T t\|u_\alpha'(t)\|^2\,dt + T\psi\big(u_\alpha(T)\big) - \int_0^T \psi\big(u_\alpha(t)\big)\,dt$$

$$= T\big(A_\alpha v, v - u_\alpha(T)\big)_H + \int_0^T \big(A_\alpha v, u_\alpha(t) - v\big)_H\,dt$$

We combine this with (3.10) to get

$$\int_0^T t\|u_\alpha'(t)\|^2\,dt \le \frac{1}{2}\big(\|v - u_0\|^2 - \|v - u_\alpha(T)\|^2\big) + T\big(A_\alpha v, v - u_\alpha(T)\big)_H$$

(3.11) $$\le \frac{1}{2}\|u_0 - v\|^2 + \frac{1}{2}T^2\|A_\alpha v\|^2 \ .$$

The function $\|u_\alpha'(t)\| = \|A_\alpha^0(u_\alpha(t))\|$ is non-increasing in t, so the left side of (3.11) dominates $(T^2/2)\|u_\alpha'(T)\|^2$ and we obtain

$$\|A_\alpha u_\alpha(T)\|^2 \le \frac{1}{T^2}\|u_0 - v\|^2 + \|A_\alpha v\|^2 \ .$$

That is, we have the estimate

(3.12) $$\|A_\alpha\big(u_\alpha(T)\big)\| \le \|A^0 v\| + \frac{1}{T}\|u_0 - v\| \ , \qquad \alpha > 0 \ .$$

Let's show $u_\alpha(T) \to S(T)u_0$ as $\alpha \to 0$. If $v_0 \in D(A)$ and v_α is the solution of (3.5) with $v(0) = v_0$, then

$$\|u_\alpha(T) - S(T)u_0\| \le \|u_\alpha(T) - v_\alpha(T)\| + \|v_\alpha(T) - S(T)v_0\| + \|S(T)v_0 - S(T)u_0\|$$

$$\le 2\|v_0 - u_0\| + \|v_\alpha(T) - S(T)v_0\| \ .$$

Given $\varepsilon > 0$, choose v_0 and then α_0 so that $0 < \alpha \le \alpha_0$ implies $\|u_\alpha(T) - S(T)u_0\| < \varepsilon$. From the estimate (3.12) and

$$\|J_\alpha\big(u_\alpha(T)\big) - u_\alpha(T)\| = \alpha\|A_\alpha\big(u_\alpha(T)\big)\|$$

it is clear that $J_\alpha(u_\alpha(T)) \to S(T)u_0$. From Proposition 1.6 we obtain (as in the proof of Proposition 3.1) $S(T)u_0 \in D(A)$ and some subsequence $A_{\alpha_n}(u_{\alpha_n}(t)) \to y \in A(u(T))$, so (3.9) follows from (3.12) by weak lower-semi-continuity of the norm. $\square$

COROLLARY 3.2. *For each $u_0 \in \overline{D(A)}$ there is a unique solution u of (3.1) with $u(0) = u_0$. This solution satisfies the following:*
   $u(t) \in D(A)$ for every $t > 0$;
   u is Lipschitz on $[a, \infty)$ for every $a > 0$ with

$$\|\frac{du}{dt}\|_{L^\infty([a,\infty),H)} \le \|A^0 v\| + \frac{1}{a}\|u_0 - v\| \ , \quad v \in D(A) \ ;$$

and the function $\varphi(u(t))$ is non-increasing and convex with right-derivative given at every $t > 0$ by

$$(3.13) \qquad D^+\varphi\big(u(t)\big) = -\|D^+u(t)\|^2 .$$

PROOF. It remains yet to verify (3.13). From this it will follow that $\varphi(u(\cdot))$ is non-increasing; since $\|D^+u(t)\| = \|A^0(u(t))\|$ is non-increasing by Proposition 3.1, $\varphi(u(\cdot))$ is also convex. Furthermore, the non-increase of $\|D^+u\|$ gives

$$\|u(t+h) - u(t)\| \leq \int_t^{t+h} \|D^+u(s)\|\, ds \leq h\|D^+u(a)\| , \quad t \geq a , \ h > 0 .$$

From (3.7), where $A^0 \subset \partial\varphi$, we obtain

$$-\big(D^+u(t+h), u(t+h) - u(t)\big)_H \geq \varphi\big(u(t+h)\big) - \varphi\big(u(t)\big)$$
$$\geq -\big(D^+u(t), u(t+h) - u(t)\big)_H$$

and this gives with the preceding estimate

$$\big|\varphi\big(u(t+h)\big) - \varphi\big(u(t)\big)\big| \leq h\|D^+u(a)\|^2 , \qquad t \geq a , \ h > 0 ,$$

so $\varphi(u(\cdot))$ is Lipschitz on $[a, \infty)$ with right-derivative given by (3.13). $\qquad\square$

The behavior of $u(t)$ and $\varphi(u(t))$ for t near zero is described by the following.

COROLLARY 3.3. *In the situation of Proposition 2, if $u_0 \in \overline{D(A)}$ then $\varphi(u) \in L^1(0, a)$ and $\sqrt{t}\frac{du(t)}{dt} \in L^2(0, a; H)$. Furthermore we have $u_0 \in \mathrm{dom}(\varphi)$ if and only if $\frac{du}{dt} \in L^2(0, a; H)$ for each $a > 0$, and in that case*

$$(3.14) \qquad \varphi(u_0) - \varphi(u(t)) = \int_0^t \|\frac{du}{dt}\|^2 , \qquad t > 0 ,$$

so $\lim_{t\to 0} \varphi(u(t)) = \varphi(u_0)$ with monotone convergence, and

$$\|u(t) - u_0\| \leq \sqrt{t}\Big(\varphi(u_0) - \varphi\big(u(t)\big)\Big)^{1/2} , \qquad t > 0 ,$$
$$\sqrt{t}\|D^+u(t)\| \leq \Big(\varphi(u_0) - \varphi\big(u(t)\big)\Big)^{1/2} .$$

PROOF. Since $J_\alpha(u_\alpha(t)) \to u(t)$, $t > 0$, and

$$\varphi\big(J_\alpha(w)\big) \leq \varphi_\alpha(w) = \frac{1}{2\alpha}\|w - J_\alpha w\|^2 + \varphi(J_\alpha w)$$

we deduce from the energy estimate (3.10) with $T = a$ by Fatou's Lemma that $\varphi(u) \in L^1(0, a)$. From the estimate (3.11) we obtain $\sqrt{t}\frac{du}{dt} \in L^2(0, a; H)$ from weak convergence of the derivative.

Let $u_0 \in \mathrm{dom}(\varphi)$. From the counterpart of (3.13) for the approximate equation (3.5) follows

$$\varphi_\alpha(u_0) - \varphi_\alpha\big(u_\alpha(a)\big) = \int_0^a \|u_\alpha'(s)\|^2\, ds$$

and, therefore, as above

$$\varphi\big(J_\alpha u_\alpha(a)\big) + \int_0^a \|u_\alpha'(s)\|^2\, ds \le \varphi_\alpha(u_0) \le \varphi(u_0)\ .$$

Since $J_\alpha u_\alpha(a) \to u(a)$ as $\alpha \to 0$ we find in the limit

$$\varphi\big(u(a)\big) + \int_0^a \|u'(s)\|^2\, ds \le \varphi(u_0)\ .$$

Conversely, if $\frac{du}{dt} \in L^2(0,a;H)$ then from (3.13) we have

$$\varphi\big(u(t)\big) = \varphi\big(u(a)\big) + \int_t^a \|u'(s)\|^2\, ds\ , \qquad 0 < t < a\ ,$$

and taking the limit $t \to 0$ shows

$$\varphi(u_0) \le \varphi\big(u(a)\big) + \int_0^a \|u'(s)\|^2\, ds$$

by lower-semi-continuity, and therefore from above follows (3.14). The remaining estimates follow easily. That is, from (3.14) and the general

$$\|u(t) - u_0\| \le \int_0^t \|u'\|\, ds \le \sqrt{t}\Big(\int_0^t \|u'\|^2\Big)^{1/2}$$

follows the first, and from non-increase of $\|u'\|$ follows

$$t\|u'(t)\|^2 \le \int_0^t \|u'(s)\|^2\, ds$$

and, hence, the second. $\qquad\qquad\square$

IV.4. Additional Topics and Evolution Equations

We begin with some *Gronwall inequalities* that are frequently used in ordinary differential equations and that were implicit in some of our earlier estimates.

LEMMA 4.1. *Let $a(\cdot)$, $b(\cdot) \in L^1(0,T)$ with $b(t) \ge 0$ a.e., and let the absolutely continuous $v : [0,T] \to \mathbb{R}^+$ satisfy*

$$(1 - \alpha)v'(t) \le a(t)v(t) + b(t)v^\alpha(t)\ , \qquad a.e.\ t \in [0,T]\ ,$$

where $0 \le \alpha < 1$. Then

$$v^{1-\alpha}(t) \le v^{1-\alpha}(0)e^{\int_0^t a} + \int_0^t e^{\int_s^t a} b(s)\, ds\ , \quad 0 \le t \le T\ .$$

PROOF. Replace the non-negative $v(t)$ by $v(t) + \varepsilon$ for some $\varepsilon > 0$, and then divide by $(v(t)+\varepsilon)^\alpha$. Setting $w(t) = e^{-\int_0^t a}(v(t)+\varepsilon)^\alpha$, we have $w'(t) \le e^{-\int_0^t a} b(t)$, and after integrating this inequality and letting $\varepsilon \to 0$, the desired estimate follows. $\square$

In the situation of Lemma 4.1 with $\alpha = 0$ we can integrate the assumed estimate to get a related weaker assumption which leads to the same resulting bound.

LEMMA 4.2. *Let $a \in L^1(0,T)$ and let $B : [0,T] \to \mathbb{R}$ be absolutely continuous; assume the bounded measurable function v satisfies*

$$v(t) \leq \int_0^t a(s)v(s)\,ds + B(t) , \qquad 0 \leq t \leq T .$$

Then we have

$$v(t) \leq B(0)e^{\int_0^t a} + \int_0^t e^{\int_s^t a}B'(s)\,ds , \qquad 0 \leq t \leq T .$$

PROOF. Define $w(t) \equiv \int_0^t a(s)v(s)\,ds + B(t)$ and note that for a.e. $t \in [0,T]$

$$w'(t) \leq a(t)w(t) + B'(t) ,$$

so Lemma 4.1 applies with $\alpha = 0$. □

Alternatively one can define $w(t) = \int_0^t a(s)v(s)\,ds$ and note that

$$w'(t) \leq a(t)w(t) + a(t)B(t) .$$

Then Lemma 4.1 applies to give the following.

LEMMA 4.2'. *Let $a \in L^1(0,T)$ and $B \in L^\infty(0,T)$. Assume the bounded measurable function v satisfies a.e.*

$$v(t) \leq \int_0^t a(s)v(s)\,ds + B(t) , \qquad 0 \leq t \leq T .$$

Then we have

$$v(t) \leq B(t) + \int_0^t e^{\int_s^t a}a(s)B(s)\,ds , \qquad 0 \leq t \leq T .$$

Note that the conclusions in Lemma 4.2 and Lemma 4.2' are equivalent; this follows from an integration-by-parts.

Next we show that by modifying the proof of Proposition 3.1 we obtain the corresponding results for the more general equation

$$(4.1) \qquad \frac{du}{dt}(t) + A(u(t)) \ni \omega u(t) + f(t) .$$

THEOREM 4.1 (KATO). *Let A be m-accretive in the Hilbert space H and $\omega \geq 0$. For each $u_0 \in D(A)$ and absolutely continuous $f : [0,T] \to H$, there is a unique absolutely continuous $u : [0,T] \to H$, such that $u(0) = u_0$ and (4.1) holds at a.e. $t > 0$. Also, u is Lipschitz continuous and right-differentiable with $u(t) \in D(A)$ at every $t \geq 0$.*

PROOF. For any two solutions u_1, u_2 of (4.1) we have

$$\frac{1}{2}\frac{d}{dt}\|u_1(t) - u_2(t)\|^2 \leq \omega\|u_1(t) - u_2(t)\|^2 , \qquad t \geq 0 ,$$

since A is accretive, and from here follows from Lemma 4.1 with $\alpha = 0$

$$\|u_1(t) - u_2(t)\| \leq e^{\omega t}\|u_1(0) - u_2(0)\| , \qquad t \geq 0 .$$

Uniqueness is now immediate from the initial condition.

To obtain existence, let u_α be the unique solution for each $\alpha > 0$ of

$$(4.2) \qquad \frac{du_\alpha}{dt}(t) + A_\alpha(u_\alpha(t)) = \omega u_\alpha(t) + f(t) , \qquad 0 \leq t \leq T ,$$

with $u_\alpha(0) = u_0$. If $h > 0$, then $u_\alpha(t + h)$ is a solution of (4.2) with $f(t)$ replaced by $f(t + h)$, and the accretive estimate on A_α gives

$$\frac{1}{2}\frac{d}{dt}\|u_\alpha(t+h) - u_\alpha(t)\|^2 \leq \omega\|u_\alpha(t+h) - u_\alpha(t)\|^2 + \|f(t+h) - f(t)\| \, \|u_\alpha(t+h) - u_\alpha(t)\| .$$

Applying Lemma 4.1 with $\alpha = \frac{1}{2}$ to the preceding estimate and letting $h \to 0$ give

$$(4.3) \qquad \|u_\alpha'(t)\| \leq e^{\omega t}\| - A_\alpha(u_0) + \omega u_0 + f(0)\| + \int_0^t e^{\omega(t-s)}\|f'(s)\|\, ds .$$

Since $\|A_\alpha(u_0)\| \leq \|A^0(u_0)\|$, it follows with (4.2) that u_α', u_α, and $A_\alpha(u_\alpha)$ are bounded in $C([0, T], H)$.

To show $\{u_\alpha\}$ is Cauchy in $C([0, T], H)$, let $\alpha, \beta > 0$ and use (4.2) to obtain

$$\frac{1}{2}\frac{d}{dt}\|u_\alpha(t) - u_\beta(t)\|^2 = \omega\|u_\alpha(t) - u_\beta(t)\|^2 - (A_\alpha(u_\alpha(t)) - A_\beta(u_\beta(t)), u_\alpha(t) - u_\beta(t)) .$$

Writing $u_\alpha = \alpha A_\alpha u_\alpha + J_\alpha u_\alpha$ and similarly for u_β and using the accretive property of A, we obtain as in the proof of Proposition 3.1

$$\frac{1}{2}\frac{d}{dt}\|u_\alpha(t) - u_\beta(t)\|^2 \leq \omega\|u_\alpha(t) - u_\beta(t)\|^2 + \frac{\alpha + \beta}{4}K^2 ,$$

where $K \equiv \sup\{\|A_\alpha(u_\alpha(t)\| : 0 \leq t \leq T , \alpha > 0\}$. Thus we have

$$\|u_\alpha(t) - u_\beta(t)\|^2 \leq K^2\frac{\alpha + \beta}{2} \cdot \frac{e^{2\omega t} - 1}{2\omega} , \qquad 0 \leq t \leq T ,$$

so $\{u_\alpha\}$ is uniformly Cauchy and converges to a $u \in C([0, T], H)$ with

$$(4.4) \qquad \|u_\alpha(t) - u(t)\|^2 \leq K^2 \cdot \frac{\alpha}{2} \cdot \frac{e^{2\omega t} - 1}{2\omega} , \qquad 0 \leq t \leq T , \; \alpha > 0 ;$$

from (4.3) it follows that $u' \in L^\infty(0, T; H)$. Also $u_\alpha - J_\alpha u_\alpha = \alpha A_\alpha(u_\alpha) \to 0$ in $C([0, T], H)$. Since $J_\alpha u_\alpha \to u$ and u_α' is bounded in $L^2(0, T; H)$, from Proposition 1.6 we obtain (4.1). The additional properties of this solution are verified just as in Proposition 3.1. $\qquad\square$

COROLLARY 4.1. *Let A be an operator on H such that for some $\omega_1 \geq 0$, $A + \omega_1 I$ is m-accretive. Let $B : \overline{D(A)} \to H$ be a Lipschitz function such that for some $\omega_2 > 0$*

$$\|B(u) - B(v)\| \leq \omega_2 \|u - v\| , \qquad u, v \in \overline{D(A)} .$$

Then for each $\omega \geq 0$, $u_0 \in D(A)$ and absolutely continuous $f : [0,T] \to H$, there is a unique solution (as in Theorem 4.1) of

$$(4.5) \qquad \frac{du}{dt}(t) + A(u(t)) + B(u(t)) \ni \omega u(t) + f(t)$$

with $u(0) = u_0$.

PROOF. It suffices to add $(\omega_1 + \omega_2)u(t)$ to both sides of (4.5), and then note that $B + \omega_2 I$ is accretive. Thus by Lemma 2.1 we are in the situation of Theorem 4.1 with ω replaced by $\omega + \omega_1 + \omega_2$. $\qquad\square$

We consider the problem of finding a *periodic* solution to the equation (4.1), that is,

$$(4.6.\text{a}) \qquad \frac{du}{dt}(t) + A(u(t)) + \omega u(t) \ni f(t) , \qquad 0 < t < T ,$$

$$(4.6.\text{b}) \qquad u(0) = u(T) .$$

This is equivalent to finding a periodic solution of (4.6.a) on all of $\mathbb{R}$ with period $T > 0$. Note that there is a solution of (4.6) with $A = 0$, $\omega = 0$ only if $\int_0^T f(t)\,dt = 0$, and in that case any two solutions differ by a constant. Thus, we expect existence or uniqueness only with additional positivity hypotheses on $A + \omega I$.

PROPOSITION 4.1. *Let A be m-accretive, $\omega > 0$, and $f : [0,T] \to H$ be absolutely continuous. Then there exists a unique absolutely continuous solution of (4.6).*

PROOF. We shall consider (4.6) in $\mathcal{H} = L^2(0,T;H)$. Thus, let $\mathcal{A}$ be the realization of A in $\mathcal{H}$ (see Example 2.C) and let $L = \frac{d}{dt}$ with domain $D(L) = \{u \in \mathcal{H} : u' \in \mathcal{H} , u(0) = u(T)\}$. Then $\mathcal{A}$ and L are m-accretive on $\mathcal{H}$ and the uniqueness of a solution follows immediately. In order to show the existence of a solution we consider the approximating problem (cf., (2.5))

$$(4.7) \qquad Lu_\alpha + \mathcal{A}_\alpha(u_\alpha) + \omega u_\alpha = f$$

By Proposition 2.1 it suffices to show that $\{Lu_\alpha\} = \{u'_\alpha\}$ is bounded in $\mathcal{H}$. Extend each u_α and f to $\mathbb{R}$ as T-periodic and denote these extensions the same. If $h > 0$ we obtain from (4.7)

$$\frac{1}{2}\frac{d}{dt}\|u_\alpha(t+h) - u_\alpha(t)\|^2 + \omega\|u_\alpha(t+h) - u_\alpha(t)\|^2 \leq \|f(t+h) - f(t)\| \, \|u_\alpha(t+h) - u_\alpha(t)\| .$$

Thus for $0 \leq s < s + h \leq T$, $t = T + s$ we get from Lemma 4.1

$$e^{\omega t}\|u_\alpha(t+h) - u_\alpha(t)\| \leq e^{\omega s}\|u_\alpha(s+h) - u_\alpha(s)\| + \int_s^t e^{\omega\tau}\|f(\tau+h) - f(\tau)\|\,d\tau ,$$

and since u_α is T-periodic this gives

(4.8) $\|u_\alpha(s+h) - u_\alpha(s)\|(1 - e^{-\omega T})$

$$\leq \int_s^{T+s} \|f(\tau + h) - f(\tau)\| \, d\tau , \qquad 0 \leq s < s + h \leq T .$$

Denote the *variation* of f on $[0, t]$ by

$$V(t) = \sup\left\{ \sum_{j=0}^{n-1} \|f(t_{j+1}) - f(t_j)\| : 0 = t_0 < t_1 < \ldots < t_n = t \right\} ,$$

where the supremum is taken over all partitions of $[0, t]$; then note

$$V(\tau) + \|f(\tau + h) - f(\tau)\| \leq V(\tau + h) , \qquad \tau \geq 0 ,$$

so from (4.8) follows by monotonicity of V

$$\|u_\alpha(s+h) - u_\alpha(s)\|(1 - e^{-\omega T}) \leq \int_s^{T+s} (V(\tau + h) - V(\tau)) \, d\tau$$

$$\leq h(V(T + s + h) - V(s)) , \quad 0 \leq s < s + h \leq T .$$

Finally we note that since f is absolutely continuous on $[0, T]$ and on $[T, 2T]$ we have

$$V(T + s + h) - V(s) \leq \int_s^T \|f'\| \, d\tau + \|f(T) - f(0)\| + \int_T^{T+s+h} \|f'\| \, d\tau ,$$

so we let $h \searrow 0$ above to obtain

(4.9) $\|u_\alpha'(s)\| \leq (1 - e^{-\omega T})^{-1} \left(\|f'\|_{L^1(0,T)} + \|f(0) - f(T)\| \right)$, a.e. $s \in [0, T]$.

Thus $\{u_\alpha'\}$ is bounded in $\mathcal{H}$ and Proposition 2.1 gives existence. The solution u of (4.6) satisfies (4.9), so it is Lipschitz continuous; it is a solution of the Cauchy problem on $[\tau, T]$ for a.e. $\tau \in [0, T]$, so it has the additional properties described in Theorem 4.1. $\square$

We obtained in Theorem 4.1 the existence of a solution of the Cauchy problem

(4.10) $$\frac{du}{dt} + A(u) \ni f + \omega u , \qquad u(0) = u_0 ,$$

whenever f is absolutely continuous and $u_0 \in D(A)$. Moreover it follows that if u_1, u_2 are solutions of respective Cauchy problems with data f_1, f_2 in $L^1(0, T; H)$ and u_0^1, u_0^2 in $\overline{D(A)}$, then from accretivity of A follows

$$\frac{1}{2} \frac{d}{dt} e^{-2\omega t} \|u_1(t) - u_2(t)\|^2 \leq e^{-2\omega t} (f_1(t) - f_2(t), u_1(t) - u_2(t)) .$$

This leads directly to

$$\frac{1}{2} e^{-2\omega t} \|u_1(t) - u_2(t)\|^2 \leq \frac{1}{2} e^{-2\omega s} \|u_1(s) - u_2(s)\|^2$$

(4.11) $$+ \int_s^t e^{-2\omega \tau} (f_1 - f_2, u_1 - u_2) \, d\tau , \qquad 0 \leq s \leq t \leq T ,$$

and with Lemma 4.1 it also leads to

$$e^{-\omega t}\|u_1(t) - u_2(t)\| \leq e^{-\omega s}\|u_1(s) - u_2(s)\|$$

(4.12)
$$+ \int_s^t e^{-\omega\tau}\|f_1(\tau) - f_2(\tau)\|\,d\tau\ , \qquad 0 \leq s \leq t \leq T\ .$$

The estimate (4.12) with $s = 0$ leads to the following notion.

DEFINITION. A *generalized solution* of (4.1) is a function $u \in C([0,T], H)$ for which there exists a sequence of (absolutely continuous) solutions u_n of

$$\frac{du_n}{dt} + A(u_n) \ni f_n + \omega u_n\ , n \geq 1$$

with $f_n \to f$ in $L^1(0,T;H)$ and $u_n \to u$ in $C([0,T], H)$.

Since (4.11) and (4.12) are stable with respect to such limits, they hold also for generalized solutions, and thus we obtain uniqueness for the Cauchy problem (4.10). Also, we obtain from Theorem 4.1 the following immediate extension.

THEOREM 4.1A. *If A is m-accretive on H, $\omega \geq 0$, $f \in L^1(0,T;H)$ and $u_0 \in \overline{D(A)}$, then there is a unique generalized solution of (4.10) on $[0,T]$, and any two generalized solutions of respective problems satisfy the estimates (4.11) and (4.12).*

The existence follows by approximating the given f and u_0 by a sequence of absolutely continuous $\{f_n\}$ and a sequence $\{u_0^n\}$ in $D(A)$, respectively. The estimates (4.11) and (4.12) hold in the limit as indicated above.

By the same proof we obtain a generalized solution of the Cauchy problem for (4.5) with the relaxed assumptions on u_0 and f.

COROLLARY 4.1A. *With A, B given as in Corollary 1, for each $u_0 \in \overline{D(A)}$ and $f \in L^1(0,T;H)$ there is a unique generalized solution of (4.5) with $u(0) = u_0$.*

Moreover, one obtains a generalized solution of the periodic problem (4.6) for this wider class of functions, f.

PROPOSITION 4.1A. *Let A be m-accretive, $\omega > 0$, and $f \in L^1(0,T;H)$. Then there exists a unique generalized solution of (4.6).*

PROOF. This follows directly from Proposition 4.1 by approximating f with a sequence of absolutely continuous functions.

Alternatively, one can solve (4.6) directly by a fixed-point argument based on the periodicity condition (4.6.b). For each $u_0 \in \overline{D(A)}$ there is a unique generalized solution u of (4.6.a) with $u(0) = u_0$; define $\mathcal{F} : \overline{D(A)} \to \overline{D(A)}$ by $\mathcal{F}(u_0) = u(T)$. Thus, (4.6) is equivalent to $\mathcal{F}(u_0) = u_0$, and $\mathcal{F}$ is a strict contraction if $\omega > 0$. $\square$

The notion of a generalized solution of (4.1) is based directly on the estimate (4.12) which implies the solution of Cauchy problem (4.10) depends continuously on the pair $[f, u_0]$ in $L^1(0,T;H) \times H$. We shall show the solution depends continuously on the *triple* $[A, f, u_0]$ with the appropriate notion of convergence of operators. We set $\omega = 0$ without loss of generality.

THEOREM 4.2 (BREZIS-PAZY-MIYADERA-OHARU). *Let A^n, $n \geq 1$, and A be m-accretive in the Hilbert space H, let f_n and f in $L^1(0,T;H)$, $u_0^n \in \overline{D(A^n)}$ and $u_0 \in \overline{D(A)}$ be given. Let u^n, u be the respective generalized solutions of*

$$\frac{du^n}{dt} + A^n(u^n) \ni f_n , \qquad u^n(0) = u_0^n ,$$

$$\frac{du}{dt} + A(u) \ni f , \qquad u(0) = u_0 .$$

If $u_0^n \to u_0$ in H, $f_n \to f$ in $L^1(0,T;H)$, and if

(4.13) $$(I + \alpha A^n)^{-1} w \to (I + \alpha A)^{-1} w , \qquad w \in H , \ \alpha > 0 ,$$

then $u^n \to u$ in $C([0,T],H)$.

PROOF. We shall proceed as follows. For the case of $f_n = f = 0$, we regularize the initial data to get nearby (absolutely continuous) solutions, regularize the operator to get the Lipschitz case, then estimate directly the Lipschitz case with the Gronwall type inequalities. The corresponding forced equations, with f_n and f as given, are thereafter easily obtained.

Consider the case $f_n = f = 0$ and let $u, v_\alpha, w_\alpha, w_\alpha^n, v_\alpha^n, u^n$ be the generalized solutions of the following Cauchy problems with $n \geq 1$, $\alpha > 0$,

$$\frac{du}{dt} + A(u) \ni 0 , \qquad u(0) = u_0 \in \overline{D(A)} ,$$

$$\frac{dv_\alpha}{dt} + A(v_\alpha) \ni 0 , \qquad v_\alpha(0) = (I + \sqrt{\alpha}A)^{-1} u_0 \in D(A) ,$$

$$\frac{dw_\alpha}{dt} + A_\alpha(w_\alpha) = 0 , \qquad w_\alpha(0) = (I + \sqrt{\alpha}A)^{-1} u_0 ,$$

$$\frac{dw_\alpha^n}{dt} + A_\alpha^n(w_\alpha^n) = 0 , \qquad w_\alpha^n(0) = (I + \sqrt{\alpha}A^n)^{-1} u_0^n ,$$

$$\frac{dv_\alpha^n}{dt} + A^n(v_\alpha^n) \ni 0 , \qquad v_\alpha^n(0) = (I + \sqrt{\alpha}A^n)^{-1} u_0^n \in D(A^n) ,$$

$$\frac{du^n}{dt} + A^n(u^n) \ni 0 , \qquad u^n(0) = u_0^n \in \overline{D(A^n)} .$$

Since A and A^n are accretive we obtain from (4.12)

$$\|u - v_\alpha\|_{C([0,T],H)} \leq \|u_0 - J_{\sqrt{\alpha}} u_0\|_H ,$$
$$\|u^n - v_\alpha^n\|_{C([0,T],H)} \leq \|u_0^n - J_{\sqrt{\alpha}}^n u_0^n\|_H .$$

Using the convergence-rate estimate from the proof of Proposition 3.1, and then observing $A_{\sqrt{\alpha}} \subset A \circ J_{\sqrt{\alpha}}$, one finds

$$\|v_\alpha - w_\alpha\|_{C([0,T],H)} \leq \left(\frac{\alpha T}{2}\right)^{1/2} \|A^0(J_{\sqrt{\alpha}} u_0)\|$$

$$\leq \left(\frac{\alpha T}{2}\right)^{1/2} \|\frac{1}{\sqrt{\alpha}} (I - J_{\sqrt{\alpha}}) u_0\| ,$$

and thus we have

$$\|v_\alpha - w_\alpha\|_{C([0,T],H)} \le \left(\frac{T}{2}\right)^{1/2} \|u_0 - J_{\sqrt{\alpha}} u_0\| \ ,$$

$$\|v_\alpha^n - w_\alpha^n\|_{C([0,T],H)} \le \left(\frac{T}{2}\right)^{1/2} \|u_0^n - J_{\sqrt{\alpha}}^n u_0^n\| \ .$$

In order to estimate the remaining link, $w_\alpha - w_\alpha^n$, we subtract their corresponding equations, integrate over the interval $[0,t]$, and then use the Lipschitz continuity of A_α^n to obtain

$$\|w_\alpha(t) - w_\alpha^n(t)\| \le \|w_\alpha(0) - w_\alpha^n(0)\| + \int_0^t \|A_\alpha(w_\alpha) - A_\alpha^n(w_\alpha^n)\| \, ds$$

$$\le \|w_\alpha(0) - w_\alpha^n(0)\| + \int_0^T \|A_\alpha(w_\alpha) - A_\alpha^n(w_\alpha)\| \, ds$$

$$+ \frac{1}{\alpha} \int_0^t \|w_\alpha(s) - w_\alpha^n(s)\| \, ds \ .$$

From Lemma 4.2 and the definition of A_α, A_α^n we get

$$\|w_\alpha - w_\alpha^n\|_{C([0,T],H)} \le \left(\|J_{\sqrt{\alpha}} u_0 - J_{\sqrt{\alpha}}^n u_0^n\| + \frac{1}{\alpha}\|J_\alpha w_\alpha - J_\alpha^n w_\alpha\|_{L^1(0,T;H)}\right) e^{T/\alpha} \ .$$

From (4.13) and the Lebesgue Theorem it follows that $\|J_\alpha w_\alpha - J_\alpha^n w_\alpha\|_{L^1(0,T;H)} \to 0$ as $n \to \infty$. In summary, we have for every $\alpha > 0$,

$$\limsup_{n\to\infty} \|u - u^n\|_{C([0,T],H)} \le 2\left(1 + \left(\frac{T}{2}\right)^{1/2}\right) \|u_0 - J_{\sqrt{\alpha}} u_0\| \ ,$$

so it follows that $u^n \to u$ in $C([0,T], H)$.

For the case of $f_n = f = \xi$, a constant vector in H, the result follows directly from above by considering the translated operators $A(u) - \xi$, $A^n(u) - \xi$; one need only check that (4.13) holds. Thus the result holds also for the case $f_n = f = g$, where g is a piece-wise constant function in $L^1(0,T;H)$. If v, v^n are the generalized solutions of

$$\frac{dv}{dt} + A(v) \ni g \ , \qquad v(0) = u_0 \ ,$$

$$\frac{dv^n}{dt} + A^n(v^n) \ni g \ , \qquad v^n(0) = u_0^n \ ,$$

then from (4.13) follows

$$\|u - u^n\|_{C([0,T],H)} \le \|u - v\| + \|v - v^n\| + \|v^n - u^n\|$$

$$\le \|f - g\|_{L^1(0,T;H)} + \|v - v^n\|_{C([0,T],H)} + \|g - f_n\|_{L^1(0,T;H)} \ ,$$

so we have

$$\limsup_{n\to\infty} \|u - u^n\|_{C([0,T],H)} \le 2\|f - g\|$$

for every piece-wise constant function g. But these are dense in $L^1(0,T;H)$, so the desired result follows. $\qquad\square$

We consider next those m-accretive operators which are subgradients and the effects of the *regularizing property* of (4.1). From Theorem 4.1A we know there is a unique generalized solution u of (4.1) when $u_0 \in \overline{D(A)}$ and $f \in L^1(0,T;H)$. It follows from Theorem 4.1 that if $u_0 \in D(A)$ and f is absolutely continuous, then u is absolutely continuous. However, Corollary 3.2 shows that for the case $f = 0$, if A is a subgradient, the generalized solution is *always* absolutely continuous. We shall extend this last result to the equation (4.1) with $f \in L^2(0,T;H)$.

In the proof of Corollary 3.2 we made our estimates on the smooth approximations to the operator. In order to estimate directly on the equation with general data, we begin with the following *chain rule*.

LEMMA 4.3. *Let $\varphi : H \to \mathbb{R}_\infty$ be proper, convex, and lower-semi-continuous on the Hilbert space H. Denote the subgradient by $\partial\varphi$. If $u, \frac{du}{dt} \in L^2(0,T;H)$ and if there exists a $g \in L^2(0,T;H)$ with $g \in \partial\varphi(u)$ a.e. on $[0,T]$, then the function $\varphi \circ u$ is absolutely continuous on $[0,T]$ and*

$$\frac{d}{dt}\varphi(u(t)) = \left(h(t), \frac{du}{dt}(t)\right), \qquad a.e. \ \ t \in [0,T]$$

for any function h with $h \in \partial\varphi(u)$ a.e. on $[0,T]$.

PROOF. Set $A = \partial\varphi$ and recall from Proposition 1.9 that $A_\alpha = \varphi'_\alpha$, the Frechet derivative of the convex φ_α, for each $\alpha > 0$. By the usual chain rule we have

$$\frac{d}{dt}\varphi_\alpha(u(t)) = \left(A_\alpha(u(t)), \frac{du}{dt}(t)\right), \qquad \text{a.e. } t \in [0,T],$$

so there follows

$$\varphi_\alpha(u(t)) - \varphi_\alpha(u(s)) = \int_s^t \left(A_\alpha(u), \frac{du}{dt}\right) d\tau, \qquad s,t \in [0,T].$$

Letting $\alpha \to 0$ yields

$$\varphi(u(t)) - \varphi(u(s)) = \int_s^t \left(A^0(u), \frac{du}{dt}\right) d\tau, \qquad s,t \in [0,T],$$

since the existence of the function g as above shows the integrand belongs to $L^1(0,T;H)$, so $\varphi \circ u$ is absolutely continuous. Pick a $t \in (0,T)$ for which $u(t) \in D(A)$ and $u'(t), (\varphi \circ u)'(t)$ exist. For any vector $\xi \in \partial\varphi(u(t))$ we have

$$(\xi, v - u(t)) \leq \varphi(v) - \varphi(u(t)), \qquad v \in H.$$

By setting $v = u(t \pm \varepsilon)$, dividing by ε, and letting $\varepsilon \to 0$, we obtain

$$(\xi, u'(t)) = \frac{d}{dt}\varphi(u(t))$$

as desired. $\qquad\qquad\square$

THEOREM 4.3 (BREZIS). *Let $\varphi : H \to \mathbb{R}_\infty$ be proper, convex, and lower-semi-continuous on the Hilbert space H and set $A = \partial\varphi$. Let u be the generalized solution of*

$$\frac{du}{dt} + A(u) \ni f \ \text{a.e. on } [0, T] \ , \qquad u(0) = u_0$$

with $f \in L^2(0, T; H)$ and $u_0 \in \overline{D(A)}$. Then

$$\varphi \circ u \in L^1(0, T) \ , \quad \sqrt{t}\,\frac{du}{dt} \in L^2(0, T; H) \ , \quad \text{and } u(t) \in D(A) \ , \ \text{a.e. } t \in [0, T] \ .$$

If in addition $u_0 \in \operatorname{dom}(\varphi)$, then

$$\varphi \circ u \in L^\infty(0, T) \quad , \quad \frac{du}{dt} \in L^2(0, T; H) \ .$$

PROOF. We may assume without loss of generality that

$$\varphi(x_0) = 0 = \min\{\varphi(x) : x \in H\} \ .$$

Otherwise, pick any $[x_0, y_0] \in \partial\varphi$ and set $\psi(u) = \varphi(u) - \varphi(x_0) - (y_0, u - x_0)$. Then our equation is equivalent to $\frac{du}{dt} + \partial\psi(u) \ni f - y_0$.

We proceed as in Proposition 3.2. At first we assume $u_0 \in D(A)$ and $f' \in L^2(0, T; H)$. Since $f - u' \in \partial\varphi(u)$ we have

$$\varphi(u(t)) \le (f(t) - u'(t) \, , u(t) - x_0) \ , \quad \text{a.e. } t \in [0, T] \ .$$

so we obtain

$$\int_0^T \varphi(u(t))\, dt \le \frac{1}{2}\|u_0 - x_0\|^2 + \int_0^T \|f(t)\|\|u(t) - x_0\|\, dt \ .$$

Also, by applying $u(t) - x_0$ to our evolution equation and using Lemma 4.1 we get

$$\|u(t) - x_0\| \le \|u_0 - x_0\| + \int_0^T \|f(t)\|\, dt$$

so this gives with the above

$$(4.14) \qquad \int_0^T \varphi(u(t))\, dt \le \left(\|u_0 - x_0\| + \int_0^T \|f\|\, dt\right)^2 \ .$$

This is the *energy estimate*. Next we take the scalar product of our equation with $tu'(t)$ and use Lemma 4.3 to obtain

$$t\|u'(t)\|^2 + t\frac{d}{dt}\varphi(u(t)) = t(f(t), u'(t)) \ ,$$

and therefore

$$\int_0^T t\|u'(t)\|^2\, dt + T\varphi(u(T)) = \int_0^T t(f(t), u'(t))\, dt + \int_0^T \varphi(u(t))\, dt \ .$$

Since φ is non-negative this leads to the *rate estimate*

$$(4.15) \qquad \int_0^T t\|u'(t)\|^2\, dt \le \int_0^T t\|f(t)\|^2\, dt + 2\int_0^T \varphi(u(t))\, dt \ .$$

By combining (4.14) and (4.15) we have

$$(4.16) \qquad \int_0^T t\|u'(t)\|^2 \, dt \le \int_0^T t\|f(t)\|^2 \, dt + 2\Big(\|u_0 - x_0\| + \int_0^T \|f(t)\| \, dt\Big)^2 \, .$$

Assume $u_0 \in \overline{D(A)}$ and that $f \in L^2(0,T;H)$. Let $\{u_0^n\}$ be a sequence in $D(A)$ with $u_0^n \to u_0$ in H and let $\{f_n\}$ be a sequence with each $f_n' \in L^2(0,T;H)$ and with $f_n \to f$ in $L^2(0,T;H)$. Then it follows from (4.12) that the solutions u^n corresponding to the data u_0^n, f_n, converge uniformly to u on $[0,T]$. From (4.16) applied to $\{u^n\}$ it follows that $\{\sqrt{t}\frac{du^n}{dt}\}$ converges weakly in $L^2(0,T;H)$ to $\sqrt{t}\frac{du}{dt}$. Thus u is absolutely continuous on each $[\delta,T]$, $\delta > 0$, and we show as before that u satisfies the equation (4.1) with $\omega = 0$ in each $L^2(\delta,T;H)$ since $A = \partial\varphi$ is m-accretive in that space.

Finally we assume $u_0 \in \text{dom}(\varphi)$. Take the scalar product of the equation with $u'(t)$ and use Lemma 4.3 to obtain

$$\|u'(t)\|^2 + \frac{d}{dt}\varphi(u(t)) \le \|f(t)\| \, \|u'(t)\| \, ,$$

and therefore at a.e. $t \in [0,T]$

$$\frac{1}{2}\|u'(t)\|^2 + \frac{d}{dt}\varphi(u(t)) \le \frac{1}{2}\|f(t)\|^2 \, .$$

By integrating over $[s,t] \subset (0,T]$ and then letting $s \to 0^+$ we find

$$\max\{\tfrac{1}{2}\|u'\|_{L^2(0,T;H)} \, , \|\varphi \circ u\|_{C([0,T],H)}\} \le \varphi(u_0) + \tfrac{1}{2}\|f\|_{L^2(0,T;H)} \, ,$$

and this finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

We consider finally the *singular-in-time* evolution equation

$$(4.17) \qquad \frac{1}{b(\tau)}\frac{dv(\tau)}{d\tau} + \omega v(\tau) + A(v(\tau)) \ni g(\tau) \, , \qquad -\infty \le \tau_0 < \tau < \tau_1 \, ,$$

where b is a locally-integrable positive function on the interval (τ_0, τ_1). The effect of the singularity at τ_0 and at τ_1 can be determined from the change-of-variable $t = B(\tau)$ where B is absolutely continuous with $B'(\tau) = b(\tau)$ a.e. on (τ_0, τ_1). Thus v is a solution of (4.17) if and only if the function $u = v \circ B^{-1}$ satisfies

$$(4.18) \qquad \frac{du(t)}{dt} + \omega u(t) + A(u(t)) \ni f(t)$$

on the interval $\big(B(\tau_0), B(\tau_1)\big)$ with $f = g \circ B^{-1}$. If b is *weakly singular* at τ_0, i.e., if $b(\cdot)$ is integrable at τ_0, so $B(\tau_0) > -\infty$, then the Cauchy problem is appropriate for (4.17) and one obtains a well-posed problem by specifying initially $v(\tau_0) = v_0$. But if b is *strongly singular* at τ_0, so that $B(\tau_0) = -\infty$, then we are led to a *history-value problem*: find a solution of (4.18) on $(-\infty, 0]$. Of course one can use the preceding results on the Cauchy problem to uniquely extend this solution to all of $(-\infty, B(\tau_1))$.

We shall work in the Hilbert space $\mathcal{H} \equiv L^2(-\infty, 0; H)$. Let L be the operator given by $Lu = \frac{du}{dt}$ on the domain $D(L) = \{u \in \mathcal{H} : \frac{du}{dt} \in \mathcal{H}\}$.

LEMMA 4.4. *L is m-accretive on $\mathcal{H}$ and for each $u \in D(L)$, $\lim_{t \to -\infty} u(t) = 0$. For each $\alpha > 0$, $u + \alpha L u = f$ if and only if*

$$(4.19) \qquad u(t) = \int_{-\infty}^{t} f(s) \frac{1}{\alpha} e^{s-t/\alpha} \, ds \;, \qquad -\infty < t < 0 \;.$$

PROOF. If $(I + \alpha L)u = f$, then

$$e^{t/\alpha} u(t) - e^{s/\alpha} u(s) = \int_{s}^{t} f(r) \frac{1}{\alpha} e^{r/\alpha} \, dr \;, \qquad s < t \leq 0 \;,$$

and this gives

$$\|e^{t/\alpha} u(t) - e^{s/\alpha} u(s)\|^2 \leq \int_{s}^{t} \|f\|^2 \, dr \frac{1}{2\alpha} \left(e^{2t/\alpha} - e^{2s/\alpha} \right) \;.$$

This shows that $\lim_{t \to -\infty} e^{t/\alpha} u(t) \equiv h$ exists in H and that

$$u(t) = h e^{-t/\alpha} + \int_{-\infty}^{t} f(s) \frac{1}{\alpha} e^{s-t/\alpha} \, ds \;, \qquad t \leq 0 \;.$$

We shall see below that the second term belongs to $\mathcal{H}$; the first is in $\mathcal{H}$ only if $h = 0$, so it follows that $u(-\infty) = 0$.

Next let $f \in \mathcal{H}$, $\alpha > 0$, and define u by (4.19). The integral converges because $e^{t/\alpha} \in \mathcal{H}$. Note also that

$$\int_{-\infty}^{t} \frac{1}{\alpha} e^{s-t/\alpha} \, ds = 1$$

so the convexity of the norm-squared shows that

$$(4.20) \qquad \|u(t)\|^2 \leq \int_{-\infty}^{t} \|f(s)\|^2 \frac{1}{\alpha} e^{s-t/\alpha} \, ds \;.$$

From Fubini's Theorem we get

$$\|u\|_{\mathcal{H}}^2 \leq \int_{-\infty}^{0} \int_{s}^{0} \|f(s)\|^2 \frac{1}{\alpha} e^{s-t/\alpha} \, dt \, ds \leq \|f\|_{\mathcal{H}}^2 \;,$$

so $u \in D(L)$ and L is m-accretive. $\qquad \square$

COROLLARY 4.2. *For each $u \in D(L)$, $\alpha > 0$,*

$$\sup\{\|u(t)\| : t \leq 0\} \leq \frac{1}{\sqrt{2\alpha}} \|(I + \alpha L)u\|_{\mathcal{H}} \;.$$

PROOF. This is immediate from (4.19). $\qquad \square$

PROPOSITION 4.2. *Let A be m-accretive on H, $A(0) \ni 0$, and $\omega > 0$. Then for each $f \in D(L)$ there exists a unique $u \in D(L)$ for which (4.18) holds at a.e. $t \leq 0$, that is, $Lu + \omega u + A(u) \ni f$ in $\mathcal{H}$.*

PROOF. Let A_α be the accretive Lipschitz approximation of A and consider for each $\alpha > 0$ the solution u_α (guaranteed by Lemma 2.1) of

(4.21) $$L u_\alpha + \omega u_\alpha + A_\alpha(u_\alpha) = f \ .$$

From Proposition 2.1 and Lemma 2.2 it follows that (4.18) has a solution if (and only if) $\{L u_\alpha\}$ is bounded in $\mathcal{H}$. To verify this, let $h > 0$ and extend f and the solutions u_α to $(-\infty, h)$. By the accretive estimate on A_α we obtain successively

$$\frac{1}{2}\frac{d}{dt}\|u_\alpha(t+h) - u_\alpha(t)\|^2 + \omega\|u_\alpha(t+h) - u_\alpha(t)\|^2$$
$$\leq \big(f(t+h) - f(t) \, , u_\alpha(t+h) - u_\alpha(t)\big)_H$$
$$\leq \frac{1}{2}\Big\{\frac{1}{\omega}\|f(t+h) - f(t)\|^2 + \omega\|u_\alpha(t+h) - u_\alpha(t)\|^2\Big\} \ ,$$

and then

$$\|u_\alpha(t+h) - u_\alpha(t)\|^2 e^{\omega t} \leq \|u_\alpha(\tau+h) - u_\alpha(\tau)\|^2 e^{\omega\tau} + \frac{1}{\omega}\int_\tau^t \|f(s+h) - f(s)\|^2 e^{\omega s}\, ds \ ,$$

Since $\lim_{\tau \to -\infty}(u_\alpha(\tau)e^{\omega\tau} = 0$, we get

$$\|u_\alpha(t+h) - u_\alpha(t)\|^2 \leq \frac{1}{\omega}\int_{-\infty}^t \|f(s+h) - f(s)\|^2 e^{\omega(s-t)}\, ds \ ,$$

and then by Fubini's Theorem

$$\int_{-\infty}^0 \|u_\alpha(t+h) - u_\alpha(t)\|^2\, dt \leq \frac{1}{\omega}\int_{-\infty}^0 \|f(s+h) - f(s)\| \int_s^0 e^{\omega(s-t)}\, dt\, ds$$
$$\leq \frac{1}{\omega^2}\int_{-\infty}^0 \|f(s+h) - f(s)\|^2\, ds \ .$$

After dividing by h^2 and letting $h \to 0$ we have

$$\Big\|\frac{du_\alpha}{dt}\Big\|_\mathcal{H} \leq \frac{1}{\omega}\Big\|\frac{df}{dt}\Big\|_\mathcal{H} \ , \qquad \alpha > 0 \ ,$$

and this completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

When the operator A is a subgradient, we can as usual relax the hypotheses on the data.

PROPOSITION 4.3. *Let* $\varphi : H \to \mathbb{R}_\infty$ *be convex, lower-semi-continuous, and* $\varphi(0) = 0 = \min\{\varphi(x) : x \in H\}$. *Then for each* $\omega > 0$ *and* $f \in \mathcal{H}$ *there is a unique* $u \in D(L)$ *for which* (4.18) *holds at a.e.* $t \leq 0$.

PROOF. We shall apply Proposition 2.2 to show $L + \partial\Phi$ is m-accretive, where

$$\Phi(u) \equiv \int_{-\infty}^0 \varphi(u(t))\, dt \ , \qquad u \in \mathcal{H} \ .$$

Let $u \in \mathcal{H}$ and $(I + \alpha L)u_\alpha = u$, $\alpha > 0$. Using the representation (4.19) and the convexity of φ we get

$$\varphi(u_\alpha(t)) \leq \int_\infty^t \varphi(u(s))\frac{1}{\alpha}e^{\frac{s-t}{\alpha}}\, ds \ .$$

From this follows

$$\Phi(u_\alpha) \le \int_{-\infty}^{0} \varphi(u(s)) \int_{s}^{0} \frac{1}{\alpha} e^{\frac{s-t}{\alpha}} \, dt \, ds$$

by Fubini, and hence $\Phi(u_\alpha) \le \Phi(u)$ as desired. $\square$

IV.5. Parabolic Equations and Inequalities

We shall apply the preceding results on the solvability of the abstract Cauchy problem (4.10) to the case of operators A constructed from monotone operators which correspond to elliptic boundary-value problems. Many such examples were given in II.5 and we shall refer to them in the following. As we have seen above, sharper results are possible for the class of subgradient operators, and we briefly explore how these are obtained from corresponding subdifferentials; see II.8 and IV.2 for examples. The results of this section should be compared with those of III.4; there we resolved the Cauchy problems in somewhat more generality, whereas, here we shall obtain solutions which are more regular.

Assume V is a reflexive and separable Banach space which is dense and continuously imbedded in a Hilbert space H. We identify H with its dual H' by the Riesz map and thereby identify H as a subspace of V' with

$$f(v) = (f, v)_H \ , \qquad f \in H \ , \ v \in V \ .$$

Assume we are given a function $\mathcal{A} : V \to V'$ which is monotone and demicontinuous. Then define an operator A on H with domain $D(A) = \{u \in V : \mathcal{A}(u) \in H\}$ and values $Au = \mathcal{A}(u)$, $u \in D(A)$. Since

$$(Au - Av, u - v)_H = \langle \mathcal{A}u - \mathcal{A}v, u - v \rangle \ , \qquad u, v \in D(A) \ ,$$

and since $\mathcal{A}$ is monotone, it follows that A is accretive on H. We define an operator A on H to be *regular accretive* if it is so constructed from $\mathcal{A}, V$ and H as above.

LEMMA 5.1. *Let A be regular accretive on H and assume*

$$(5.1) \qquad \lim_{\|v\| \to +\infty} \frac{|v|_H^2 + \mathcal{A}v(v)}{|v|_H} = +\infty \ .$$

Then A is m-accretive.

PROOF. Let $f \in H$. In order to show that $f \in Rg(I + \mathcal{A})$ it suffices by Theorem II.2.2 to show there is a $\rho > 0$ such that $(I + \mathcal{A})v(v) > (f, v)_H$ for all $v \in V$ with $\|v\| > \rho$. Suppose on the contrary that $v \in V$ with

$$|v|_H^2 + \mathcal{A}v(v) \le |f|_H |v|_H \ .$$

That is, $\big(|v|_H^2 + \mathcal{A}v(v)\big)/|v|_H \le |f|_H$. Then (5.1) shows there is a ρ for which $\|v\| \le \rho$. Thus, there is a $u \in V : u + \mathcal{A}u = f$, and it is clear that $u \in D(A)$, so $(I + A)u = f$. $\square$

REMARK. Assume there is a seminorm $[\cdot]$ on V and numbers $\lambda > 0$, $\alpha > 0$ such that

$$[v] + \lambda |v|_H \geq \alpha \|v\| , \qquad \text{and}$$
$$\mathcal{A}v(v) \geq \alpha [v]^p - \lambda |v|_H , \qquad v \in V .$$

Then (5.1) holds. Recall that these assumptions were used in Proposition III.4.1 and Corollary III.4.1 to resolve the Cauchy problem. Lemma 5.1 leads to the following result which gives a more regular solution when the data is appropriately more regular.

PROPOSITION 5.1. *If A is regular accretive and (5.1) holds, then for each $u_0 \in D(A)$ and $f \in W^{1,1}(0,T;H)$ there exists a unique $u \in W^{1,\infty}(0,T;H)$ with $u(t) \in D(A)$ for all $0 \leq t \leq T$, $u(0) = u_0$, and*

$$(5.2) \qquad \frac{du}{dt}(t) + A\big(u(t)\big) = f(t) , \qquad a.e. \ t \in (0,T) .$$

Also, the right-derivative satisfies

$$D^+ u(t) + A\big(u(t)\big) = f(t) , \qquad t \in [0,T) .$$

PROOF. This follows directly from Lemma 5.1 and Theorem 4.1. $\square$

It is instructive to compare Proposition 5.1 with Proposition III.4.1. In the latter we are permitted a time-dependent family of operators, but with an explicit uniform growth rate from V to V'. With more general data, $f \in L^{p'}(0,T;V')$ and $u_0 \in H$, we obtain from Proposition III.4.1 a solution $u \in L^p(0,T;V)$ for which the equation (5.2) holds in $L^{p'}(0,T;V)$. By comparison, we need for Proposition 5.1 more restrictions on the data but we obtain a solution u satisfying stronger properties; in particular, the equation (5.2) holds in H. The requirement that $u(t) \in D(A)$ is specifically related to boundary conditions when $\mathcal{A}$ corresponds to an elliptic boundary-value problem.

For some examples of initial-boundary-value problems we recall the following model problem from III.4. Let G be a bounded domain in $\mathbb{R}^n$ whose boundary ∂G is a C^1 manifold, $2 \leq p < \infty$, for simplicity take $a(\xi) = |\xi|^{p-1} \operatorname{sgn} \xi$, and let $b \in L^\infty(G)$, and $c \in L^\infty(\partial G)$ be non-negative. Let V be a closed subspace of $W^{1,p}(G)$ and define

$$(5.3) \qquad \mathcal{A}u(v) = \int_G \left\{ \sum_{j=1}^n a\big(\partial_j u(x)\big)\partial_j v(x) + b(x)a\big(u(x)\big)v(x) \right\} dx$$
$$+ \int_{\partial G} c(s)a\big(\gamma u(s)\big)\gamma v(s)\, ds , \qquad u,v \in V .$$

From Lemma 5.1, its following Remark, and II.5, we find that A is m-accretive on $H = L^2(G)$. Moreover, $\mathcal{A}u = f \in L^2(G)$ if and only if

$$u \in V : \mathcal{A}u(v) = \int_G fv\, dx , \qquad v \in V .$$

We shall assume $C_0^\infty(G) \subset V$, so this implies

$$(5.4.\text{a}) \qquad A(u) = -\sum_{j=1}^n \partial_j a(\partial_j u) + ba(u) \quad \text{in } \mathcal{D}^* .$$

Moreover, we shall show below that

$$(5.5) \qquad D(A) = \{u \in V : A(u) \in L^2(G) \quad \text{and } \partial_A u = 0\} ,$$

where ∂_A is the abstract boundary operator given by

$$(5.4.\text{b}) \qquad \partial_A u = \sum_{j=1}^n a(\partial_j u)\nu_j + ca(\gamma u)$$

on smooth functions $u \in D(A)$. Here $\vec{\nu} = (v_1, \ldots, \nu_n)$ is the unit outward normal on ∂G. It follows, then, that the value of Au is determined by the *formal operator* (5.4.a), the restriction of $\mathcal{A}(u)$ to $C_0^\infty(G)$, and the domain $D(A)$ is determined by both the stable boundary conditions imposed by $u \in V$ and the variational boundary conditions $\partial_A u = 0$ obtained from the abstract Green's formula as the extension of (5.4.b).

We shall use the abstract Green's theorem to verify (5.5) and, hence, that the domain of A consists of those vectors which satisfy both the stable boundary conditions specified by V and the variational or complementary boundary conditions specified by ∂_A. First we recall the Green's formula from II.5. Let V, B be $\mathcal{D}$ Banach spaces and $\gamma : V \to B$ a strict homomorphism with kernel V_0. Then γ is an abstract *trace operator* and its dual γ^* is an isomorphism of B' onto the annihilator $V_0^\perp$ in V'. Let H be a Hilbert space for which we identify $H = H'$, let $V \subset H$ with a continuous injection, and let V_0 be dense in H. Thus we have $H \subset V'$ and $H \subset V_0'$ by restriction. Assume $\mathcal{A} : V \to V'$ is given and define the corresponding *formal operator* $Au = \mathcal{A}u \mid_{V_0}$ as the indicated restriction. Set $D = \{u \in V : Au \in H\}$; this is the domain of the abstract *boundary operator*. For each $u \in D$, we find $\mathcal{A}u - Au \in V_0^\perp$, so there is a unique $\partial_A u \in B'$ for which $\mathcal{A}u - Au = \gamma^*(\partial_A u)$ in V'. This defines $\partial_A : D \to B'$, and we have

$$(5.6) \qquad \mathcal{A}u(v) = (Au, v)_H + \partial_A u(\gamma v) , \qquad u \in D , \ v \in V .$$

This is the *abstract Green's formula* for $\mathcal{A}$.

Next we use (5.6) to characterize the restriction of $\mathcal{A}$ to H. Recall that we defined $D(A) = \{u \in V : \mathcal{A}u \in H\}$. If $u \in D(A)$, then there is an $f \in H$ for which $\mathcal{A}u = f$ in V', hence, by (5.5)

$$(Au, v)_H + \partial_A u(\gamma v) = (f, v)_H , \qquad v \in V .$$

This holds for each $v \in V_0$, and V_0 is dense in H, so we obtain successively $Au = f$ in H and $\partial_A u = 0$ in B'. Conversely, if $u \in D$ and $\partial_A u = 0$, it follows from (5.5) that $u \in D(A)$ and $\mathcal{A}u = Au$ in H. This completes the proof of (5.5).

We illustrate this further with the examples above. V is a subspace of $W^{1,p}(G)$ and $\mathcal{A} : V \to V'$ is given by (5.3). Let γ be the trace operator from V into $L^p(\partial G)$ and let B be the range of γ (restricted to V) with the inherited quotient norm. Then the kernel of γ is $V_0 = W_0^{1,p}(G)$, and this contains $C_0^\infty(G)$ and so is dense in $H = L^2(G)$. Note also that $B \hookrightarrow L^p(\partial G)$ and $L^{p'}(\partial G) \subset B'$. The formal part

of $\mathcal{A}$ is computed from (5.3), and it is given by (5.4). The boundary operator is computed on the sufficiently smooth functions $u \in D$ to be given by

$$\partial_A u(\psi) = \int_G \left(\sum_{j=1}^u a(\partial_j u)\nu_j + c(s)a\big(u(s)\big) \right) \psi(s)\, ds \,, \qquad \psi \in Rg(\gamma) \,.$$

This follows directly from (5.3), (5.4) and (5.6). When we apply Proposition 5.1 in this situation we obtain a generalized solution u of the nonlinear parabolic equation

$$(5.7.\text{a}) \qquad \partial_t u - \sum_{j=1}^n \partial_j a(\partial_j u) + ba(u) = f \,, \qquad x \in G \,,\ t > 0 \,,$$

and this solution satisfies $u(t) \in D(A)$ at each $t \geq 0$. If $V = W_0^{1,p}(G)$, then $\partial_A = 0$ so the last condition in (5.5) is vacuous and the boundary conditions are obtained solely from $u(t) \in V$; that is,

$$(5.7.\text{b}) \qquad u(s,t) = 0 \,, \quad \text{a.e. } s \in \partial G \,,\ t > 0 \,.$$

If $V = W^{1,p}(G)$, then the boundary conditions are obtained solely from $\partial_A u(t) = 0$, so we have

$$(5.7.\text{c}) \qquad \sum_{j=1}^n a\big(\partial_j u(s,t)\big)\nu_j(s) + c(s)a\big(u(s,t)\big) = 0 \,, \quad \text{a.e. } s \in \partial G \,,\ t > 0 \,.$$

Thus, we can solve (5.7.a) subject to the boundary condition (5.7.b) of *Dirichlet type*, and we can likewise solve (5.7.a) subject to the *Neumann type* boundary condition (5.7.c). Of course, one needs also to specify the appropriate initial data $u_0 \in D(A)$ in either problem in order to obtain a well-posed problem. Other types of boundary conditions, for example, of mixed or non-local type, can be achieved by a corresponding choice of V. For each such choice, $W_0^{1,p} \subset V \subset W^{1,p}$, there will be a *pair* of boundary constraints implicit in $D(A)$. See II.5 for some examples.

The preceding examples can be easily modified so as to obtain first-order derivatives in the formal operator (5.4). Such examples show that in some sense the Proposition 5.1 is sharp: the Cauchy problem can be *reversible*, hence, there exists a solution *only if* the initial data u_0 belongs to $D(A)$. (See I.5.) By contrast, those Cauchy problems associated with a derivative (see Proposition III.4.2), or more generally with a subgradient, are well-posed for any u_0 in the space H, hence, they are *non-reversible*. This suggests we consider the extension of the notion of regular accretive operators to those which are multi-valued.

Assume the spaces V and H are given as above and let $\mathcal{A}$ be a monotone relation from V to V', i.e., $\mathcal{A} \subset V \times V'$. Then we define a relation A on H by $A = \mathcal{A} \cap (V \times H)$, so its domain is $D(A) = \{u \in V : \mathcal{A}(u) \cap H \neq \varphi\}$. Such an operator A is accretive on H. An important example is the case that arises from a subdifferential.

LEMMA 5.2. *Let V and H be given as above, and let $\varphi : V \to \mathbb{R}_\infty$ be proper, convex and lower-semi-continuous. Extend φ to all of H as $\varphi(v) = +\infty$ if $v \in H$, $v \notin V$. Assume there is a $q > 1$ such that*

$$(5.8) \qquad \lim_{\|v\| \to \infty} \Big(|v|_H^q + \varphi(v) \Big) = +\infty \,.$$

Then $\operatorname{dom} \varphi \subset V$ *and* φ *is lower-semi-continuous on* H.

PROOF. The first conclusion is trivial and the second follows easily. If $v_n \to v$ in H and $\varphi(v_n) \to a$, then (5.8) shows some subsequence of $\{v_n\}$ converges weakly in V to v. This leads directly to a proof of the second claim. □

The Lemma 5.2 gives easy sufficient conditions for the hypotheses in the following result.

PROPOSITION 5.2. *Let the spaces* V *and* H *be given as above, and let* $\varphi : H \to \mathbb{R}_\infty$ *be convex and lower-semi-continuous with* $\operatorname{dom} \varphi \cap V \neq \varphi$.

(a) *If* $\partial_H \varphi$ *is the subgradient and* $\partial \varphi$ *is the subdifferential of the restriction* $\varphi : V \to \mathbb{R}_\infty$, *then*

$$\partial_H \varphi \cap (V \times H) \subset \partial \varphi \cap (V \times H) \ .$$

(b) *If* $\operatorname{dom}(\partial_H \varphi) \subset V$, *then* $\partial_H \varphi = \partial \varphi \cap (V \times H)$. *Thus, for each* $f \in L^2(0, T; H)$ *and* $u_0 \in \overline{\operatorname{dom}(\varphi)}$, *there exists a unique* $u \in C([0, T], H)$ *which satisfies* $u \in W^{1,2}(\delta, T; H)$ *for* $0 < \delta < T$, $u(t) \in D(A)$ *at a.e.* $t \in (0, T)$, $\sqrt{t}\frac{du}{dt} \in L^2(0, T; H)$, $\varphi(u(\cdot)) \in L^1(0, T)$, $u(0) = u_0$ *and*

(5.9)
$$\frac{du}{dt}(t) + \partial\varphi\big(u(t)\big) \ni f(t) \ , \qquad a.e. \ t \in (0, T) \ ,$$

Also, if $u_0 \in \operatorname{dom}(\varphi)$ *then* $u \in W^{1,2}(0, T; H)$.

PROOF.

(a) If $f \in H$, $u \in V$ and $f \in \partial_H \varphi(u)$, then

$$(f, v - u)_H \leq \varphi(v) - \varphi(u) \ , \qquad v \in H$$

and so it follows, from $f \in V'$ and by applying this to $v \in V$, that $f \in \partial\varphi(u)$.

(b) In this case, $\partial_H \varphi = \partial_H \varphi \cap (V \times H)$ is maximal accretive and $\partial\varphi \cap (V \times H)$ is accretive in H, so by part (a) they are equal. The last part follows from Theorem 4.3. □

The case (b) holds in the situation of Lemma 5.2, and then the subgradient $A = \partial_H \varphi$ is exactly the accretive operator on H obtained from the subdifferential $\mathcal{A} = \partial\varphi$. In the case where $\mathcal{A}$ is a G-differential, we have already obtained such results as Proposition III.4.2. For examples of parabolic initial-boundary-value problems which are resolved by Proposition 5.2, we have (5.7) with initial data any u_0 in $L^2(G)$. The details follow easily, as was described at the end of III.4. Moreover, Proposition 5.2 applies to problems with a multi-valued operator A, such as those described in II.8 and in IV.5. These Cauchy problems are *strictly parabolic* in the sense that initial data u_0 in the large space $\overline{\operatorname{dom}(\varphi)}$, the closure in H of $\operatorname{dom}(\varphi)$, leads to a solution u whose value $u(t)$ at any $t > 0$ belongs to the smaller set $D(A)$ of smoother functions which satisfy the boundary conditions. This is a *regularizing property* of the parabolic problem.

We turn now to evolution problems associated with the stationary problem of Theorem II.4 or Proposition 2.2, that is, for equations of the form

(5.10)
$$\frac{du}{dt}(t) + \mathcal{A}\big(u(t)\big) + \partial\varphi\big(u(t)\big) \ni f(t) \ , \quad a.e. \ t \in (0, T) \ ,$$

where $\mathcal{A}$ is a function from V to V' and φ is a convex function on V. This can be regarded as a *perturbation* of either (5.2) or (5.9). First we record the conditions sufficient to give a solution of (5.10) with $f(t) - u'(t) \in H$ at a.e. $t \in (0, T)$. Then we show that this solution satisfies $u(t) \in D(A)$, i.e., $\mathcal{A}(u(t)) \in H$, under hypotheses of *compatibility* of $\mathcal{A}$ and φ. This is an abstract *regularity* result which has strong implications for various semilinear parabolic equations and for related variational inequalities. The point here is to separate the operator in (5.10) into a smoothing or regularizing part and a singular part which does not destroy the smoothing effect of the other.

LEMMA 5.3. *Let V and H be given as above, let $\mathcal{A} : V \to V'$ be bounded, monotone and demicontinuous, let $\varphi : V \to \mathbb{R}_\infty$ be convex, lower-semi-continuous, assume $\varphi(0) = 0$ and*

$$(5.11) \qquad \lim_{\|v\| \to \infty} \frac{|v|_H^2 + \mathcal{A}v(v) + \varphi(v)}{|v|_H} = +\infty .$$

Then $(\mathcal{A} + \partial\varphi) \cap (V \times H)$ is m-accretive on H.

PROOF. The restriction of $\mathcal{A} + \partial\varphi$ to H is clearly accretive, so it suffices to show that for each $f \in H$ there is a

$$u \in V : u + \mathcal{A}u + \partial\varphi(u) \ni f .$$

For this it suffices by Theorem II.4 that there be a $R > 0$ such that $\|v\| > R$ imply $\langle v + \mathcal{A}v - f, v \rangle + \varphi(v) > 0$. But $f \in H$, so this follows from (5.11). $\qquad\square$

PROPOSITION 5.3. *Let the spaces V and H be given as above, and let $\varphi : V \to \mathbb{R}_\infty$ be convex and lower-semi-continuous with $\varphi(0) = 0$. Assume $\mathcal{A} : V \to V'$ is bounded, monotone and demicontinuous.*

(a) *If (5.11) holds, then for each $u_0 \in V$ with $(\mathcal{A}u_0 + \partial\varphi(u_0)) \cap H \neq \varphi$ and each $f \in W^{1,1}(0, T; H)$ there exists a unique $u \in W^{1,\infty}(0, T; H)$ with $u(0) = u_0$ and (5.10), and we also have*

$$f(t) - D^+u(t) \in \left(\mathcal{A}(u(t)) + \partial\varphi(u(t)) \right) \cap H , \qquad t \in [0, T] .$$

(b) *If A is m-accretive and if*

$$(5.12) \qquad 0 \leq \varphi\left((I + \varepsilon A)^{-1}v\right) \leq \varphi(v) , \qquad v \in V , \ \varepsilon > 0 ,$$

then $u(t) \in D(A)$ for $0 \leq t \leq T$, hence,

$$f(t) - \frac{du}{dt}(t) - A(u(t)) \in \partial\varphi(u(t)) \cap H , \qquad a.e. \ t \in (0, T) .$$

PROOF. Part (a) follows from Theorem 4.1, since Lemma 5.3 shows $(\mathcal{A} + \partial\varphi) \cap V \times H$ is m-accretive on H. To prove (b), it suffices to show that if there exists a $g \in (\mathcal{A}(u) + \partial\varphi(u)) \cap H$, then $u \in D(A)$. Thus, let $g - \mathcal{A}(u) \in \partial\varphi(u)$ with $g \in H$, i.e.,

$$u \in V : \langle g - \mathcal{A}u, v - u \rangle \leq \varphi(v) - \varphi(u) , \qquad v \in V .$$

For each $\varepsilon > 0$, there is a unique $u_\varepsilon = (I + \varepsilon A)^{-1}u$, and we set $v = u_\varepsilon$ in the above to obtain

$$\langle g - \mathcal{A}u, u_\varepsilon - u\rangle \le 0 \ , \qquad \varepsilon > 0 \ .$$

Since $\mathcal{A}$ is monotone, this implies

$$\langle g - \mathcal{A}u_\varepsilon, u_\varepsilon - u\rangle = \big(g - A(u_\varepsilon), -\varepsilon A(u_\varepsilon)\big)_H \le 0 \ .$$

This shows $|A(u_\varepsilon)|_H \le |g|_H$, $\varepsilon > 0$, so (by passing to a subsequence and changing notation)

$$|u_\varepsilon - u|_H \le \varepsilon |g|_H \ , \qquad A(u_\varepsilon) \rightharpoonup h \text{ in } H \ .$$

Since A is m-accretive, Proposition 1.6 shows that $h = A(u)$, hence, $u \in D(A)$ as desired. $\qquad\square$

REMARKS. By taking the case of $\varphi \equiv 0$ we see that Proposition 5.3 contains Proposition 5.1. However it does not contain Proposition 5.2 since it does not imply the regularizing property and in fact does apply to some reversible evolution equations. The estimate (5.12) is the *compatibility condition* between $\mathcal{A}$ and $\partial\varphi$; see Lemma 5.4 below for alternative statements of it and compare (2.6).

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We shall develop two examples to illustrate the application of Proposition 5.3 to initial-boundary-value problems for nonlinear parabolic partial differential equations. These examples contain variational inequalities as special cases, and it will be clear from the constructions that much more general problems could be obtained similarly. Also see II.8 and IV.2 above.

EXAMPLE 5.A. Assume G is a bounded domain in $\mathbb{R}^n$, ∂G is a C^1 manifold with unit outward normal ν, $2 \le p < \infty$, and $\varphi_0 : \mathbb{R} \to \mathbb{R}_\infty$ is non-negative, convex, lower-semi-continuous, with $\varphi(0) = 0$. Set $V = W^{1,p}(G)$ and define

$$\varphi(v) = \frac{1}{p}\int_G \sum_{j=1}^n |\partial_j v(x)|^p \, dx + \int_{\partial G} \varphi_0\big(\gamma v(s)\big)\, ds \ , \qquad v \in V \ .$$

The principle part of φ is continuous so as in II.8 we can compute the subdifferential $\partial\varphi$ termwise by Proposition II.7.7. The formal operator, obtained by restricting $\partial\varphi(u)$ to $C_0^\infty(G)$, is

$$B(u) = -\sum_{j=1}^n \partial_j a(\partial_j u) \ , \qquad u \in V \ ,$$

where $a(\xi) = |\xi|^{p-1}\, \text{sgn}(\xi)$. If $Bu \in H = L^2(G)$, we obtain the boundary condition

$$\partial_B(u) + \partial\varphi_0(\gamma u) \ni 0 \quad \text{in } L^{p'}(\partial G)$$

from the abstract Green's Theorem, where ∂_B is the boundary operator given by

$$\partial_B u = \sum_{j=1}^n a(\partial_j u)\nu_j$$

on smooth functions, and $\partial\varphi_0$ is the realization of $\partial\varphi_0 \subset \mathbb{R} \times \mathbb{R}$ in $L^{p'}(\partial G)$.

Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a monotone, continuous function with $\alpha(0) = 0$ and

$$|\alpha(\xi)| \le C(|\xi|^{p-1} + 1) , \qquad \xi \in \mathbb{R} .$$

The operator $\mathcal{A} : L^p(G) \to L^{p'}(G)$ defined by

$$\mathcal{A}(u)(x) = \alpha\big(u(x)\big) , \qquad \text{a.e. } x \in G ,$$

is bounded, continuous and monotone; see II.3 for such substitution or Nemitskyi operators. It follows that A, the restriction to H, is m-accretive on H; see Proposition II.8.1. From Proposition II.5 follows the estimate

$$\left(\sum_{j=1}^{n} \|\partial_j v\|_{L^p(G)}^p \right)^{1/p} + \|v\|_{L^2(G)} \ge c_0 \|v\|_{W^{1,p}(G)} , \quad v \in W^{1,p}(G) ;$$

this shows that (5.11) holds. Moreover, $(I + \varepsilon A)^{-1}(v) = (I + \varepsilon \alpha)^{-1}(v)$, and $(I + \varepsilon \alpha)^{-1}$ is Lipschitz and monotone, so (5.12) is true and Proposition 3(b) holds. Assume $u_0 \in V$ satisfies

$$B(u_0) \in L^2(G) , \qquad -\partial_B(u_0) \in \partial \varphi_0(\gamma u_0) , \qquad \alpha(u_0) \in L^2(G) ,$$

and that $f \in W^{1,1}(0,T; L^2(G))$ is given. Then there exists a unique

$$u \in W^{1,\infty}(0,T; L^2(G))$$

which is a generalized solution of the initial-boundary-value problem

(5.13.a) $\dfrac{\partial u}{\partial t}(x,t) - \displaystyle\sum_{j=1}^{n} \partial_j a\big(\partial_j u(x,t)\big) + \alpha\big(u(x,t)\big) = f(x,t) , \quad \text{a.e. } x \in G , t > 0 ,$

(5.13.b) $\displaystyle\sum_{j=1}^{n} a(\partial_j u)\nu_j(s) + \partial \varphi_0\big(u(s,t)\big) \ni 0 , \quad \text{a.e. } s \in \partial G , t > 0 ,$

(5.13.c) $u(x,0) = u_0(x) , \quad \text{a.e. } x \in G .$

The parabolic equation (5.13.a) and the boundary condition (5.13.b) follow from (5.10); the regularizing effect is that $\alpha(u(\cdot,t)) \in L^2(G)$ for each $t > 0$. Hence, each of the four terms in (5.13(a)) individually belongs to $L^2(G)$ at each $t > 0$, so (5.13.b) has both terms in $L^{p'}(\partial G) \subset B'$, and it can be interpreted as above by the abstract Green's Theorem.

In Example 5.A it is the higher-order operator $\partial \varphi$ through which the additional regularity is most effective. The operator $\mathcal{A}$ is more regular and thereby makes it easier to verify the compatibility condition (5.12). Much more general Dirichlet integrands could be used here, as well as other useful and interesting convex functions such as those in II.8. Moreover, we remark that the restrictions on the data, u_0 and f, can be relaxed by using Proposition 5.2. This follows since $\mathcal{A} + \partial \varphi$ is actually a subdifferential in Example 5.A; thus, it suffices for *existence* of a solution to require that

$$u_0 \in L^2(G) , \qquad f \in L^2\big(G \times (0,T)\big) .$$

The first condition follows from the observation that $C_0^\infty(G) \subset \text{dom}(\varphi) \cap D(A)$ and $C_0^\infty(G)$ is dense in $L^2(G)$. Therefore (5.13) is a *quasilinear parabolic problem* with the *regularizing property*: at a.e. $\delta > 0$, $u(\delta)$ is in the domain of $(\mathcal{A} + \partial \varphi) \cap V \times H$

and then Proposition 5.3(b) applies. For this we need only require additionally that $f \in W^{1,1}(\delta, T; L^2(G))$ for every $0 < \delta < T$.

REMARKS.
(1) In the special situation of (5.13) a solution of (5.10) satisfies $\alpha(u(\cdot, t)) \in L^{p'}(G)$ for each $t > 0$, even without the regularizing effect obtained from Proposition 5.3(b). Thus we have $B(u(t)) \in L^{p'}(G)$; this is sufficient to construct the boundary operator $\partial_B(u(t)) \in B'$. This follows from the discussion of II.5 of the Green's Theorem; we merely choose $\| \cdot \|_{L^p(G)}$ as the continuous seminorm on V, and thus $L^{p'}(G)$ is obtained as the pivot space which determines the domain of the boundary operator.
(2) Variational inequalities are included in Example 5.A. To illustrate this, consider the convex function

$$(5.14) \qquad \varphi_0(x) = \begin{cases} 0 , & x \geq 0 , \\ +\infty , & x < 0 . \end{cases}$$

Its subgradient is given by

$$r \in \partial\varphi_0(s) \iff s \geq 0 , \ r \leq 0 , \ rs = 0 .$$

Thus the boundary condition (5.13.b) is equivalent to

$$\gamma(u) \geq 0 , \qquad \partial_B(u) \geq 0 , \qquad \partial_B u(\gamma u) = 0 ,$$

and this has a "pointwise a.e." characterization since $B \subset L^p(\partial G)$. This can be regarded as a *boundary control* problem in which one seeks to show there is a non-negative flux, $\partial_B u$, which will maintain a non-negative boundary temperature, γu, and for which no heat flux is supplied at those points on the boundary where the temperature is positive. By taking similar functions in the Dirichlet integrand defining φ, one could also obtain such variational inequalities with constraints on the gradient, ∇u.

Before considering the next example, we give a useful characterization of the *compatibility condition* (5.12) in the special case where φ is given on H.

PROPOSITION 5.4 (BREZIS-PAZY). *Let A be an m-accretive function on H and $\varphi : H \to \mathbb{R}_\infty$ be proper, convex and lower-semi-continuous. The following are equivalent:*
(i) $\varphi((I + \varepsilon A)^{-1} x) \leq \varphi(x)$, $x \in H$, $\varepsilon > 0$.
(ii) $(A_\varepsilon x, w) \geq 0$, $w \in \partial\varphi(x)$, $\varepsilon > 0$.
(iii) $(A_\varepsilon x, \varphi'_\alpha(x)) \geq 0$, $x \in H$, $\varepsilon > 0$, $\alpha > 0$.
(iv) $(Ax, \varphi'_\alpha(x)) \geq 0$, $x \in D(A)$, $\alpha > 0$.
(v) $\varphi_\alpha((I + \varepsilon A)^{-1} x) \leq \varphi_\alpha(x)$, $x \in H$, $\alpha > 0$.

PROOF. Assume (i). If $w \in \partial\varphi(x)$ then

$$-\varepsilon(A_\varepsilon x, w) = \big((I + \varepsilon A)^{-1} x - x, w\big) \leq \varphi\big((I + \varepsilon A)^{-1} x\big) - \varphi(x) \leq 0$$

so (ii) holds. Assume (ii). Since A_ε is monotone

$$\big(A_\varepsilon x - A_\varepsilon\big((I + \alpha\partial\varphi)^{-1} x\big) , \quad x - \big((I + \alpha\partial\varphi)^{-1}\big)\big) \geq 0 ,$$

and $x - (I + \alpha\partial\varphi)^{-1}x = \alpha\varphi'_\alpha(x)$, we obtain

$$\big(A_\varepsilon x, \varphi'_\alpha(x)\big) \geq \Big(A_\varepsilon\big((I + \alpha\partial\varphi)^{-1}x\big), \varphi'_\alpha(x)\Big) \geq 0$$

from $\varphi'_\alpha(x) \in \partial\varphi((I + \alpha\partial\varphi)^{-1}x)$ and (ii). Letting $\varepsilon \to 0$ shows that (iii) implies (iv). For any $x \in H$ we have for $y = (I + \varepsilon A)^{-1}x$

$$\varphi_\alpha(x) - \varphi_\alpha(y) \geq (\varphi'_\alpha(y), x - y) = (\varphi'_\alpha(y), \varepsilon A_\varepsilon x)$$
$$= \varepsilon(\varphi'_\alpha(y), Ay) \geq 0$$

by (iv), so (iv) implies (v). Finally, note that (i) follows by letting $\alpha \to 0$ in (v).$\quad\square$

We have already seen examples of such estimates in Example 2.F and Example 2.G, and in Proposition II.9.3 where A is a linear elliptic operator in divergence form, and φ is a convex integrand on $L^p(G)$. This can be extended to nonlinear A as we illustrate with the following simple case.

EXAMPLE 5.B. Assume that the domain G, the functions $a(\cdot)$, φ_0, and the spaces $V = W^{1,p}(G)$, $H = L^2(G)$ are just as given in Example 5.A. Assume $b \in L^\infty(G)$ and $c \in L^\infty(\partial G)$ are both non-negative, and define $\mathcal{A} : V \to V'$ by (5.3). Furthermore, define the convex integrand

$$\varphi(v) = \int_G \varphi_0\big(v(x)\big)\, dx \ \ \text{if} \ \ \varphi_0(v) \in L^1(G) \ , \ +\infty \ \text{otherwise} \ ,$$

as in Proposition II.8.1. To check that (5.12) is satisfied, it suffices by Proposition 5.4(iv) to verify that

$$\big(Au, \sigma(u)\big)_{L^2(G)} \geq 0 \ , \qquad u \in D(A)$$

for any monotone Lipschitz function $\sigma : \mathbb{R} \to \mathbb{R}$ with $\sigma(0) = 0$. Recall that the Yosida approximation $\sigma = (\varphi_0)_\alpha$ has such properties; see Proposition 1.9. But for $u \in D(A) \subset V$ we have $v = \sigma(u) \in V$, and in (5.3) we obtain

$$\big(Au, \sigma(u)\big)_{L^2(G)} = \int_G \left\{ \sum_{j=1}^n (\partial_j u)^p \sigma'\big(u(x)\big) + b(x)\big(u(x)\big)^p \right\} dx$$
$$+ \int_{\partial G} c(s)\big(\gamma u(s)\big)^p\, ds \geq 0 \ .$$

The estimate (5.1) holds, so A is m-accretive by Lemma 5.1 and we have the situation of Proposition 5.3(b). Assume $u_0 \in V$ satisfies

$$A(u_0) \in L^2(G) \ , \qquad \partial_A(u_0) = 0 \ , \qquad u_0 \in \text{dom}(\partial\varphi_0) \ ,$$

and that $f \in W^{1,1}(0,T;L^2(G))$ is given. Then there exists a unique generalized solution $u \in W^{1,\infty}(0,T;L^2(G))$ of the initial-boundary-value problem

(5.15.a) $\dfrac{\partial u}{\partial t}(x,t) - \displaystyle\sum_{j=1}^{n} \partial_j a\big(\partial_j u(x,t)\big) + b(x)a\big(u(x)\big) + v(x,t) = f(x,t)$,

(5.15.b) $v(x,t) \in \partial\varphi_0\big(u(x,t)\big)$,   a.e. $x \in G$, $t > 0$,

(5.15.c) $\displaystyle\sum_{j=1}^{n} a(\partial_j u)\nu_j(s) + c(s)a\big(\gamma u(s)\big) = 0$,   a.e. $s \in \partial G$, $t > 0$,

(5.15.d) $u(x,0) = x_0(x)$,   a.e. $x \in G$.

From the fact that the solution of (5.10) satisfies $u(t) \in D(A)$, with $D(A)$ given by (5.5), we obtain the boundary condition (5.15.c) from (5.4.b). This also implies that (5.15.b) holds with $v(\cdot,t) \in L^2(G)$ for each $t \geq 0$. (From Proposition 5.3(a) we could only have deduced that $v(t) \in V'$, and then $\partial_A(u(t))$ and (5.15.c) would be meaningless.) Finally we note that the operator $\mathcal{A}$ above is actually a (sub) differential, so we can apply Proposition 5.2 and obtain a solution from more general data, u_0, f.

For an example of a *variational inequality* we choose φ_0 as in (5.14). Then (5.15.b) is equivalent to

$$u(x,t) \geq 0 \;,\; v(x,t) \leq 0 \;,\; u(x,t)v(x,t) = 0 \;,\qquad \text{a.e. } x \in G \;,\; t > 0 \;.$$

As before this can be regarded as a distributed control problem in which a non-negative source, $-v(x,t)$, will maintain a non-negative temperature, $u(x,t)$, and the source is active (non-zero) only where the temperature is zero.

It is clear that there can be many different ways to obtain a specific initial-boundary-value problem in the abstract form (5.10). For example, each of (5.13) and (5.15) can be modified slightly to contain the other if $\partial\varphi_0$ is a function of the form of α. In Example 5.B the regularity, $u(t) \in D(A)$, is effective on both operators: on the higher-order $\mathcal{A}$ because it gives meaning to (5.15.c), and on $\partial\varphi$ because (5.15.b) contains functions with a pointwise meaning. The compatibility condition (5.12) is easy to verify because of the simple pointwise form of φ. It is clear that more general operators $\mathcal{A}$ could have been used. For example, with appropriate boundary conditions one could introduce first-order derivatives in $\mathcal{A}$; see Proposition II.9.3 for an example of linear $\mathcal{A}$ of second-order elliptic type with first-order derivatives.

IV.6. Semilinear Degenerate Evolution Equations

We have previously considered implicit evolution equations, i.e., those in which the solution is not necessarily differentiated, or even differentiable, but rather some function of the solution is differentiated in the equation. Such equations were resolved directly in III.3 and III.6. Here we give an algebraic construction which directly extends the preceding results to a rather large and very useful class of such equations.

Let E be a real vector space and denote its algebraic dual space by E^*. Let $\mathcal{B} : E \to E^*$ be linear, symmetric and non-negative, i.e., *monotone*. This determines

a semi-scalar-product

$$b(x, y) = \mathcal{B}x(y), \qquad x, y \in E ,$$

and we denote the corresponding seminorm space by E_b. Its continuous dual E_b' is a Hilbert space. Finally, let $A \subset E \times E_b'$ be a relation with domain $D = \{x \in E : A(x) \neq \varphi\}$; this will be regarded as a multi-valued operator as before, i.e., $[x, w] \in A$ if and only if $w \in A(x)$. A *solution* of the semilinear equation in E_b'

$$(6.1) \qquad \frac{d}{dt}\big(\mathcal{B}u(t)\big) + A\big(u(t)\big) \ni f(t) , \qquad 0 < t < T ,$$

is a function $u : [0, T] \to E$ for which $\mathcal{B}u \in C([0, T], E_b')$, $\mathcal{B}u(\cdot)$ is absolutely continuous on each $[\delta, T]$, $0 < \delta < T$, hence, differentiable a.e., and (6.1) holds a.e. on $(0, T)$. The *Cauchy problem* for (6.1) is to find a solution u for which $(\mathcal{B}u)(0) = w_0$ is specified in E_b'. We shall show that (6.1) is equivalent to the standard evolution equation (4.1) and, from this correspondence, determine conditions on $\mathcal{B}$, A, f and w_0 for which the Cauchy problem is well-posed.

For the moment let's consider the special case in which $b(\cdot, \cdot)$ is a scalar-product and E_b is complete, i.e., E_b is a Hilbert space. Then $\mathcal{B}$ is the Riesz isomorphism onto E_b'. Furthermore if A is monotone then any two solutions u_1, u_2 of (6.1) with respective data f_1, f_2 satisfy

$$\frac{d}{dt}\frac{1}{2}\|u_1(t) - u_2(t)\|_b^2 \leq \|f_1(t) - f_2(t)\|_{E_b'}\|u_1(t) - u_2(t)\|_b , \qquad 0 \leq t \leq T ,$$

and thus Lemma 4.1 implies that

$$\|u_1(t) - u_2(t)\|_b \leq \|u_1(0) - u_2(0)\|_b + \int_0^t \|f_1(s) - f_2(s)\|_{E_b'}\, ds , \qquad 0 \leq t \leq T .$$

This estimate shows that when A is monotone E_b is the appropriate space in which solutions are estimated, hence, the space in which to seek solutions. Also, the solution of the Cauchy problem should be represented by a semigroup which is generated by some realization of $-\mathcal{B}^{-1}A$. These observations will be developed below. Note that if we identify $E_b \cong E_b'$ then $\mathcal{B}$ is the identity and (6.1) is the standard evolution equation studied previously.

In this special case where E_b is Hilbert space, (6.1) is equivalent to the equation in E_b

$$\frac{du}{dt} + \mathcal{B}^{-1} \circ A(u(t)) \ni \mathcal{B}^{-1}f(t) , \qquad 0 < t < T ,$$

so we are led to ask whether the composite operator $\mathcal{A} \equiv \mathcal{B}^{-1} \circ A$ is m-accretive on E_b. Here we define $[x, y] \in \mathcal{A}$ if and only if $\mathcal{B}y = w$ for some $[x, w] \in A$, and then we have

$$(y, z)_{E_b} = b(y, z) = \mathcal{B}y(z) = \langle w, z \rangle , \qquad z \in E .$$

From here it follows that $\mathcal{A}$ is accretive in E_b if A is monotone in $E \times E_b'$. Also the range condition, $Rg(I + \mathcal{A}) = E_b$, is fulfilled if and only if $Rg(\mathcal{B} + A) = E_b'$. Thus from Theorem 4.1 it follows that for each absolutely continuous $f : [0, T] \to E_b'$ and each $u_0 \in D$ there is a unique absolutely continuous solution $u : [0, T] \to E_b$ of (6.1) with $u(0) = u_0$.

———————

We give two examples to illustrate this situation where $\mathcal{B}$ is a Riesz isomorphism.

EXAMPLE 6.A PSEUDOPARABOLIC EQUATION. Let $V = W_0^{1,p}(G)$, $p \geq 2$, and suppose the given operator $A : V \to V'$ is monotone and continuous. Let $b(\cdot) \in L^\infty(G)$ with $b(x) \geq 0$, a.e. $x \in G$, and define

$$\langle \mathcal{B}u, v \rangle = \int_G (u(x)v(x) + b(x)\nabla u(x) \cdot \nabla v(x))\, dx, \quad u,\ v \in V\ .$$

The completion of V under the continuous seminorm $\langle \mathcal{B}\cdot, \cdot \rangle^{\frac{1}{2}}$ defines the space E_b, and the imbeddings $H_0^1(G) \hookrightarrow E_b \hookrightarrow L^p(G)$ are continuous. It follows that $V \hookrightarrow E_b$ is continuous and dense and that $E_b' \subset (H_0^1)' = H^{-1}$. We will have $Rg(\mathcal{B}+A) \supset E_b'$ if $\mathcal{B}+A : V \to V'$ is *onto*, and this is the case here since $\mathcal{B}+A$ is type M, bounded and coercive. It follows from Theorem 4.1 that the Cauchy problem for (6.1) has a unique solution for each $u_0 \in E_b$ with $A(u_0) \in E_B'$, i.e., for each $u_0 \in \text{dom}(\mathcal{A})$.

The operator $\mathcal{B} : V \to \mathcal{D}(G)^*$ is given by $\mathcal{B}u = u - \nabla \cdot b(\cdot)\nabla u$. A solution $u(t)$ of the Cauchy problem for (6.1) corresponds to a solution of the the mixed *parabolic-pseudoparabolic* partial differential equation with initial and first type boundary conditions

$$u(t) \in W_0^{1,p}(G) : \frac{\partial}{\partial t}(u(t) - \nabla \cdot b(\cdot)\nabla u(t)) + A(u(t)) = f(t), \quad t \in (0, T),$$

$$u(x, 0) = u_0(x).$$

Here the initial condition $u_0(\cdot)$ is given as above, and the equation holds in $L^\infty(0, T; E_b')$. The operator A can be chosen as in Example II.5.A, and corresponding problems with boundary conditions of other types can as easily be treated. See Example III.6.A. Note that the equation here holds in the much smaller space E_b', i.e., $A(u(\cdot)) \in L^\infty(0, T; E_b')$, and this is a *regularity* result for the solution. The condition on the initial data above will be relaxed in Theorem 6.1 below when A is a subdifferrential.

EXAMPLE 6.B POROUS MEDIUM EQUATION. We saw in Example III.6.C how to choose the spaces and operators to realize the porous medium equation III.6.10. We repeat the construction here in order to use Theorem 4.1 to obtain the H^{-1}-solution in a more general setting. Let G be a bounded domain in $\mathbb{R}^n$ and assume $\frac{2n}{n+2} \leq p < \infty$. Let $\varphi : \mathbb{R} \to \mathbb{R}_\infty$ be proper, convex, lower semi-continuous and define the convex integrand $\Phi : L^p(G) \to \mathbb{R}_\infty$, $1 \leq p < \infty$, by

$$\Phi(u) = \int_G \varphi\big(u(x)\big)\, dx \ \text{ if } \ \varphi(u) \in L^1(G)\ , \quad +\infty \text{ otherwise.}$$

According to Proposition II.8.1, the function Φ is proper, convex, lower semi-continuous, and its subdifferential is given by $f \in \partial\Phi(u)$ if and only if

$$f \in L^{p'}(G)\ , u \in L^p(G) \text{ and } f(x) \in \partial\varphi\big(u(x)\big)\ , \text{ a.e. } \ x \in G\ .$$

Define $V = L^p(G)$ and set

$$A(u) = \partial\Phi(u), \qquad u \in L^p(G)\ .$$

The Riesz map $\mathcal{R} : H_0^1 \to H^{-1}$ is the isomorphism defined by the scalar product

$$\mathcal{R}\varphi(\psi) = (\varphi, \psi)_{H_0^1} \equiv \int_G \nabla\varphi \cdot \nabla\psi \, dx \,,$$

so we have $\mathcal{R} = -\Delta$. The corresponding scalar product on H^{-1} is given by

$$(f, g)_{H^{-1}} \equiv (\mathcal{R}^{-1}f, \mathcal{R}^{-1}g)_{H_0^1} \,, \quad f, g \in H^{-1} \,,$$

and it satisfies

$$(f, g)_{H^{-1}} = \langle f, \mathcal{R}^{-1}g \rangle = \langle \mathcal{R}^{-1}f, g \rangle \,, \quad f, g \in H^{-1}.$$

Define

$$\mathcal{B}u(v) = (u, v)_{H^{-1}}, \quad u, v \in H^{-1}.$$

The completion of V under the continuous seminorm $\langle \mathcal{B}\cdot, \cdot \rangle^{\frac{1}{2}}$ determines the space $E_b = H^{-1}$. We have the continuous imbedding $V \hookrightarrow E_b$, i.e., $E_b' = H_0^1(G) \hookrightarrow L^{p'}(G) = V'$ by the Sobolev imbedding Theorem II.4.3 since $p' \leq 2n/(n-2)$.

The multi-valued operator A is monotone, so the composition $\mathcal{A} = \mathcal{B}^{-1} \cdot A$ is accretive, and it remains to show that $Rg(\mathcal{B} + A) = E_b$. That is, for each $f \in E_b$ we want to solve

$$u \in H^{-1}: \quad \mathcal{B}u + A(u) \ni f \text{ in } H_0^1 \,,$$

and this is equivalent to solving

$$w \in H_0^1: \quad A^{-1}(w) - \Delta w \ni -\Delta f \text{ in } H^{-1} \,.$$

Now if the monotone graph $A^{-1}(\cdot)$ is a *function* which satisfies the growth condition

$$|A^{-1}(\xi)| \leq K(1 + |\xi|^{p'-1}), \quad \xi \in \mathbb{R},$$

then the methods of Section II.5 suffice to show that $Rg(A^{-1}(\cdot) - \Delta) = H^{-1}$. More generally, if the convex function φ satisfies a lower bound

$$\varphi(\xi) \geq K(\frac{|\xi|^p}{p} - 1), \quad \xi \in \mathbb{R},$$

then the *conjugate* convex function φ^*, whose subgradient is A^{-1}, satisfies the dual upper bound

$$\varphi^*(\xi) \leq K^*(\frac{|\xi|^{p'}}{p'} + 1), \quad \xi \in \mathbb{R}.$$

The corresponding convex integrand on $L^{p'}(G)$ is then continuous, and the methods of Section II.8 suffice. Note that either condition implies that $A(\cdot)$ is necessarily *onto*.

We shall assume that $F \in W^{1,\infty}(0, T; H^{-1}(G))$ and define

$$f \in W^{1,\infty}(0, T; H_0^1(G))$$

by

$$f(t)(v) = \int_G ((-\Delta)^{-1}F)(x, t)v(x) \, dx \,, \qquad v \in V \,.$$

Assume that $u_0 \in L^p(G)$ and there is a $w \in A(u_0) \cap H_0^1(G)$. From the preceding discussion and Theorem 4.1 we obtain a solution $u : [0, T] \to L^p(G)$ of the *porous medium equation*

$$\frac{\partial u(t)}{\partial t} - \Delta w(t) = F(t) \text{ in } H^{-1}(G), \quad w(t) \in \partial \Phi(u(t)) \text{ for a.e. } t \in (0, T),$$

$$w \in L^\infty(0, T; H_0^1(G)), \qquad \lim_{t \to 0} u(t) = u_0 \text{ in } H^{-1}(G),$$

with each term of the evolution equation in $L^\infty(0, T; H^{-1})$. This procedure requires more of the initial function than does Example III.6.C, but it yields a somewhat stronger solution.

Next, let $\mathcal{B}$ be as originally given above, i.e., linear, symmetric and monotone on the vector space E. From the Cauchy-Schwartz inequality

$$|b(x, y)|^2 \le b(x, x)b(y, y), \qquad x, y \in E$$

it follows that $\mathcal{B}$ is continuous from E_b into E_b'. Denote the kernel of $\mathcal{B}$ by

$$K = \{x \in E : \mathcal{B}x = 0\} = \{x \in E : b(x, x) = 0\}$$

and the corresponding quotient space by E/K. Thus, each element of E/K is a coset, $\widetilde{x} = x + K = \{x + y : y \in K\}$, and we can define a scalar-product on this space by

$$(6.2) \qquad \widetilde{b}(\widetilde{x}, \widetilde{y}) = b(x, y), \qquad x, y \in E.$$

The completion of E/K with the corresponding norm is a Hilbert space W whose scalar product is an extension of $\widetilde{b}(\cdot, \cdot)$. Let q denote the canonical quotient map, $q(x) = \widetilde{x}, x \in E$. Then q is a strict homomorphism of E_b into W, with a dense range, and so its dual map $q' : W' \to E_b'$, given by

$$q'(g)(x) = g(q(x)), \qquad g \in W', \ x \in E_b,$$

is an isomorphism. Let $\mathcal{B}_0 : W \to W'$ be the Riesz isomorphism given by

$$\mathcal{B}_0 v(w) = (v, w)_W, \qquad v, w \in W.$$

For each pair $x, y \in E_b$ we have by (6.2)

$$\mathcal{B}x(y) = \mathcal{B}_0(q(x))(q(y)) = q'\mathcal{B}_0 q(x)(y),$$

so we have factored the given operator into the form

$$(6.3) \qquad \mathcal{B} = q'\mathcal{B}_0 q.$$

This will play a fundamental role below.

To obtain a similar factorization of the (possibly multi-valued) operator $A : D \to E_b'$ we define $A_0 \subset W \times W'$ by

$$g \in A_0(\widetilde{x}) \text{ if and only if there exists}$$

$$\text{an } x \in D \text{ with } q(x) = \widetilde{x} \text{ and } q'(g) \in A(x).$$

Thus, we obtain a relation A_0 with domain $q[D] = \{\widetilde{x} : x \in D\}$ for which

$$(6.4) \qquad\qquad A = q' A_0 q \ .$$

Note that q is not one-to-one, so A_0 need not be single-valued, even if A is a function! The connections between these operators are given as follows.

LEMMA 6.1.
(a) *For each $\widetilde{x} = q(x) \in W$, $q'(g) \in A(x)$ if and only if $g \in A_0(\widetilde{x})$, and in this case, we have $q'(g)(x) = g(\widetilde{x})$.*
(b) *The inclusion $q'(g) \in (\mathcal{B} + A)(x)$ holds if and only if $g \in (\mathcal{B}_0 + A_0)(\widetilde{x})$.*

COROLLARY 6.1.
(a) $A : D \to E'_b$ *is monotone if and only if $A_0 : q[D] \to W'$ is monotone.*
(b) q' *is a bijection of $Rg(\mathcal{B}_0 + A_0)$ onto $Rg(\mathcal{B} + A)$.*

Let the operators $\mathcal{B}$ and A be given as in Lemma 6.1 and suppose that u is a solution of (6.1). Since q' is an isomorphism of W' onto E'_b, it follows from (6.3) and (6.4) that $\widetilde{u} \equiv q \cdot u$ is a solution of

$$(6.5) \qquad\qquad \frac{d}{dt}\big(\mathcal{B}_0 \widetilde{u}(t)\big) + A_0\big(\widetilde{u}(t)\big) \ni \widetilde{f}(t) \ , \qquad 0 < t < T \ ,$$

where $\widetilde{f}(t) = (q')^{-1} f(t)$. Conversely, if $\widetilde{u}$ is a solution of (6.5), then for a.e. $t \in (0, T)$, $\widetilde{u}(t) \in q[D]$, so there is a $u(t) \in D$ with $q\big(u(t)\big) = \widetilde{u}(t)$. Since q' is an isomorphism and $\mathcal{B}u(t) = q'\mathcal{B}_0\widetilde{u}(t)$, $0 < t < T$, it follows that u is a solution of (6.1). This proves the following.

LEMMA 6.2. *A function $u : [0, T] \to E$ is a solution of (6.1) if and only if $\widetilde{u} = q \cdot u : [0, T] \to W$ is a solution of (6.5).*

From Theorem 4.1 and the correspondence between the operators in (6.1) and (6.5) and between the solutions of these equations as given in Lemma 6.1 and Lemma 6.2, we obtain the first two parts of our main result.

THEOREM 6.1. *Let the linear, symmetric and monotone operator $\mathcal{B}$ be given from the real vector space E to its algebraic dual E^*, and let E'_b be the Hilbert space which is the dual of E with the seminorm*

$$|x|_b = \mathcal{B}x(x)^{1/2} \ , \qquad x \in E \ .$$

Let $A \subset E \times E'_b$ be a relation with domain $D = \{x \in E : A(x) \neq \varphi\}$.
(a) *Assume A is monotone. If u_j is a solution of*

$$\frac{d}{dt}\big(\mathcal{B}u_j(t)\big) + A\big(u_j(t)\big) \ni f_j(t) \ , \qquad 0 < t < T \ ,$$

for $j = 1, 2$, then it follows that

$$(6.6) \qquad |u_1(t) - u_2(t)|_b \leq |u_1(0) - u_2(0)|_b$$

$$+ \int_0^t \|f_1(s) - f_2(s)\|_{E'_b} \, ds \ , \qquad 0 \leq t \leq T \ .$$

> *If $f_1 = f_2$ and if $\mathcal{B}u_1(0) = \mathcal{B}u_2(0)$ then $\mathcal{B}u_1(t) = \mathcal{B}u_2(t)$ for all $0 \le t \le T$. Furthermore, if $\mathcal{B}+A$ is strictly monotone then there is at most one solution of the Cauchy problem for (6.1).*

(b) *Assume A is monotone and $Rg(\mathcal{B} + A) = E_b'$. Then, for each $u_0 \in D$ and each $f \in W^{1,1}(0, T; E_b')$, there is a solution u of (6.1) with*

$$\mathcal{B}u \in W^{1,\infty}(0, T; E_b') \ , \quad u(t) \in D \ , \quad \text{all } 0 \le t \le T \ , \quad \text{and } \ \mathcal{B}u(0) = \mathcal{B}u_0 \ .$$

(c) *Let A be the subdifferential, $\partial\varphi$, of a convex lower-semi-continuous function $\varphi : E_b \to [0, +\infty]$ with $\varphi(0) = 0$. Then for each u_0 in the E_b-closure of $\mathrm{dom}(\varphi)$ and each $f \in L^2(0, T; E_b')$ there is a solution u of (6.1) with*

$$\varphi \circ u \in L^1(0, T), \sqrt{t}\frac{d}{dt}\mathcal{B}u(\cdot) \in L^2(0, T; E_b') \ , \quad u(t) \in D \ , \quad a.e. \ t \in [0, T] \ ,$$

and $\mathcal{B}u(0) = \mathcal{B}u_0$. If in addition $u_0 \in \mathrm{dom}(\varphi)$ then

$$\varphi \circ u \in L^\infty(0, T) \quad , \quad \frac{d}{dt}\mathcal{B}u \in L^2(0, T; E_b') \ .$$

PROOF. In view of the preceding development it suffices to show that part (c) follows from Theorem 4.3. Now φ is lower-semi-continuous on E_b so it follows that φ is constant on each coset $\widetilde{x} \in E/K$. This means that a function $\widetilde{\varphi} : W \to [0, \infty]$ is defined implicitly by $\varphi = \widetilde{\varphi} \circ q$ and the effective domain of $\widetilde{\varphi}$ is $\mathrm{dom}(\widetilde{\varphi}) = q[\mathrm{dom}(\varphi)]$. It is a direct consequence of the definitions that

$$(6.7) \qquad\qquad\qquad\qquad \partial\varphi = q' \circ \partial\widetilde{\varphi} \circ q \ .$$

Specifically, $f \in \partial\varphi(x)$ if and only if $f = q'(g)$ with $g \in \partial\widetilde{\varphi}(\widetilde{x})$. Thus, comparing (6.4) and (6.7) we find $A_0 = \partial\widetilde{\varphi}$. The proof now follows from the preceding case. $\square$

Next we describe how the situation of Theorem 1 can be attained by operators of the type determined by nonlinear elliptic boundary-value problems. The following is typical.

DEFINITION. Assume X is a reflexive Banach space, $\mathcal{A} : X \to X'$ is monotone and demicontinuous, $\mathcal{B} : X \to X'$ is continuous, linear, symmetric and monotone. Then $\mathcal{B}, \mathcal{A}$ is a *regular pair* on X.

For such a pair it follows from Theorem II.2.2 that if $f \in X'$ and there is an $R > 0$ such that

$$\mathcal{B}v(v) + \mathcal{A}v(v) > f(v)$$

for all v with $\|v\| > R$, then $f \in Rg(\mathcal{B}+\mathcal{A})$. This proves the following result, which yields the situation of Theorem 1(b).

LEMMA 6.3. *If $\mathcal{B}, \mathcal{A}$ is a regular pair on X, and if*

$$(6.8) \qquad\qquad\qquad \lim_{\|v\|_X \to \infty} \frac{\mathcal{B}v(v) + \mathcal{A}v(v)}{\mathcal{B}v(v)^{1/2}} = +\infty \ ,$$

then $Rg(\mathcal{B}+\mathcal{A}) \supset E_b'$, where E_b is the space X with the seminorm $|v|_b = \mathcal{B}v(v)^{1/2}$, $v \in X$.

To obtain the situation of Theorem 1(b), we define $D = \{v \in X : \mathcal{A}(v) \in E_b'\}$ and $A(x) = \mathcal{A}(x)$ for $x \in D$.

PROPOSITION 6.1. *Assume $\mathcal{B}, \mathcal{A}$ is a regular pair on X and (6.8) holds. Then for each $u_0 \in D$ and $f \in W^{1,1}(0, T; E_b')$ there exists a solution u of (6.1) with $\mathcal{B}u \in W^{1,\infty}(0, T; E_b')$, $u(t) \in D$ for all $t \in [0, T]$, and $\mathcal{B}u(0) = \mathcal{B}u_0$.*

A typical application of Proposition 6.1 would be elliptic-parabolic initial-boundary-value problems of the form III.6.9 where $\mathcal{B}$ corresponds to multiplication by a function $b(x) \geq 0$. When $X = W^{1,p}(G)$, such a $\mathcal{B}$ is continuous if $b \in L^{q^*}(G)$, where $q^* = p^*/(p^* - 2)$ and $W^{1,p}(G) \hookrightarrow L^{p^*}(G)$ is continuous.

The boundedness of $\mathcal{B}$ can be relaxed as follows.

LEMMA 6.4. *Assume V is a reflexive Banach space, $\mathcal{A} : V \to V'$ is monotone and demicontinuous; $E_b = \{E, b(\cdot, \cdot)\}$ is a semi-scalar-product space, and $\mathcal{B} : E \to E_b'$ is the canonical operator, $\mathcal{B}u(v) = b(u, v)$; and $E \cap V$ is dense in V and dense in E_b. Set $X = E \cap V$, $\|v\|_X = \|v\|_V + b(v, v)^{1/2}$, $v \in X$. If X is complete, then $\mathcal{B}, \mathcal{A}$ is a regular pair on X.*

PROOF. Since the embeddings $V' \hookrightarrow X'$, $E_b' \hookrightarrow X'$, $X \hookrightarrow V$, $X \hookrightarrow E_b$ are all continuous, it suffices to show X is reflexive. Consider the map $x \mapsto [x, \widetilde{x}] : X \to V \times W$, where W is the completion of E_b/K as in the proof of Theorem 6.1. This is an isomorphism of X onto a closed subspace of a product of reflexive spaces, so X is reflexive. $\square$

COROLLARY 6.2. *If E_b is complete then $\mathcal{B}, \mathcal{A}$ is a regular pair on X.*

PROOF. Let X_0 be the completion of X. Then the imbeddings $X_0 \hookrightarrow V$ and $X_0 \hookrightarrow E_b$ are continuous, so $X_0 \subset V \cap E_b = X$ and X is complete. $\square$

We consider a related class of *second-order evolution equations*. Let V_a be a semi-scalar-product space whose semi-scalar-product is given by the linear symmetric $\mathcal{A} : V_a \to V_a'$ and whose seminorm is $|x|_a = \mathcal{A}x(x)^{1/2}$. Similarly let $\mathcal{C} : V_c \to V_c'$ be the linear symmetric operator determined by the semi-scalar-product on V_c, and denote the seminorm by $|x|_c = \mathcal{C}x(x)$. Set $W = V_a \cap V_c$ with seminorm $|x|_W \equiv (\mathcal{A} + \mathcal{C})x(x)^{1/2}$. We shall assume

$$W \text{ is dense in } V_a \text{ and in } V_c$$

so we can identify $V_a' \hookrightarrow W'$ and $V_c' \hookrightarrow W'$ by restriction. Suppose we are given a relation B from W to W', i.e., $B \subset W \times W'$, and a pair of functions $f_a : [0, T] \to V_a'$, $f_c : [0, T] \to V_c'$. Then we consider the evolution system

(6.9.a)
$$\frac{d}{dt}\mathcal{A}u(t) - \mathcal{A}v(t) = f_a(t), \qquad 0 < t < T,$$

(6.9.b)
$$\frac{d}{dt}\mathcal{C}v(t) + \mathcal{A}u(t) + B(v(t)) \ni f_c(t).$$

A *solution* of (6.9) is a pair of functions $u : [0, T] \to V_a$, $v : [0, T] \to W$ such that each of $\mathcal{A}u : [0, T] \to V_a'$ and $\mathcal{C}v : [0, T] \to V_c'$ are continuous on $[0, T]$, absolutely continuous on $[\delta, T]$ for each $0 < \delta < T$, and (6.9) holds at a.e. $t \in (0, T)$. Note that necessarily $v(t) \in \text{dom}(B)$ and that $\mathcal{A}u(t) + B(v(t))$ contains an element of

V_c' at each such $t \in (0, T)$. The *Cauchy problem* for (6.9) is to find a solution for which

$$\mathcal{A}u(0) = w_1 \in V_a' , \quad \mathcal{C}v(0) = w_2 \in V_c'$$

where w_1 and w_2 are given. We shall obtain the following result as a direct consequence of Theorem 6.1.

THEOREM 6.2. *Let the spaces V_a, V_c, W, the linear, symmetric, monotone operators $\mathcal{A}$, $\mathcal{C}$, and the relation B be given as above.*

(a) *Assume B is monotone. If u^j, v^j are solutions of*

$$(6.10.a) \qquad \frac{d}{dt}\mathcal{A}u^j(t) - \mathcal{A}v^j(t) = f_a^j(t)$$

$$(6.10.b) \qquad \frac{d}{dt}\mathcal{C}v^j(t) + \mathcal{A}u^j(t) + B(v^j(t)) \ni f_c^j(t) , \qquad 0 < t < T$$

for $j = 1, 2$, then it follows that

$$(6.11) \quad (|u^1(t) - u^2(t)|_a^2 + |v^1(t) - v^2(t)|_c^2)^{1/2}$$
$$\leq (|u^1(0) - u^2(0)|_a^2 + |v^1(0) - v^2(0)|_c^2)^{1/2}$$
$$+ \int_0^t (|f_a^1(s) - f_a^2(s)|_{V_a'}^2 + |f_c^1(s) - f_c^2(s)|_{V_c'}^2)^{1/2} \, ds , \quad 0 \leq t \leq T .$$

If u^j, v^j are solutions of (6.9) for $j = 1, 2$ and if $\mathcal{A}u^1(0) = \mathcal{A}u^2(0)$, $\mathcal{C}v^1(0) = \mathcal{C}v^2(0)$, then for a.e. $t \in [0, T]$ we have

$$(6.12) \qquad \mathcal{A}u^1(t) = \mathcal{A}u^2(t) , \quad \mathcal{A}v^1(t) = \mathcal{A}v^2(t) , \quad \mathcal{C}v^1(t) = \mathcal{C}v^2(t)$$

and the selections $w^j(t) = f_c(t) - (\mathcal{C}v^j(t))' - \mathcal{A}u^j(t) \in B(v^j(t))$ satisfy $\langle w^1(t) - w^2(t), v^1(t) - v^2(t) \rangle = 0$. Thus, if $\mathcal{A} + B + \mathcal{C}$ is strictly-monotone, then $v^1(t) = v^2(t)$, and if $\mathcal{A}$ is strictly-monotone, then also $u^1(t) = u^2(t)$.

(b) *Assume B is monotone, that the seminormed space V_a is complete, and $Rg(\mathcal{A} + B + \mathcal{C}) = W'$. Then for each pair $w_1 \in V_a'$, $u_2 \in W$ with $(w_1 + B(u_2)) \cap V_c'$ non-empty, and each pair $f_a : [0, T] \to V_a'$, $f_c : [0, T] \to V_c'$ of absolutely-continuous functions, there exists a solution of (6.9) with*

$$\mathcal{A}u(0) = w_1 , \qquad \mathcal{C}v(0) = \mathcal{C}u_2 .$$

PROOF. In order to obtain Theorem 6.2 as a consequence of Theorem 6.1, set $E = V_a \times V_c$, the indicated product space, and define $\mathcal{B}\vec{u} = [\mathcal{A}u , \mathcal{C}v] \in V_a' \times V_c' = E_b'$ for $\vec{u} = [u, v] \in E$. Next define the relation $A \subset E \times E_b'$ by $\vec{g} \in A(\vec{u})$ if for some $w \in B(v)$, $\vec{g} = [-\mathcal{A}v , \mathcal{A}u + w] \in V_a' \times V_c'$. Thus the domain of A is

$$D = \{\vec{u} \in V_a \times V_a : \mathcal{A}u + B(v)) \cap V_c' \text{ is non-empty } \} .$$

In case (a) it is clear that A is monotone so (6.11) follows from (6.6). The first and last equalities in (6.12) follow from (6.11), and the second follows from (6.10.a). This finishes the proof of (a).

To prove (b) we need to verify the range condition on $\mathcal{B} + A$. For this purpose, note the equivalence of the equation

$$\vec{u} \in D : (\mathcal{B} + A)\vec{u} \ni \vec{f} , \qquad \vec{f} \in E_b'$$

and the system

$$v \in W : (\mathcal{A} + B + \mathcal{C})v = f_c - f_a \in W' \ ,$$

$$u \in V_a : \mathcal{A}u = \mathcal{A}v + f_a \in V_a' \ .$$

The first has a solution by the range condition, and the second is solvable because V_a is complete, hence, $Rg(\mathcal{A}) = V_a'$. Note that any such solution u, v satisfies $\mathcal{A}u + B(v) \ni f_c - \mathcal{C}v \in V_c'$, so $[u, v] \in D$. Thus (b) is obtained from the corresponding part of Theorem 6.1.

The system (6.9) is actually equivalent to a single equation of second-order. By eliminating the first component we obtain the equation

$$(6.13) \qquad \frac{d}{dt}\Big(\frac{d}{dt}\mathcal{C}v(t) + Bv(t) - f_c(t)\Big) + \mathcal{A}v(t) = f_a(t) \ , \qquad 0 < t < T \ .$$

A *solution* of (6.13) is a function $v : [0, T] \to V_c$ such that $\mathcal{C}v : [0, T] \to V_c'$ is absolutely continuous with $v(t) \in D(B)$ at a.e. $t \in [0, T]$, $\frac{d}{dt}\mathcal{C}v + B(v) - f_c :$ $[0, T] \to V_a$ is absolutely continuous and (6.13) holds at a.e. $t \in (0, T)$. The *Cauchy problem* for (6.13) is to find a solution for which

$$\mathcal{C}v(0) = v_1 \in V_c' \ , \qquad \Big(\frac{d}{dt}\mathcal{C}v + B(v) - f_c)\Big)\Big|_{t=0} = v_2 \in V_a'$$

are specified. It is clear that the Cauchy problems for (6.9) and for (6.13) are equivalent, so Theorem 6.2 gives the following.

COROLLARY 6.3. *Let the spaces V_a, V_c, W, the linear operator $\mathcal{A}, \mathcal{C}$ and the relation B be given as above.*
(a) *If B is monotone and $\mathcal{A} + B + \mathcal{C}$ is strictly monotone, then there is at most one solution of the Cauchy problem for (6.13).*
(b) *If B is monotone, V_a is complete, and $Rg(\mathcal{A} + B + \mathcal{C}) = W'$, then for each pair $w_0 \in W$, $v_0 \in V_a'$ with $(v_0 + B(w_0)) \cap V_c'$ non-empty, and each pair $f_a : [0, T] \to V_a'$, $f_c : [0, T] \to V_c'$ of absolutely continuous functions, there exists a solution of (6.13) with*

$$\mathcal{C}v(0) = \mathcal{C}v_0 \ , \qquad \Big(\frac{d}{dt}(\mathcal{C}v) + B(v) - f_c\Big)(0) = v_o \ .$$

REMARKS. If v_1 and v_2 are solutions of (6.13) corresponding to two pairs of data

$$w_0^j, v_0^j \ , \qquad f_a^j , f_c^j \ , \qquad j = 1, 2 \ ,$$

then an estimate like (6.11) holds for

$$|v^1(t) - v^2(t)|_c^2 + \Big|\frac{d}{dt}C(v^1(t) - v^2(t)) + B^0 v^1(t) - B^0 v^2(t) + (f_c^2(t) - f_c^1(t)\Big|_{V_a'}^2$$

where $B^0 v^j(t)$ denotes the selection from $B(v^j(t))$ determined by the solution v^j.

Theorem 6.1 is obtained as the special case of Corollary 6.3 or Theorem 6.2 that results from the choice of $\mathcal{A} = 0$. Thus the three results are equivalent!

In the special case where $\mathcal{A}$ is an isomorphism, i.e., the Riesz map of the Hilbert space V_a onto its dual, we obtain an equivalent second-order equation which

characterizes (essentially) the first component u in (6.9). This is a smoother notion of solution and requires correspondingly smoother data for existence. A *solution* of

$$(6.14) \qquad \frac{d}{dt}\left(\mathcal{C}\left(\frac{dw}{dt}\right)\right) + B\left(\frac{dw}{dt}\right) + \mathcal{A}w(t) = \mathfrak{f}_a(t) + f_c(t) , \qquad 0 < t < T$$

is an absolutely continuous $w : [0, T] \to V_a$ such that $\frac{dw}{dt}(t) \in V_c$ at a.e. $t \in [0, T]$, $\mathcal{C}(\frac{dw}{dt}) : [0, T] \to V_c'$ is absolutely continuous and (6.14) holds at a.e. $t \in (0, T)$. The Cauchy problem for (6.14) is to find a solution of (6.14) which satisfies

$$w(0) = w_0 , \qquad \mathcal{C}\left(\frac{dw}{dt}\right)(0) = \mathcal{C}w_1$$

where $w_0 \in V_a$ and $w_1 \in V_c$ are given. From the change-of-variable

$$w(t) = u(t) - \mathcal{A}^{-1}\mathfrak{f}_a(t) , \qquad f_a(t) = \frac{d}{dt}\mathfrak{f}_a(t)$$

we obtain the following.

COROLLARY 6.4. *Let the spaces* V_a, V_c, W, *the linear operators* $\mathcal{A}, \mathcal{C}$ *and relation* B *be given as above.*
(a) *If* B *is monotone and* $\mathcal{A} + B + \mathcal{C}$ *is strictly-monotone, then there is at most one solution of* (6.14).
(b) *If* V_a *is Hilbert space, so* $\mathcal{A}$ *is isomorphism,* B *is monotone, and* $Rg(\mathcal{A} + B + \mathcal{C}) = W'$ *then for each pair* $w_0 \in V_a, w_1 \in D(B)$ *with* $(\mathcal{A}w_0 + B(w_1) + \mathfrak{f}_a(0)) \cap V_c'$ *non-empty and each pair* $\mathfrak{f}_a[0, T] :\to V_a'$ *with* $\frac{d}{dt}\mathfrak{f}_a$ *absolutely continuous and* $f_c : [0, T] \to V_c'$ *absolutely continuous, there exists a solution of the Cauchy problem for* (6.14).

IV.7. Accretive Operators in Banach Space

Here we begin by extending the notion of accretive operators from Hilbert space to Banach space. Many of the properties of such operators hold in this more general setting, and we list some of them. Then we briefly indicate how the proofs of existence and uniqueness given previously in Hilbert space for the evolution equation carry over to those special Banach spaces for which the duality map plays the role of the scalar product, namely, those Banach spaces whose dual is uniformly convex. The well-posedness of the Cauchy problem in a general Banach space is a much more difficult topic, and we shall develop it in Section 8.

Let X be a real Banach space. For any non-empty subset C of X we define $|C| \equiv \inf\{\|x\| : x \in C\}$. A *relation* or graph on X is a subset of $X \times X$. If A is a relation on X, we define its domain $D(A) = \{x : [x, y] \in A\}$, range $Rg(A) = \{y : [x, y] \in A\}$ and inverse $A^{-1} = \{[y, x] : [x, y] \in A\}$. Such a relation can be regarded as a function into the subsets of X with $A(x) = \{y : [x, y] \in A\}$, and then A is a (graph of a) function exactly when $A(x)$ is single-valued. We define the usual linear combinations by

$$\lambda A = \{[x, \lambda y] : [x, y] \in A\}$$
$$A + B = \{[x, y + z] : [x, y] \in A \text{ and } [x, z] \in B\} .$$

DEFINITION. The relation A on X is *accretive* if $[x_j, y_j] \in A$ for $j = 1, 2$ implies

$$\|x_1 - x_2\| \leq \|(x_1 + \alpha y_1) - (x_2 + \alpha y_2)\|$$

for all $\alpha > 0$.

This is clearly equivalent to requiring that $J_\alpha \equiv (I + \alpha A)^{-1}$ be a contraction on $Rg(I + \alpha A)$ for each $\alpha > 0$. We shall also define $A_\alpha \equiv \frac{1}{\alpha}(I - J_\alpha)$ on $Rg(I + \alpha A)$.

PROPOSITION 7.1. *Let A be accretive and $\alpha > 0$.*
(a) A_α *is Lipschitz with constant* $\frac{2}{\alpha}$ *and accretive on* $Rg(I + \alpha A)$.
(b) $A_\alpha(x) \in A \circ J_\alpha(x)$ *for* $x \in Rg(I + \alpha A)$.
(c) $\|A_\alpha(x)\| \leq |A(x)|$, $x \in D(A) \cap Rg(I + \alpha A)$.
(d) $\lim\limits_{\alpha \to 0^+} J_\alpha(x) = x$, $x \in \overline{D(A)} \cap \bigcap\limits_{\alpha > 0} Rg(I + \alpha A)$

PROOF.
(a) For $j = 1, 2$ let $y_j = (I + \beta A_\alpha)(x_j)$. Then $y_j = (I + \frac{\beta}{\alpha}(I - J_\alpha))(x_j)$ so

$$y_1 - y_2 + \frac{\beta}{\alpha}(J_\alpha x_1 - J_\alpha x_2) = \left(1 + \frac{\beta}{\alpha}\right)(x_1 - x_2) \ ,$$

hence,

$$\left(1 + \frac{\beta}{\alpha}\right)\|x_1 - x_2\| \leq \|y_1 - y_2\| + \frac{\beta}{\alpha}\|J_\alpha x_1 - J_\alpha x_2\| \ ,$$

and we need only note that J_α is a contraction.
(b) $A_\alpha x \in \frac{1}{\alpha}\left((I + \alpha A)J_\alpha x - J_\alpha x\right) = A(J_\alpha x)$.
(c) $A_\alpha(x) = \frac{1}{\alpha}\left(J_\alpha(I + \alpha A)x - J_\alpha x\right) = \frac{1}{\alpha}\left(J_\alpha(x + \alpha y) - J_\alpha(x)\right)$ for each $y \in Ax$, so $\|A_\alpha x\| \leq \|y\|$.
(d) $\|J_\alpha x - x\| = \alpha\|A_\alpha x\| \leq \alpha|Ax|$ for $x \in D(A) \cap Rg(I + \alpha A)$. The convergence extends by uniform continuity to $x \in \overline{D(A)}$.   □

LEMMA 7.1. *If A is accretive, then $Rg(I + \alpha A) = X$ for all $\alpha > 0$ if it holds for some $\alpha > 0$.*

PROOF. If $Rg(I + \alpha A) = X$ for some $\alpha > 0$ and $\beta > \alpha/2$, then for $w \in X$ we have $(I + \beta A)x \ni w$, or equivalently, $(I + \alpha A)x \ni \frac{\alpha}{\beta}w + (1 - \frac{\alpha}{\beta})x$, and this is equivalent to $T(x) = x$ for the strict contraction

$$T(x) \equiv (I + \alpha A)^{-1}\left(\frac{\alpha}{\beta}w + \left(1 - \frac{\alpha}{\beta}\right)x\right) \ ,   x \in X \ .$$

Thus $Rg(I + \beta A) = X$ for all $\beta > \alpha/2$ and by induction for all $\beta > 0$.   □

We define the relation A to be *m-accretive* if it is accretive and the above range condition holds.

PROPOSITION 7.2. *Let A be m-accretive. Then* (a) *A is maximal accretive,* (b) *A is closed, and* (c) *if $x_\alpha \to x$ and $A_\alpha x_\alpha \to y$, then $[x, y] \in A$.*

PROOF.

(a) Suppose $[x_0, y_0] \in X \times X$ and

$$\|x_0 - x\| \leq \|(x_0 + \alpha y_0) - (x + \alpha y)\| , \qquad \alpha > 0 , \ [x, y] \in A .$$

Choose $[x, y] \in A$ such that $x + y = x_0 + y_0$. Then $x_0 = x$ and $y_0 = y$, so $[x_0, y_0] \in A$.

(b) Let $A \ni [x_n, y_n] \to [x_0, y_0]$ in $X \times X$. Since $\|x_n - x\| \leq \|(x_n + \alpha y_n) - (x + \alpha y)\| , \quad \alpha > 0 , [x, y] \in A$, we have $\|x_0 - x\| \leq \|(x_0 + \alpha y_0) - (x + \alpha y)\| , \quad \alpha > 0 , [x, y] \in A$. Part (a) then shows $[x_0, y_0] \in A$.

(c) Since $\{A_\alpha x_\alpha\}$ is bounded, $J_\alpha x_\alpha \to x$. But $A_\alpha x_\alpha \in A(J_\alpha x_\alpha)$, so the result follows by (b). $\qquad\square$

The accretiveness of the relation A on X can be determined by the normalized duality map $J : X \to X'$. Indeed, comparing Proposition II.8.4 with Proposition II.8.6, we find that A is accretive if and only if $[x_j, y_j] \in A$ for $j = 1, 2$ implies there is an $f \in J(x_1 - x_2)$ such that $f(y_1 - y_2) \geq 0$. By definition of J, this f is characterized by

$$f \in X' : f(x_1 - x_2) = \|f\|^2 = \|x_1 - x_2\|^2 .$$

This characterization clearly extends the Hilbert space case where J is the identity with the identification $H \cong H'$, and it permits the following extension of Theorem 4.1 to the case of uniformly convex X'.

THEOREM 7.1 (KATO). *Let A be m-accretive on the Banach space X, and assume that X' is uniformly convex. Let $u_0 \in D(A)$, $f \in W^{1,1}(0, T; X)$, and $\omega \in \mathbb{R}$. Then there exists a unique function $u \in W^{1,\infty}(0, T; X)$ for which $u(0) = u_0$ and*

(7.1) $$\frac{du}{dt}(t) + A(u(t)) \ni \omega u(t) + f(t) , \qquad a.e. \ t \in [0, T] .$$

PROOF. Suppose u_1 and u_2 are both solutions of (7.1); then we have

$$\frac{d}{dt}(u_1(t) - u_2(t)) + Au_1(t) - Au_2(t) \ni \omega(u_1(t) - u_2(t)) .$$

By Proposition II.8.5 the function $\varphi(t) \equiv \|u_1(t) - u_2(t)\|$ is differentiable a.e., and

$$\|u_1(t) - u_2(t)\| \cdot \frac{d}{dt}\|u_1(t) - u_2(t)\| = f(u_1'(t) - u_2'(t)) , \qquad f = J(u_1(t) - u_2(t)) .$$

(See Lemma 7.2 below for a direct proof.) Thus applying f to the difference above yields

$$\|u_1(t) - u_2(t)\| \cdot \frac{d}{dt}\|u_1(t) - u_2(t)\| \leq \omega\|u_1(t) - u_2(t)\|^2 ,$$

and from here we get

$$\|u_1(t) - u_2(t)\| \leq e^{\omega t}\|u_1(0) - u_2(0)\| , \qquad t \geq 0 ;$$

this implies *uniqueness*.

The following is a direct verification of the differentiation-of-norm rule used above.

LEMMA 7.2. *Let u be weakly-differentiable at s, i.e., $\frac{d}{dt} f(u(t))|_{t=s} = f(u'(s))$ for every $f \in X'$, and assume $t \mapsto \|u(t)\|$ is G-differentiable at $t = s$. Then*

$$\|u(s)\| \frac{d}{ds} \|u(s)\| = f(u'(s)) \text{ for each } f \in J(u(s)) .$$

PROOF. For each $h > 0$ we have

$$f(u(s+h) - u(s)) \leq (\|u(s+h)\| - \|u(s)\|) \|f\| , \qquad f \in J(u(s)) .$$

Since u is weakly differentiable at s_1 we obtain

$$f(u'(s)) \leq \left(\frac{d}{ds} \|u(s)\| \right) \|u(s)\| ;$$

the reverse inequality follows similarly. $\square$

We continue with the proof of *existence* of a solution of (7.1). Consider the *approximate equations*

$$(7.2) \qquad \frac{du_\alpha}{dt}(t) + A_\alpha(u_\alpha(t)) = \omega u_\alpha(t) + f(t) , \qquad 0 \leq t \leq T$$

with $u_\alpha(0) = u_0$ for each $\alpha > 0$. Here A_α is prescribed by Proposition 7.1; specifically, each A_α is Lipschitz continuous on all of X, so (7.2) has a unique solution $u_\alpha \in C^1([0,T]; X)$ with the indicated initial value.

Let $\alpha, \beta > 0$ be given. From Lemma 7.2 we obtain

$$\frac{1}{2} \frac{d}{dt} \|u_\alpha(t) - u_\beta(t)\|^2 + J(u_\alpha(t) - u_\beta(t)) (A_\alpha u_\alpha(t) - A_\beta u_\beta(t)) = \omega \|u_\alpha(t) - u_\beta(t)\|^2 ,$$

and this gives

$$\|u_\alpha(t) - u_\beta(t)\|^2 \leq -2 \int_0^t e^{2\omega(t-s)} J(u_\alpha - u_\beta)(A_\alpha u_\alpha - A_\beta u_\beta) \, ds .$$

Since $A_\alpha u_\alpha \in A(J_\alpha u_\alpha)$ and A is accretive, we get

$$\|u_\alpha(t) - u_\beta(t)\|^2$$
$$\leq -2 \int_0^t e^{2\omega(t-s)} \left(J(u_\alpha - u_\beta) - J(J_\alpha u_\alpha - J_\beta u_\beta) \right) (A_\alpha u_\alpha - A_\beta u_\beta) \, ds$$

and, hence,

$$\begin{aligned}
&\|u_\alpha(t) - u_\beta(t)\|^2 \\
(7.3) \quad &\leq 2 \int_0^t e^{2\omega(t-s)} \|J(u_\alpha - u_\beta) - J(J_\alpha u_\alpha - J_\beta u_\beta\| \cdot \|A_\alpha u_\alpha - A_\beta u_\beta\| \, ds .
\end{aligned}$$

On the other hand, by differentiating (7.2) we get

$$\frac{d^2}{dt^2} u_\alpha + \frac{d}{dt} A_\alpha u_\alpha = \omega \frac{d}{dt} u_\alpha + \frac{df}{dt} , \qquad \text{a.e. on } [0, T] .$$

The accretiveness of A yields $J(\frac{du_\alpha}{dt})(\frac{d}{dt}A_\alpha u_\alpha) \geq 0$, so applying $J(\frac{du_\alpha}{dt})$ to this equation leads to

$$\frac{1}{2}\frac{d}{dt}\|u'_\alpha(t)\|^2 \leq \omega\|u'_\alpha(t)\|^2 + \|u'_\alpha(t)\| \cdot \|f'(t)\| \ .$$

Applying Lemma 4.1 then gives

(7.4) $$\|\frac{du_\alpha(t)}{dt}\| \leq e^{\omega t}\|\frac{du_\alpha(0)}{dt}\| + \int_0^t e^{\omega(t-s)}\|f'(s)\| \, ds \ .$$

From (7.2) at $t = 0$ and $\|A_\alpha(u_0)\| \leq |A(u_0)|$ it follows that for some $k > 0$, $\|A_\alpha u_\alpha(t)\| \leq k$ for $0 \leq t \leq T$. Thus

$$\|u_\alpha(t) - J_\alpha u_\alpha(t)\| = \alpha\|A_\alpha u_\alpha(t)\| \leq \alpha k \ ,$$

so $\{u_\alpha(t) - J_\alpha u_\alpha(t)\}$ converges to zero uniformly on $[0, T]$. Since J is uniformly continuous on bounded sets, (7.3) then shows that $u(t) = \lim_{\alpha \to 0} u_\alpha(t)$ exists in $C(0, T; X)$. Also (7.4) shows that u is Lipschitz on $[0, T]$, hence, absolutely continuous.

It remains to show that u is a solution of (7.1). Let $[x, y] \in A$ and set $x_\alpha = x + \alpha y$, so $y = A_\alpha(x_\alpha)$. Since A_α is accretive it follows from (7.2) that

$$\|u_\alpha(t) - x_\alpha\|^2 \leq \|u_\alpha(t_0) - x_\alpha\|^2 + 2\int_{t_0}^t J(u_\alpha(s) - x_\alpha)(\omega u_\alpha(s) + f(s) - y) \, ds \ .$$

By letting $\alpha \to 0$ we get

(7.5) $$\|u(t) - x\|^2 - \|u(t_0) - x\|^2 \leq 2\int_{t_0}^t J(u(s) - x)(\omega u(s) + f(s) - y) \, ds \ .$$

Note that for any pair $v, w \in X$ we have

$$J(v)(w - v) \leq \|w\|\|v\| - \|v\|^2 \leq \frac{1}{2}(\|w\|^2 - \|v\|^2) \ ,$$

so from (7.5) we obtain

$$J(u(t_0) - x)\left(\frac{u(t) - u(t_0)}{t - t_0}\right) \leq \frac{1}{t - t_0}\int_{t_0}^t J(u(s) - x)(\omega u(s) + f(s) - y) \, ds \ .$$

Now if t_0 is a point in $(0, T)$ at which $\frac{du}{dt}(t_0)$ exists, then by letting $t \to t_0$ we obtain

$$J(u(t_0) - x)\left(-\frac{du}{dt}(t_0) + \omega u(t_0) + f(t_0) - y\right) \geq 0 \ .$$

Since this holds for any $[x, y] \in A$ and A is maximal accretive by Proposition 7.2.a, we have

$$-\frac{du}{dt}(t_0) + \omega u(t_0) + f(t_0) \in A(u(t_0)) \ .$$

Since u is differentiable at a.e. $t_0 \in [0, T]$ we have (7.1). $\square$

LEMMA 7.3. *If A is m-accretive and B is accretive and Lipschitz, then $A + B$ is m-accretive.*

PROOF. Since B is single-valued, the characterization of accretive by the normalized duality map shows that $A + B$ is accretive. To show $A + B$ is m-accretive, it suffices to solve

$$x + \alpha Bx + \alpha A(x) \ni y \text{ for } y \in X \text{ and some } \alpha > 0 .$$

We may choose α such that αB is a strict contraction. But a solution of this equation is characterized by $x = (I + \alpha A)^{-1}(y - \alpha B(x))$, the fixed point of a strict contraction, so there always exists such a solution. $\quad\square$

COROLLARY 7.1. *Let X be a Banach space with uniformly convex dual, X'. Let A be a relation on X such that for some $\omega_1 \geq 0$, $A + \omega_1 I$ is m-accretive. Let B be a Lipschitz function: for some $\omega_2 > 0$*

$$\|B(x) - B(y)\| \leq \omega_2 \|x - y\| , \qquad x, y \in X .$$

Then for each $u_0 \in D(A)$, $f \in W^{1,1}(0, T; X)$ and $\omega \in \mathbb{R}$ there exists a unique $u \in W^{1,\infty}(0, T; X)$ for which $u(0) = u_0$ and

$$(7.6) \qquad \frac{du}{dt}(t) + A(u(t)) + B(u(t)) \ni \omega u(t) + f(t) , \quad a.e. \ t \in [0, T] .$$

PROOF. Add $(\omega_1 + \omega_2)u(t)$ to both sides of (7.6) and note that $B + \omega_2 I$ is accretive. $\quad\square$

One can also resolve the *periodic problem* when the operator is strongly accretive.

PROPOSITION 7.3. *Let X be a Banach space with uniformly convex dual, X'. Let A be an m-accretive relation on X for which $A - \omega I$ is accretive for some $\omega > 0$. Then for each $f \in W^{1,1}(0, T; X)$, there is a unique solution $u \in W^{1,\infty}(0, T; X)$ of*

$$(7.7) \qquad \frac{du}{dt}(t) + A(u(t)) \ni f(t) , \qquad a.e. \ t \in [0, T] ,$$
$$u(0) = u(T) .$$

PROOF. Define $K : D(A) \to \overline{D(A)}$ by $K(u_0) = u(T)$ where u is the solution of (7.7) with $u(0) = u_0$ given by Theorem 7.1. Then K is a strict contraction with $\|K(u_1) - K(u_2)\| \leq e^{-\omega T}\|u_1 - u_2\|$, so it extends by continuity to all of $\overline{D(A)}$. Moreover, it has a unique fixed point $u_0 \in \overline{D(A)}$. Furthermore, there is a sequence $u_n \in W^{1,\infty}(0, T; X)$ of solutions of (7.7) for which $u_n \to u$ in $C(0, T; X)$ and $u(0) = u(T) = u_0$.

We shall show that u is a solution of (7.7). Since $A - \omega I$ is accretive, for each n we have for $h > 0$

$$\|u_n(t + h) - u_n(t)\| \leq e^{-\omega t}\|u_n(h) - u_n(0)\| + \int_0^t e^{-\omega(t-s)}\|f(s + h) - f(s)\| \, ds .$$

By taking the limit as $n \to \infty$ we get

$$(7.8) \quad \|u(t + h) - u(t)\| \leq e^{-\omega t}\|u(h) - u(0)\| + \int_0^t e^{-\omega(t-s)}\|f(x + h) - f(s)\| \, ds$$

for all $t \in [0, T)$ and $h > 0$ with $t + h \in [0, T]$. By extending f and u to $t \in (T, 2T]$ by $f(t - T)$ and $u(t - T)$, respectively, we obtain

$$\|u(T + h) - u(T)\| \leq e^{-\omega T} \|u(h) - u(0)\|$$
$$+ \int_0^{T-h} e^{-\omega(T-s)} \|f(s + h) - f(s)\| \, ds$$
$$+ \int_{T-h}^T e^{-\omega(T-s)} \|f(s + h) - f(T) + f(0) - f(s)\| \, ds \ .$$

due to the jump in variation of f at T. The periodicity of u then gives

$$(1 - e^{-\omega T})\|u(h) - u(0)\| \leq \int_0^T e^{-\omega(T-s)} \|f(s + h) - f(s)\| \, ds$$
$$+ \|f(T) - f(0)\| \frac{1 - e^{-\omega h}}{\omega} \ ,$$

and using this in (7.8) yields

$$\|u(t + h) - u(t)\| \leq e^{-\omega t}(1 - e^{-\omega T})^{-1} \left\{ \int_0^T e^{-\omega(T-s)} \|f(s + h) - f(s)\| \, ds \right.$$
$$\left. + \|f(T) - f(0)\| \frac{1 - e^{-\omega h}}{\omega} \right\} + \int_0^t e^{-\omega(t-s)} \|f'(s)\| \, ds \ .$$

Thus u is absolutely continuous on $[0, T]$ with

$$\|u'(t)\| \leq e^{-\omega t}(1 - e^{-\omega T})^{-1} \left\{ \int_0^T e^{-\omega(T-s)} \|f'(s)\| \, ds \right.$$
$$\left. + \|f(T) - f(0)\| \right\} + \int_0^t e^{-\omega(t-s)} \|f'(s)\| \, ds \ .$$

This shows $u \in W^{1,\infty}(0, T; X)$, and the technique of the proof of Theorem 7.1 will show that it is a solution of (7.7). $\qquad \square$

We shall apply the preceding results to *semilinear evolution equations* of the form

$$\frac{du}{dt} + Au + \alpha(u) \ni f$$

in which A is a linear operator on $L^p(G)$ and α is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$. This is a direct application of the results in II.9 which were motivated by the example of the *elliptic* operator

(7.9)
$$A_1 u = - \sum_{i,j=1}^n \partial_j(a_{ij}\partial_i u) + \sum_{i=1}^n \partial_i(a_i u) + au$$

with Dirichlet boundary conditions.

In order to show that the operator "$A + \alpha$" is m-accretive in $L^p(G)$, we need to consider the corresponding stationary problem

$$u + \varepsilon(Au + v) = f \ , v \in \alpha(u)$$

for each $\varepsilon > 0$. In order to apply Theorem II.9.5, we shall assume the following:
  (i) G is a bounded domain in $\mathbb{R}^n$ and $A : D(A) \to L^1(G)$ is linear, $D(A)$ is dense in $L^1(G)$, and $(I + \lambda A)^{-1}$ is a contraction on $L^1(G)$ for every $\lambda > 0$.
  (ii) For each $f \in L^1(G)$ and $\lambda > 0$,

$$\sup_G (I + \lambda A)^{-1} f \leq (\sup_G f)^+ \ .$$

  (iii) α is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \alpha(0)$.
In this situation it follows by Theorem II.9.5 that for each $\varepsilon > 0$ and $f \in L^1(G)$ there is a unique pair $u \in D(A)$, $v \in L^1(G)$ such that

(7.10) $$u + \varepsilon(Au + v) = f \ , \quad v(x) \in \alpha(u(x)) \ , \quad \text{a.e. } x \in G \ .$$

Moreover we can apply Proposition II.9.3 to obtain estimates on any such solution. If $f \in L^p(G)$, $1 < p < \infty$, and if u is the solution of (7.10), then we multiply the equation by $\sigma_p(u(x))$, where $\sigma_p(s) = |s|^{p-1} \operatorname{sgn}(s)$ is the monotone duality map for $L^p(G)$, and integrate to obtain $\|u\|_{L^p}^p \leq \|f\|_{L^p} \|u\|_{L^p}^{p-1}$, hence, $\|u\|_{L^p(G)} \leq \|f\|_{L^p(G)}$.

Similarly, multiply by $\sigma_p(v) \in (\sigma_p \circ \alpha)(u)$ and note that $\sigma_p \circ \alpha$ is maximal monotone so from Proposition II.9.3 we obtain $\|v\|_{L^p(G)} \leq \|f\|_{L^p(G)}$. We conclude that if $f \in L^p(G)$, the solution u satisfies

$$u \in D(A), Au \in L^p(G), \text{ and for some } v \in L^p(G) v(x) \in \alpha(u(x)) \ , \quad \text{a.e. } x \in G \ .$$

We define $D(A + \alpha)_p$ to be the set of all such u. Suppose $f_1, f_2 \in L^p(G)$, $\varepsilon > 0$, and u_j, v_j are the corresponding solutions of (7.10) for $j = 1, 2$. Subtracting the equations, multiplying by $\sigma_p(u_1 - u_2)$, and integrating the products shows that $\|u_1 - u_2\|_{L^p(G)} \leq \|f_1 - f_2\|_{L^p(G)}$. This shows that the operator $A + \alpha$ with domain $D(A + \alpha)_p$ is m-accretive on $L^p(G)$.

Example. Let G be a bounded domain in $\mathbb{R}^n$, $1 < p < \infty$, and let $A = A_1$, the operator (7.9) in $L^1(G)$ constructed in Proposition II.9.1 with domain $D(A) = \{u \in W_0^{1,1}(G) : A_1 u \in L^1(G)\}$. Then $u \in L^p(G) \cap D(A)$ and $A_1 u \in L^p(G)$ imply that $u \in W_0^{1,p}(G) \cap W^{2,p}(G)$. According to Theorem 7.1, for each $u_0 \in D(A + \alpha)_p$ and $f \in W^{1,1}(0, T; L^p(G))$ there is a unique solution $u \in W^{1,\infty}(0, T; L^p(G))$ of the *semilinear parabolic* initial-boundary-value problems

(7.11.a) $$\frac{\partial u(t)}{\partial t} + A_1 u(t) + v(t) = f \ , \quad v(t) \in \alpha(u)(t)) \text{ in } L^p(G)$$

(7.11.b) $$u(t) \in W_0^{1,p}(G) \cap W^{2,p}(G) \ , \quad \text{a.e. } t \in (0, T) \ ,$$

(7.11.c) $$u(0) = u_0 \text{ in } L^p(G) \ .$$

This provides a rather strong notion of solution, even though α is very general. One can similarly apply Proposition 7.3 to get periodic solutions of (7.11.a) and (7.11.b) if $a(x)I + \alpha(\cdot)$ is strongly monotone, *i.e.*, there is an $\omega > 0$ for which

$$a(x)(r - s)^2 + (\xi - \eta)(r - s) \geq \omega(r - s)^2 \ , \qquad [r, \xi], [s, \eta] \in \alpha \ .$$

IV.8. The Cauchy Problem in General Banach Space

Let A be an operator in the Banach space X, possibly multi-valued, and let $f \in L^1(a, b; X)$. We shall consider the evolution equation

$$(8.1) \qquad u'(t) + A(u(t)) \ni f(t) , \qquad a < t < b .$$

DEFINITION. An ε-*solution* of (8.1) is a discretization

$$\mathcal{D} \equiv \{a = t_0 < t_1 < \ldots < t_N = b; f_1, f_2, \ldots, f_N \in X\}$$

and a step function $v(t) = \begin{cases} v_0 , & t = t_0 \\ v_j , & t \in (t_{j-1}, t_j] \end{cases}$ for which

$$t_j - t_{j-1} \leq \varepsilon \text{ for } 1 \leq j \leq N ,$$

$$\sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \|f(t) - f_j\| \, dt < \varepsilon , \text{ and}$$

$$\frac{v_j - v_{j-1}}{t_j - t_{j-1}} + A(v_j) \ni f_j , \qquad 1 \leq j \leq N .$$

We note that if A is m-accretive, then v is determined by $\mathcal{D}$.

DEFINITION. A C^0-*solution* of (8.1) is a $u \in C([a, b], X)$ such that for every $\varepsilon > 0$ there is an ε-solution $\mathcal{D}, v$ of (8.1) with

$$\|u(t) - v(t)\| \leq \varepsilon , \qquad a \leq t \leq b .$$

PROPOSITION 8.1.
(a) *If u is a C^0-solution on $[a, b]$ and $[c, d] \subset [a, b]$, then u is a C^0-solution on $[c, d]$.*
(b) *If $u \in C([a, b], X)$ is a C^0-solution on $[a, c]$ and on $[c, b]$, then u is a C^0-solution on $[a, b]$.*
(c) *If u is a C^0-solution in X, and X is continuously embedded in the Banach space Y, then u is a C^0-solution in Y.*
(d) *If each u_n is a C^0-solution of (8.1) with f_n, if $u_n \to u$ in $C([a, b], X)$ and $f_n \to f$ in $L^1(a, b; X)$, then u is a C^0-solution of (8.1) with f.*
(e) *If u is a C^0-solution of (8.1) with A_1, and $A_1 \subset A_2$, then u is a C^0-solution of (8.1) with A_2.*
(f) *Let $\overline{A}$ be the closure of A in $X \times X$. If u is a C^0-solution of (8.1) with $\overline{A}$, then u is a C^0-solution of (8.1) with A.*

PROOF. Parts (a) through (e) are straightforward. To prove (f), let $\overline{v}(t)$ be an ε-solution, i.e.,

$$\frac{\overline{v}_j - \overline{v}_{j-1}}{t_j - t_{j-1}} + \overline{A}(\overline{v}_j) \ni \overline{f}_j , \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \|f(t) - \overline{f}_j\| < \varepsilon .$$

From the definition of $\overline{A}$ we obtain $[v_j, w_j] \in A$ such that

$$\|v_j - \overline{v}_j\| < \varepsilon , \|w_j - \left(\overline{f}_j - \frac{\overline{v}_j - \overline{v}_{j-1}}{t_j - t_{j-1}}\right)\| < \varepsilon .$$

Set $f_j \equiv w_j + \frac{v_j - v_{j-1}}{t_j - t_{j-1}}$. Then we have

$$\|f_j - \overline{f}_j\| \leq \|w_j - \overline{f}_j + \frac{\overline{v}_j - \overline{v}_{j-1}}{t_j - t_{j-1}}\| + \|\frac{v_j - v_{j-1} - (\overline{v}_j - \overline{v}_{j-1})}{t_j - t_{j-1}}\| < \varepsilon + \frac{2\varepsilon}{t_j - t_{j-1}} \ ,$$

and this gives

$$\sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \|f_j - \overline{f}_j\| < \varepsilon(b - a) + 2\varepsilon \ ,$$

so we have

$$\sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \|f(t) - f_j\| \, dt < \varepsilon(b - a) + 3\varepsilon \ .$$

If we have $\|u(t) - \overline{v}(t)\| < \varepsilon$, then $\|u(t) - v(t)\| < 2\varepsilon$. Since ε is arbitrary, we are done. $\qquad \square$

DEFINITION. A *strong solution* of (8.1) is a $u \in W^{1,1}(a, b; X)$ such that (8.1) holds at a.e. $t \in (a, b)$.

We note that from the definition of $L^1(a, b; X)$ it follows that any $g \in L^1(a, b; X)$ is the L^1-limit of measurable step functions. Specifically, for any $\varepsilon > 0$ there is a partition $\{t_j\}$ with each $t_j - t_{j-1} < \varepsilon$ and selections $\tau_j \in [t_{j-1}, t_j]$ with

$$\sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \|g(t) - g_j\| \, dt < \varepsilon \ , g_j = g(\tau_j) \ , \qquad 1 \leq j \leq N \ .$$

Moreover, we can achieve this with $\tau_j = t_j$.

PROPOSITION 8.2. *Every strong solution is a C^0-solution.*

PROOF. Let $\varepsilon > 0$. By the preceding remark with $g = f - u'$, we choose a partition $\{t_j\}$ and selections $\{f_j\}$, $\{w_j\}$ in X with

$$\sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} (\|f(t) - f_j\| + \|u'(t) - w_j\|) \, dt < \varepsilon \ , \quad w_j + A(u(t_j)) \ni f_j \ , \quad 1 \leq j \leq N \ .$$

Set $v_j = u(t_j)$ and

$$g_j = f_j + \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} (u'(t) - w_j) \, dt \ .$$

Then we have

$$\frac{v_j - v_{j-1}}{t_j - t_{j-1}} + A(v_j) \ni g_j \ , \qquad 1 \leq j \leq N \ ,$$

and

$$\int_{t_{j-1}}^{t_j} \|f(t) - g_j\| \, dt \leq \int_{t_{j-1}}^{t_j} \|f(t) - f_j\| \, dt + \int_{t_{j-1}}^{t_j} \|u'(t) - w_j\| \, dt < \varepsilon \ ,$$

so v is an ε-solution on $\mathcal{D} = \{\{t_j\}, \{g_j\}\}$. Also, since u is uniformly continuous,

$$\|u(t) - v(t)\| = \|u(t) - u(t_i)\| \to 0 \ , \qquad t_{j-1} \leq t \leq t_j \ ,$$

uniformly in t, j as the mesh converges to zero, so u is a C^0-solution. $\qquad \square$

We have previously considered the differentiability of the norm, $\varphi(x) = \|x\|$, in a general Banach space. For example, for Lemma II.8.2 we observed that the map $t \mapsto \frac{1}{t}(\varphi(x + ty) - \varphi(x))$ is monotone in $t > 0$ and it is bounded,

$$-\|y\| \leq \frac{1}{t}(\varphi(x + ty) - \varphi(x)) \leq \|y\| ,$$

so the *directional derivative*

$$\varphi'(x, y) \equiv \lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t} = \inf_{t > 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists at every x and in every direction y. Moreover, in Proposition II.8.4 we noted that $\varphi'(x, y) \geq 0$ if and only if

$$\|x\| \leq \|x + ty\| , \qquad t > 0 ,$$

so it follows that the operator A on X is accretive if and only if for $[u_j, v_j] \in A, j = 1, 2$, we have

$$\varphi'(u_1 - u_2, v_1 - v_2) \geq 0 .$$

Additional properties of φ' are collected here.

LEMMA 8.1.
(a) $\varphi'(x, ax) = a\|x\|, a \in \mathbb{R}$, *and*
   $\varphi'(ax, y) = \varphi'(x, y), \varphi'(x, ay) = a\varphi'(x, y), a > 0$.
(b) $\varphi'(x, y_1 + y_2) \leq \varphi'(x, y_1) + \varphi'(x, y_2)$,
   $|\varphi'(x, y_1) - \varphi'(x, y_2)| \leq \|y_1 - y_2\|$.
(c) $\varphi'(x_1, x_2 - x_1) \leq \|x_2\| - \|x_1\| \leq -\varphi'(x_2, x_1 - x_2)$.
(d) *If* $x_n \to x, y_n \to y$ *in* X, *then*

$$\limsup_{n \to \infty} \varphi'(x_n, y_n) \leq \varphi'(x, y) .$$

(e) $-\|y\| \leq -\varphi'(x, -y) \leq \varphi'(x, y) \leq \|y\|$.

PROOF. Parts (a), (b) and (c) follow directly from the definition of φ'. For (d), note that for $t > 0$ we have

$$\varphi'(x_n, y_n) \leq \frac{1}{t}(\|x_n + ty_n\| - \|x_n\|)$$

so we obtain

$$\limsup_{n \to \infty} \varphi'(x_n, y_n) \leq \frac{1}{t}(\|x + ty\| - \|x\|) , \qquad t > 0 .$$

Finally, part (e) follows from the successive estimates

$$2\|x\| \leq \|x + ty\| + \|x - ty\| ,$$

$$-\|y\| \leq \frac{\|x\| - \|x - ty\|}{t} \leq \frac{\|x + ty\| - \|x\|}{t} \leq \|y\| . \qquad \square$$

We note that the lower limit

$$\lim_{t \to 0^-} \frac{1}{t}(\|x + ty\| - \|x\|) = \lim_{t \to 0^+} \frac{1}{t}(\|x\| - \|x - ty\|) = -\varphi'(x, -y)$$

is the *left derivative* and this corresponds to estimates that are *stronger* than accretive. Finally, corresponding to Proposition II.8.5 we have the following.

LEMMA 8.2. *If* $u : [0, T] \to X$ *is right-differentiable (left-differentiable) at* $t \in (0, T)$, *then* $\|u(\cdot)\|$ *is also right (respectively, left)-differentiable at* t *and*

$$D^+ \|u(t)\| = \varphi'(u(t), D^+ u(t)) \ ,$$

respectively, $D^- \|u(t)\| = -\varphi'(u(t), -D^- u(t))$.

COROLLARY 8.1. *If* $u \in W^{1,1}(0, T; X)$, *then* $\|u(\cdot)\| \in W^{1,1}(0, T)$ *and*

$$\frac{d}{dt} \|u(t)\| = \varphi'(u(t), u'(t)) = -\varphi'(u(t), -u'(t)) \ , \quad a.e. \ t \in (0, T) \ .$$

PROOF. A composition of Lipschitz and absolutely continuous functions is then absolutely continuous, hence a.e. differentiable, and there the left and right derivatives agree. □

This leads to the fundamental a-priori estimate for strong solutions of (8.1).

PROPOSITION 8.3. *Assume* A *is accretive on* X *and that for* $j = 1, 2$ *we have*

$$u_j \in W^{1,1}(0, T; X) \ , f_j \in L^1(0, T; X)$$
$$u_j'(t) + A(u_j(t)) \ni f_j(t) \ , \quad a.e. \ t \in (0, T) \ .$$

Then $\|u_1(\cdot) - u_2(\cdot)\| \in W^{1,1}(0, T)$ *and*

$$(8.2) \qquad \frac{d}{dt} \|u_1(t) - u_2(t)\| \le \varphi'(u_1(t) - u_2(t), f_1(t) - f_2(t)) \ , \quad a.e \ t \in (0, T) \ .$$

PROOF. Set $v_j(t) = f_j(t) - u_j'(t) \in A(u_j(t))$ so that $\varphi'(u_1(t) - u_2(t), v_1(t) - v_2(t)) \ge 0$. From Corollary 8.1 and Lemma 8.1(b) we obtain

$$\frac{d}{dt} \|u_1(t) - u_2(t)\| = -\varphi'\left(u_1(t) - u_2(t), -(u_1'(t) - u_2'(t))\right)$$
$$\le \varphi'\left(u_1(t) - u_2(t), f_1(t) - f_2(t)\right)$$
$$- \varphi'\left(u_1(t) - u_2(t), v_1(t) - v_2(t)\right) \ .$$

a.e. on $(0, T)$, so the estimate follows. □

COROLLARY 8.2. *If* u_1, u_2 *are strong solutions of* (8.1) *on* $[0, T]$ *with data* f_1, f_2, *respectively, then*

$$\|u_1(t) - u_2(t)\| \le \|u_1(0) - u_2(0)\| + \int_0^t \|f_1(s) - f_2(s)\| \, ds \ , \quad 0 \le t \le T \ .$$

We can obtain a *discrete* version of (8.2) in the following form.

PROPOSITION 8.4. *Assume A is accretive on X and that an ε-solution of (8.1) is given by*

$$v(t) = v_j \ ,$$

$$t_{j-1} < t \leq t_j \ ,$$

$$\frac{v_j - v_{j-1}}{t_j - t_{j-1}} + A(v_j) \ni f_j \ , \qquad 1 \leq j \leq N \ .$$

Then we have for each $[x, y] \in A$

$$(8.3) \qquad \frac{\|v_j - x\| - \|v_{j-1} - x\|}{t_j - t_{j-1}} \leq \varphi'(v_j - x, f_j - y) \ , \qquad 1 \leq j \leq N \ .$$

PROOF. From Lemma 8.1 we have successively

$$\frac{\|v_j - x\| - \|v_{j-1} - x\|}{t_j - t_{j-1}} \leq \frac{-\varphi'(v_j - x, v_{i-1} - v_i)}{t_j - t_{j-1}}$$

$$= -\varphi'\left(v_j - x, \frac{v_{j-1} - v_j}{t_j - t_{j-1}} + f_j - y - (f_j - y)\right)$$

$$\leq -\varphi'\left(v_j - x, \left(\frac{v_{j-1} - v_j}{t_j - t_{j-1}} + f_j\right) - y\right) + \varphi'(v_j - x, f_j - y)$$

$$\leq 0 + \varphi'(v_j - x, f_j - y) \ . \qquad \square$$

Next we develop an *integral* version of (8.2).

LEMMA 8.3. *For $\varepsilon > 0$ let there be given a pair of real-valued step functions*

$$V^\varepsilon(t) = V_j^\varepsilon \ , F^\varepsilon(t) = F_j^\varepsilon \ , \qquad t_{j-1} < t \leq t_j \ ,$$

where the mesh of the partition $0 < t_1 < t_2 < \ldots < t_{N_\varepsilon} = T$ goes to zero with ε. Assume $V\varepsilon \to V$ in $C[0, T]$, $|F^\varepsilon| \leq g \in L^1(0, T)$, $F(t) = \limsup_{\varepsilon \to 0} F^\varepsilon(t)$, and that

$$\frac{V_j^\varepsilon - V_{j-1}^\varepsilon}{t_j - t_{j-1}} \leq F_j^\varepsilon \ , \qquad 1 \leq j \leq N_\varepsilon \ .$$

Then we have

$$V(t) - V(s) \leq \int_s^t F \ , \qquad 0 \leq s \leq t \leq T \ .$$

PROOF. By an easy induction there follows

$$V_m^\varepsilon - V_n^\varepsilon \leq \sum_{j=n+1}^m F_j^\varepsilon (t_j - t_{j-1}) \ , \qquad 1 \leq n \leq m \leq N_\varepsilon \ ,$$

so the desired estimate follows by dominated convergence. $\square$

Now apply Lemma 8.3 to (8.3). Since φ' is upper-semi-continuous this leads to the following.

COROLLARY 8.3. *Assume A is accretive on X, $f \in L^1(0,T;X)$ and u is a C^0-solution of (8.1). Then for each $[x,y] \in A$ we have*

$$(8.4) \quad \|u(t) - x\| \leq \|u(s) - x\| + \int_s^t \varphi'(u(\tau) - x, f(\tau) - y)\, d\tau \, , \qquad 0 \leq s \leq t \leq T \, .$$

PROPOSITION 8.5. *Assume A is accretive on X, $f_j \in L^1(0,T;X)$, and u_j is a C^0-solution of (8.1) with f_j for $j = 1,2$. Then we have*

$$
\begin{aligned}
& \int_a^b \left(\|u_1(t) - u_2(\sigma)\| - \|u_1(s) - u_2(\sigma)\| \right) d\sigma \\
(8.5) \qquad & \quad + \int_s^t \left(\|u_1(\tau) - u_2(b)\| - \|u_1(\tau) - u_2(a)\| \right) d\tau \\
& \leq \int_a^b \int_s^t \varphi'(u_1(\tau) - u_2(\sigma)\, , f_2(\tau) - f_2(\sigma))\, d\tau\, d\sigma \, , \\
& \text{for } 0 \leq s < t < T\, , 0 \leq a < b \leq T \, .
\end{aligned}
$$

PROOF. Choose $\mathcal{D} = \{0 = t_0 < t_1 < \cdots < t_N = T; f_1, \cdots f_N \in X\}$ and a corresponding ε-solution $v(t) = v_j$ on $(t_{j-1}, t_j]$ for the equation

$$u_2' + A(u_2) \ni f_2 \, .$$

From Corollary 8.3 with $u = u_1$ and $f = f_1$ we obtain from (8.4)

$$
\begin{aligned}
\|u_1(t) - v_j\| - \|u_1(s) - v_j\| &\leq \int_s^t \varphi'\left(u_1(\tau) - v_j, f_1(\tau) - f_j + \frac{v_j - v_{j-1}}{t_j - t_{j-1}} \right) d\tau \\
&\leq \int_s^t \left(\varphi'\left(u_1(\tau) - v_j, f_1(\tau) - f_j \right) + \varphi'(u_1(\tau) - v_j, \frac{v_j - v_{j-1}}{t_j - t_{j-1}}) \right) d\tau \, .
\end{aligned}
$$

The latter term in this integrand is bounded by

$$\frac{\|u_1(\tau) - v_{j-1}\| - \|u_1(\tau) - v_j\|}{t_j - t_{j-1}} \, ,$$

so we obtain

$$
\begin{aligned}
\|u_1(t) - v_j\| - \|u_1(s) - v_j\| &+ \int_s^t \frac{\|u_1(\tau) - v_j\| - \|u_1(\tau) - v_{j-1}\|}{t_j - t_{j-1}} \, d\tau \\
&\leq \int_s^t \varphi'(u_1(\tau) - v_j, f_1(\tau) - f_j)\, d\tau \, , \\
& 0 \leq s < t \leq T\, , 1 \leq j \leq N \, .
\end{aligned}
$$

Multiply by $t_j - t_{j-1}$ and sum for $j = n+1, n+2, \ldots, m$ to obtain

$$\int_{t_n}^{t_m} \left(\|u_1(t) - v(\sigma)\| - \|u_1(s) - v(\sigma)\| \right) d\sigma$$

$$+ \int_s^t \left(\|u_1(\tau) - v(t_m)\| - \|u_1(\tau) - v(t_n)\| \right) d\tau$$

$$\leq \int_{t_n}^{t_m} \int_s^t \varphi'(u_1(\tau) - v(\sigma), f_1(\tau) - f(\sigma)) \, d\tau \, d\sigma$$

where $f(\sigma) = f_j$ for $\sigma \in (t_{j-1}, t_j)$. Letting $\varepsilon \to 0$ with $v \equiv v_\varepsilon \to u_2$ and $f = f_\varepsilon \to f_2$ leads to (8.5). □

REMARK. Define the functions

$$u(t, s) \equiv \begin{cases} \|u_1(t) - u_2(s)\| \,, & 0 \leq s \leq t \leq T \\ 0 \,, & 0 \leq t \leq s \leq T \end{cases}$$

$$F(t, s) \equiv \begin{cases} \varphi'(u_1(t) - u_2(s) \,, f_1(t) - f_2(s)) \,, & 0 \leq s \leq t \leq T \,, \\ 0 \,, & 0 \leq t \leq s \leq T \end{cases}$$

for which we can write (8.5) in the form

$$\int_a^b \left(u(t, \sigma) - u(s, \sigma) \right) d\sigma + \int_s^t \left(u(\tau, b) - u(\tau, a) \right) d\tau$$

(8.5′)
$$\leq \int_a^b \int_s^t F(\tau, \sigma) \, d\tau \, d\sigma \,.$$

If it were true that $u(\cdot, \cdot)$ is absolutely continuous in each variable, then this would imply that

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial s} \leq F(t, s) \,,$$

hence, $\frac{d}{d\sigma} u(\sigma, \sigma) \leq F(\sigma, \sigma)$ and then

$$u(t, t) \leq u(s, s) + \int_s^t F(\sigma, \sigma) \, d\sigma \,.$$

This suggests the proof of the following.

THEOREM 8.1 (BENILAN-CRANDALL-EVANS). *Assume that A is accretive on X, $f_j \in L^1(0, T; X)$, and that u_j is a C^0-solution of (8.1) with f_j for $j = 1, 2$. Then we have*

(8.6) $\|u_1(t) - u_2(t)\| \leq \|u_1(s) - u_2(s)\|$

$$+ \int_s^t \varphi'(u_1(\sigma) - u_2(\sigma) \,, f_1(\sigma) - f_2(\sigma)) \, d\sigma \,, \quad 0 \leq s \leq t \leq T \,,$$

and therefore

$$\|u_1 - u_2\|_{C(0,T;X)} \leq \|u_1(0) - u_2(0)\|_X + \|f_1 - f_2\|_{L^1(0,T;X)} \,.$$

PROOF. The plan is to obtain (8.6) from (8.5′) by *regularization*. For this we choose a

$$\rho \in C_0^\infty(-1,1) : 0 \le \rho(x) , \int_{-1}^1 \rho(x)\,dx = 1$$

and define $\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho(\frac{x}{\varepsilon})$ for $\varepsilon > 0$. In $\mathbb{R}^2$ we define *convolutions*

$$u_\varepsilon(t,\sigma) = \iint_{\mathbb{R}^2} \rho_\varepsilon(\xi)\rho_\varepsilon(\eta)u(t-\xi,\sigma-\eta)\,d\xi\,d\eta ,$$

$$F_\varepsilon(\tau,\sigma) = \iint_{\mathbb{R}^2} \rho_\varepsilon(\xi)\rho_\varepsilon(\eta)F(\tau-\xi,\sigma-\eta)\,d\xi\,d\eta .$$

Then we obtain

$$\int_a^b (u_\varepsilon(t,\sigma) - u_\varepsilon(s,\sigma))\,d\sigma + \int_s^t (u_\varepsilon(\tau,b) - u_\varepsilon(\tau,a))\,d\tau$$

$$= \iint_{\mathbb{R}^2} \rho_\varepsilon(\xi)\rho_\varepsilon(\eta)\bigg\{ \int_a^b (u(t-\xi,\sigma-\eta) - u(s-\xi,\sigma-\eta))\,d\sigma$$

$$+ \int_s^t (u(\tau-\xi,b-\eta) - u(\tau-\xi,a-\eta))\,d\tau \bigg\} d\xi\,d\eta$$

$$= \iint_{\mathbb{R}^2} \rho_\varepsilon(\xi)\rho_\varepsilon(\eta)\bigg\{ \int_{a-\eta}^{b-\eta} (u(t-\xi,\sigma) - u(s-\xi,\sigma))\,d\sigma$$

$$+ \int_{s-\xi}^{t-\xi} (u(\tau,b-\eta) - u(\tau,a-\eta))\,d\tau \bigg\} d\xi\,d\eta$$

$$\le \iint_{\mathbb{R}^2} \rho_\varepsilon(\xi)\rho_\varepsilon(\eta)\bigg\{ \int_{a-\eta}^{b-\eta} \int_{s-\xi}^{t-\xi} F(\tau,\sigma)\,d\tau\,d\sigma \bigg\} d\xi\,d\eta$$

$$= \iint_{\mathbb{R}^2} \rho_\varepsilon(\xi)\rho_\varepsilon(\eta)\bigg\{ \int_a^b \int_s^t F(\tau-\xi,\sigma-\eta)\,d\tau\,d\sigma \bigg\} d\xi\,d\eta$$

$$= \int_a^b \int_s^t F_\varepsilon(\tau,\sigma)\,d\tau\,d\sigma ,$$

$$0 < \varepsilon \le a \le b \le T , \ 0 < \varepsilon \le s \le t \le T ,$$

where the inequality above follows from (8.5′). Now let $\lambda > 0$ and set

$$F^\lambda(t,\sigma) = \frac{1}{\lambda} \left(\|u_1(t) + \lambda f_1(t) - (u_2(\sigma) + \lambda f_2(\sigma))\| - \|u_1(t) - u_2(\sigma)\| \right)$$

and note that $F(t,\sigma) \le F^\lambda(t,\sigma)$, hence, their convolutions satisfy

$$F_\varepsilon(t,\sigma) \le F_\varepsilon^\lambda(t,\sigma) .$$

By applying our Remark to the estimate above we obtain

$$u_\varepsilon(t,t) \le u_2(s,s) + \int_s^t F_2^\lambda(\sigma,\sigma)\,d\sigma , \qquad 0 < \varepsilon \le s \le t \le T .$$

Since u is continuous, $u_\varepsilon(t,t) \to u(t,t)$ as $\varepsilon \to 0$. Once we show that

$$\int_s^t F_\varepsilon^\lambda(\sigma, \sigma)\, d\sigma \to \int_s^t F^\lambda(\sigma, \sigma)\, d\sigma$$

then (8.6) will follow and we are done.

But F^λ is of the form of the difference of two functions $\|g_1(\tau) - g_2(\sigma)\|$ with $g_j \in L^1(0, T; X)$, so it suffices to consider

$$\left| \int_s^t (\|g_1(\sigma) - g_2(\sigma)\|_\varepsilon - \|g_1(\sigma) - g_2(\sigma)\|)\, d\sigma \right|$$

$$= \left| \iint_{\mathbb{R}^2} \int_s^t \rho_\varepsilon(\xi)\rho_\varepsilon(\eta)(\|g_1(\sigma - \xi) - g_2(\sigma - \eta)\| - \|g_1(\sigma) - g_2(\sigma)\|)\, d\sigma\, d\xi\, d\eta \right|$$

$$\leq \iint_{\mathbb{R}^2} \int_s^t \rho_\varepsilon(\xi)\rho_\varepsilon(\eta)\|g_1(\sigma - \xi) - g_1(\sigma) - g_2(\sigma - \eta) + g_2(\sigma)\|\, d\sigma\, d\xi\, d\eta$$

$$\leq \int_{\mathbb{R}} \int_s^t \rho_\varepsilon(\xi)\|g_1(\sigma - \xi) - g_1(\sigma)\|\, d\sigma\, d\xi$$

$$+ \int_{\mathbb{R}} \int_s^t \rho_\varepsilon(\eta)\|g_2(\sigma - \eta) - g_2(\sigma)\|\, d\sigma\, d\eta\ ,$$

and each of these converges to zero by standard results on the mollifier in L^1. $\square$

We turn now to the question of *existence* of a C^0-solution of (8.1) and begin with the Cauchy problem for the corresponding homogeneous equation

$$(8.7) \qquad u'(t) + A(u(t)) \ni 0\ , \quad 0 < t < T\ , \quad u(0) = u_0$$

with m-accretive A and $u_0 \in D(A)$. For this case we shall show that the linear interpolants of ε-solutions for a uniform partition of $[0, T]$ are Cauchy in $C(0, T; X)$. The non-homogeneous equation will be easily resolved then by Proposition 8.1 applied to step-functions approximating a given $f \in L^1(0, T; X)$.

Given $T > 0$ and an integer $M \geq 1$, consider the sequence defined by

$$u_m = (I + sA)^{-m} u_0\ , \quad 0 \leq m \leq M\ , \quad s = T/M\ .$$

That is, we have equivalently

$$u_m + sa_m = u_{m-1}\ , \quad a_m \in A(u_m)\ , \quad 1 \leq m \leq M\ .$$

Let $a_0 \in A(u_0)$. The following two estimates are basic.

LEMMA 8.4 (LIPSCHITZ).

$$\|u_m - u_{m-1}\| \leq s\|a_0\|\ , \quad \|u_m - u_0\| \leq ms\|a_0\|\ , \qquad m = 1, 2, \dots, M\ .$$

PROOF. For $m = 1$ we note

$$u_1 - u_0 + s(a_1 - a_0) = -sa_0$$

so $\|u_1 - u_0\| \leq s\|a_0\|$. For $m > 1$ we have

$$u_m - u_{m-1} + s(a_m - a_{m-1}) = u_{m-1} - u_{m-2}$$

from which $\|u_m - u_{m-1}\| \leq \|u_{m-1} - u_{m-2}\|$ follows by accretiveness again.   □

LEMMA 8.5 (COMPARISON). *Let u_m be as above and*

$$v_n = (I + \tau A)^{-n} u_0 \ , \qquad 0 \leq n \leq N \ , \quad \tau = T/N \ ,$$

where $N > M$, hence $\theta \equiv \tau/s \in (0, 1)$. Then

$$\|u_m - v_n\| \leq \theta\|u_{m-1} - v_{n-1}\| + (1-\theta)\|u_m - v_{n-1}\| \ , \qquad 1 \leq m \leq M \ , \ 1 \leq n \leq N \ .$$

PROOF. From the inclusions

$$u_m - v_n \in u_{m-1} - v_{n-1} - sAu_m + \tau Av_n = u_{m-1} - v_{n-1} - \tau(Au_m - Av_n) - (s-\tau)Au_m$$

and

$$(s - \tau)Au_m \ni \frac{s - \tau}{s}(u_{m-1} - u_m) \ ,$$

we have

$$u_m - v_n + \tau(Au_m - Av_n) \ni u_{m-1} - v_{n-1} - (1 - \theta)(u_{m-1} - u_m)$$
$$= \theta(u_{m-1} - v_{n-1}) + (1 - \theta)(u_m - v_{n-1}) \ .$$

This yields the desired estimate.   □

Now set $a_{m,n} \equiv \|u_m - v_n\|$ and $K = \|a_0\|$. According to the preceding estimates, we are in the situation of the following.

PROPOSITION 8.6. *Consider the rectangular grid*

$$\mathcal{R} \equiv \{0, 1, 2, \ldots, M\} \times \{0, 1, 2, \ldots, N\}$$

with boundary grid

$$\partial\mathcal{R} \equiv \{0\} \times \{0, 1, \ldots, N\} \cup \{0, 1, \ldots, M\} \times \{0\} \ .$$

If the function $a : \mathcal{R} \to \mathbb{R}^+$ satisfies

$$a_{m,0} \leq msK \ , \quad 0 \leq m \leq M \ ,$$
$$a_{0,n} \leq n\tau K \ , \quad 0 \leq n \leq N \ , \ and$$
$$a_{m,n} \leq \theta a_{m-1,n-1} + (1 - \theta)a_{m,n-1} \ , \qquad 0 < m \leq M \ , \ 0 < n \leq N \ ,$$

with $\theta = \tau/s \in (0, 1)$, then

$$(8.8) \qquad a_{m,n} \leq K\left[n\tau(s - \tau) + (ms - n\tau)^2\right]^{1/2} \ .$$

Let $u, v \in C(0, T; X)$ be the piece-wise linear functions for which $u(ms) = u_m$ and $v(n\tau) = v_n$. Then for each $t \in [0, T]$ we have

$$t = ms + \xi = n\tau + \eta , \quad 0 \leq \xi < s , \quad 0 \leq \eta < \tau ,$$

and furthermore

$$u(t) = u_m + (u_{m+1} - u_m)\frac{\xi}{s} , \, v(t) = v_n + (v_{n+1} - v_n)\frac{\eta}{\tau} .$$

It follows, then, from (8.8) that

(8.9)
$$\begin{aligned}
\|u(t) - v(t)\| &\leq \|u_m - v_n\| + \|u_{m+1} - u_m\| + \|v_{n+1} - v_n\| \\
&\leq \|a_0\|[T(s - \tau) + (\xi - \eta)^2]^{1/2} + \|a_0\|(s + \tau) \\
&\leq \|a_0\|[T(s - \tau) + s^2]^{1/2} + \|a_0\|(s + \tau) ,
\end{aligned}$$

and this goes uniformly to zero as $s = \frac{1}{M} \geq \tau = \frac{1}{N} \to 0$. Thus, the piece-wise linear interpolants of $\{u_m\}$ converge uniformly to some $u \in C(0, T; X)$. Since the corresponding step functions converge uniformly to u, it is a C^0-solution of (8.7), and it is Lipschitz continuous with

$$\|u(t) - u(s)\| \leq |t - s| \inf\{\|a\| : a \in A(u_0)\} .$$

Except for the proof of Proposition 8.6, this gives the following.

PROPOSITION 8.7. *Let A be m-accretive on the Banach space X and $u_0 \in D(A)$. There is a unique $u \in C([0, \infty), X)$ such that, for every $T > 0$, u is a C^0-solution of (8.7) on $[0, T]$ and u is Lipschitz.*

The following is an immediate extension of Proposition 8.7.

THEOREM 8.2 (CRANDALL-LIGGETT). *Assume A is accretive on the Banach space X and that $Rg(I + \lambda A) \supset \overline{D(A)}$ for every $\lambda > 0$. Let $u_0 \in \overline{D(A)}$. Then $u(t) \equiv \lim_{m \to \infty} (I + \frac{t}{m}A)^{-m} u_0$ exists uniformly on each $[0, T]$, and u is the unique C^0-solution of (8.7).*

PROOF. For each $\varepsilon > 0$ there is a $[\overline{u}, \overline{v}] \in A$ with $\|\overline{u} - u_0\| < \varepsilon$. Set

$$\overline{u}_m(t) = (I + sA)^{-m}\overline{u} , \quad u_m(t) = (I + sA)^{-m}u_0$$

for $t = ms + \xi$, $0 \leq \xi < s$. Then from (8.9) we obtain

$$\|\overline{u}_m(t) - \overline{u}_n(t)\| \leq \|\overline{v}\|\sqrt{Ts + s^2} = \|\overline{v}\|T\sqrt{\frac{1}{M} + \frac{1}{M^2}} , \qquad n \geq m \geq M ,$$

so we have

$$\|u_m(t) - u_n(t)\| \leq \|\overline{v}\|T\sqrt{\frac{1}{M} + \frac{1}{M^2}} + 2\varepsilon .$$

This shows $u_m \to u$ in $C(0, T; X)$. Note that u_m is the $\frac{1}{M}$-solution for the discretization $\mathcal{D} = \{0, \frac{1}{M}, \frac{2}{M}, \dots, T; 0, \dots, 0\}$. $\square$

REMARK. The function u is not necessarily Lipschitz, but we have the estimate

$$\|u_m(t) - u(t)\| \le 2\|u_0 - \overline{u}\| + T\|\overline{v}\|\sqrt{\frac{1}{M} + \frac{1}{M^2}} \ , \qquad m \ge M \ , \ [\overline{u}, \overline{v}] \in A \ .$$

COROLLARY 8.4. *Assume A is m-accretive in the Banach space X. For each $u_0 \in \overline{D(A)}$ and $f \in L^1(0, T; X)$ there is a unique C^0-solution of the Cauchy problem*

(8.10) $$u'(t) + A(u(t)) \ni f(t) \ , \quad 0 < t < T \ , \quad u(0) = u_0 \ .$$

PROOF. If f is a constant, then $A(\cdot) - f$ is m-accretive and Theorem 8.2 applies. If f is a stepfunction, we get a C^0-solution on each interval on which f is constant, then Proposition 8.1.b gives a solution on $[0, T]$. Finally, if $f = \lim_{n \to \infty} f_n$ in $L^1(0, T; X)$ where each f_n is a stepfunction, then the corresponding solutions satisfy $\lim_{n \to \infty} u_n = u$ in $C([0, T], X)$ by Theorem 8.1, and from Proposition 8.1.d it follows that u is a C^0-solution. $\qquad\square$

We consider next the dependence of the solution u on the operator A. Let's begin with the following notion of convergence of operator.

DEFINITION. Let A and A_n, $n \ge 1$, be operators in X. Then $\{A_n\}$ is *graph-convergent* to A if for every $[u, v] \in A$ there are $[u_n, v_n] \in A_n$, $n \ge 1$, with $[u_n, v_n] \to [u, v]$ in $X \times X$.

PROPOSITION 8.8. *Let A and each $A_n, n \ge 1$, be m-accretive in X. Then $\{A_n\}$ is graph-convergent to A if and only if $(I + A_n)^{-1}f \to (I + A)^{-1}f$ for every $f \in X$.*

PROOF. If we have graph-convergence, then let $f \in X$ and set $u_n = (I + A_n)^{-1}f$, $u = (I + A)^{-1}f$, so $[u, f - u] \in A$. Then there exist $[\overline{u}_n, \overline{v}_n] \in A_n$ with $\overline{u}_n \to u$, $\overline{v}_n \to f - u$, and $\overline{u}_n = (I + A_n)^{-1}(\overline{u}_n + \overline{v}_n)$. By accretivity of A,

$$\|\overline{u}_n - u_n\| \le \|\overline{u}_n + \overline{v}_n - f\| \to 0 \ .$$

Since $\overline{u}_n \to u$ we have $u_n \to u$.

Conversely, let $[u, v] \in A$, hence, $u = (I + A)^{-1}(u + v)$, and set $u_n \equiv (I + A_n)^{-1}(u + v)$. Then $u_n \to u$, $v_n \equiv u + v - u_n \to v$, and $[u_n, v_n] \in A_n$, $n \ge 1$. $\qquad\square$

COROLLARY 8.5. *We have*

$$\lim_{n \to \infty} (I + \lambda A_n)^{-1}f = (I + \lambda A)^{-1}f \ , \qquad f \in X$$

for some $\lambda > 0$ if and only if for all $\lambda > 0$.

COROLLARY 8.6. *For each integer $m \ge 1$,*

$$\lim_{n \to \infty} (I + A_n)^{-m}f = (I + A)^{-m}f \ , \qquad f \in X \ .$$

PROOF. Suppose this holds for $m = k \geq 1$. Then

$$\|(I + A_n)^{-k-1}f - (I + A)^{-k-1}f\| =$$
$$\|(I + A_n)^{-1}(I + A_n)^{-k}f \mp (I + A_n)^{-1}(I + A)^{-k}f - (I + A)^{-1}(I + A)^{-k}f\|$$
$$\leq \|(I + A_n)^{-k}f - (I + A)^{-k}f\| + \|(I + A_n)^{-1}f - (I + A)^{-1}f\| ,$$

so the result follows by induction on m.   □

This leads to the *continuous dependence* of the C^0-solution of (8.10) on the operator A as well as u_0 and f.

THEOREM 8.3 (BENILAN). *Let A and each A_n, $n \geq 1$, be m-accretive in the Banach space X. Assume $\{A_n\}$ is graph-convergent to A. Assume $f_n \to f$ in $L^1(0, T; X)$ and let u_0, $u_0^n \in \overline{D(A)}$ with $u_0^n \to u_0$ in X. Then $u_n \to u$ in $C([0, T], X)$, where u is the C^0-solution of (8.10) and each u_n is the corresponding C^0-solution with A_n, f_n, u_0^n, $n \geq 1$.*

PROOF. Consider first the case $f_n = f = 0$. For $M > 1$ and $s = T/M$, $ms \leq t < (m+1)s$, $m = 0, 1, \ldots, M - 1$, we obtain

$$\|u_n(t) - u(t)\| \leq \|u_n(t) - (I + sA_n)^{-m}u_0^n\|$$
$$+ \|(I + sA_n)^{-m}u_0^n - (I + sA_n)^{-m}u_0\|$$
$$+ \|(I + sA_n)^{-m}u_0 - (I + sA)^{-m}u_0\|$$
$$+ \|(I + sA)^{-m}u_0 - u(t)\|$$
$$\leq \left(2\|u_0^n - u^n\| + \sqrt{\frac{2}{M}}T\|v_n\|\right) + \|u_0^n - u_0\|$$
$$+ \|(I + sA_n)^{-m}u_0 - (I + sA)^{-m}u_0\|$$
$$+ \left(2\|u_0 - u\| + \sqrt{\frac{2}{M}}T\|v\|\right)$$

for any sequence $[u_n, v_n] \in A_n$ and any $[u, v] \in A$. Pick $[u, v]$ as above so that $\|u_0 - u\|$ is small, then pick $[u_n, v_n]$ so that $u_n \to u$, $v_n \to v$, hence, for large n, $\|u_0^n - u^n\|$ is small while $\|v_n\|$ is bounded. Choose M so large that all terms are small, except possibly the third. Then letting n get large makes this last one also arbitrarily small by Corollary 8.6.   □

We have established the desired result for the case $f_n = f = 0$, and it easily follows for any constant, hence, for any step function $g = f_n = f$. Let v, v_n be the solutions of the corresponding problems for u, u_n but with f, f_n replaced by the step function g. By our preceding remark, $v_n \to v$ in $C([0, T], X)$. But we have

$$\|u_n - u\|_{L^\infty} \leq \|u_n - v_n\|_{L^\infty} + \|v_n - v\|_{L^\infty} + \|v - u\|_{L^\infty}$$
$$\leq \|f_n - g\|_{L^1} + \|v_n - v\|_{L^\infty} + \|g - f\|_{L^1}$$

and this shows $\limsup_{n\to\infty} \|u_n - u\|_{L^\infty} \leq 2\|f - g\|_{L^1}$. But g is arbitrary so we have $u_n \to u$ in $C([0, T], X)$.

Finally, we return to the proof of Proposition 8.6. Let $\mathcal{R}$ denote the rectangular grid with boundary $\partial\mathcal{R}$ as before.

LEMMA 8.6. *If the functions a, b from $\mathcal{R}$ to $\mathbb{R}^+$ satisfy*

(8.11.a) $a_{m,n} \leq \theta a_{m-1,n-1} + (1 - \theta)a_{m,n-1}$,

(8.11.b) $b_{m,n} = \theta b_{m-1,n-1} + (1 - \theta)b_{m,n-1}$, $1 \leq m \leq M$, $1 \leq n \leq N$,

and $a \leq b$ on $\partial\mathcal{R}$, then $a \leq b$ on $\mathcal{R}$.

PROOF. This is proved by induction on $\max(M, N)$, the size of the grid $\mathcal{R}$. If $\max(M, N) = 1$ then the only value in question is $a_{1,1}$, all other points being on $\partial\mathcal{R}$. However, this follows immediately from the calculation,

$$a_{1,1} \leq \theta a_{0,0} + (1 - \theta)a_{0,1}$$
$$\leq \theta b_{0,0} + (1 - \theta)b_{1,0} = b_{1,1} \ .$$

Assuming the result for grids of size less than $\max(M, N)$ it suffices to show the result for the column $\{1\} \times \{1, 2, \ldots, N\}$ and row $\{1, 2, \ldots, M\} \times \{1\}$, since this is the boundary of the smaller $(M - 1) \times (N - 1)$ grid $\{1, 2, \ldots, M\} \times \{1, 2, \ldots, N\}$. The calculation

$$a_{m+1,1} \leq \theta a_{m,0} + (1 - \theta)a_{m+1,0}$$
$$\leq \theta b_{m,0} + (1 - \theta)b_{m+1,0} = b_{m+1,1} \ ,$$

shows that the first row satisfies the result. The first column is shown to satisfy the inequality by observing that $a_{1,1} \leq b_{1,1}$, since it is also in the first row, and assuming that $a_{1,n} \leq b_{1,n}$ the relation

$$a_{1,n+1} \leq \theta a_{1,n} + (1 - \theta)a_{1,n}$$
$$\leq \theta b_{1,n} + (1 - \theta)b_{1,n} = b_{1,n+1} \ ,$$

shows the next element in the first row satisfies the same inequality. $\square$

The preceding induction argument can also be used to show that given boundary values on $\partial\mathcal{R}$ extend to a function on $\mathcal{R}$ satisfying scheme (8.11.b). Then Lemma 8.6 shows this extension to be unique.

COROLLARY 8.7. *If the functions $a, b : \mathcal{R} \to \mathbb{R}^+$ satisfy (8.11) and $a^p \leq b$ on $\partial\mathcal{R}$ for some $p \geq 1$, then $a^p \leq b$ on $\mathcal{R}$.*

PROOF. By the convexity of x^p,

$$a_{m+1,n+1}^p \leq (\theta a_{m,n} + (1 - \theta)a_{m+1,n})^p$$
$$\leq \theta a_{m,n}^p + (1 - \theta)a_{m+1,n}^p \ ,$$

so a^p satisfies (8.11.a). $\square$

LEMMA 8.7. *Let $c_{m,n} = K^2|ms - n\tau|^2$ on $\partial\mathcal{R}$. Then its extension to all of $\mathcal{R}$ satisfying scheme (8.11.b) satisfies*

$$c_{m,n} \leq K^2 \left[(n\tau)(s - \tau) + |ms - n\tau|^2 \right] \ .$$

PROOF. Define $d_{m,n} = c_{m,n} - K^2|ms - n\tau|^2$ so that

$$
\begin{aligned}
d_{m+1,n+1} &= a_{m+1,n+1} - K^2|ms - n\tau|^2 \\
&= \theta a_{m,n} + (1-\theta)a_{m+1,n} - K^2|(m+1)s - (n+1)\tau|^2 \\
&= \theta d_{m,n} + (1-\theta)d_{m+1,n} - K^2|(m+1)s - (n+1)\tau|^2 \\
&\quad + \theta K^2|ms - n\tau|^2 + (1-\theta)K^2|(m+1)s - n\tau|^2 \ .
\end{aligned}
$$

Putting $\beta = (ms - n\tau)$ enables the last three terms in the above to be written

$$
\theta|\beta|^2 + (1-\theta)|\beta + s|^2 - |\beta + (s - \tau)|^2 = \tau(s - \tau) \ ,
$$

where the definition $\theta = \tau/s$ was used to accomplish this simplification. Using this in the first estimate yields

$$
d_{m+1,n+1} = \theta d_{m,n} + (1-\theta)d_{m+1,n} + K^2\tau(s - \tau)
$$

implying

$$
\max_{0 \le m \le M} d_{m,n+1} \le \max_{0 \le m \le M} d_{m,n} + K^2\tau(s - \tau) \ .
$$

Summing on n from zero to $n - 1$ gives

$$
\begin{aligned}
d_{m,n} &\le \max_{0 \le m \le M} d_{m,n} \\
&\le \max_{0 \le m \le M} d_{m,0} + K^2(n\tau)(s - \tau) \\
&\le K^2(n\tau)(s - \tau)
\end{aligned}
$$

where the identity $d_{m,0} = 0$ was used in the last step. Inserting the definition of $d_{m,n}$ into the last estimate finishes the proof. $\qquad\square$

PROOF OF PROPOSITION 8.6. First check to see that $a_{m,n} \le K|ms - n\tau|$ for $(m,n) \in \partial\mathcal{R}$. Define $b : \mathcal{R} \to \mathbb{R}_0^+$ to be the extension satisfying scheme (8.11.b) of the boundary data $b_{m,n} = K|ms - n\tau|$, $(m,n) \in \partial\mathcal{R}$. Then $b^2 = c$ on $\partial\mathcal{R}$ where $c : \mathcal{R} \to \mathbb{R}_0^+$ is the function defined in Lemma 8.7. Lemmas 8.6 and 8.7 then guarantee

$$
\begin{aligned}
a_{m,n} &\le b_{m,n} \\
&\le (c_{m,n})^{1/2} \\
&\le K\left[n\tau(s - \tau) + (ms - n\tau)^2\right]^{1/2} \ . \qquad\square
\end{aligned}
$$

IV.9. Evolution Equations in L^1

We present here a collection of examples of the many applications of Theorem 8.2. These include a scalar conservation law, the porous medium equation, and some systems which contain either of them. Each of these equations or systems will be regarded as an abstract Cauchy problem in $L^1(G)$ of the form

(9.1)
$$
\begin{aligned}
u'(t) + \mathbb{A}(u(t)) &\ni f(t) \text{ in } L^1(G), \quad t \in [0, T], \\
u(0) &= u_0,
\end{aligned}
$$

where $\mathbb{A}$ is the realization of an appropriate m-accretive operator in $L^1(G)$.

The following result is very useful in the construction of these examples, and we repeat it here with a change of notation for reference.

THEOREM II.9.2 (BREZIS-STRAUSS). *Let α be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ and $0 \in \alpha(0)$. Let $A : D(A) \to L^1(G)$ be linear and satisfy the following:*
  (i) *$D(A)$ is dense and $(I + \lambda A)^{-1}$ is a contraction in L^1 for each $\lambda > 0$;*
  (ii) *$\sup_G(I + \lambda A)^{-1}f \leq (\sup_G f)^+ = \|f^+\|_{L^\infty}$ for $f \in L^1$ and $\lambda > 0$;*
  (iii) *there is a $c > 0$ such that*

$$c\|u\|_{L^1} \leq \|Au\|_{L^1} \quad for \quad u \in D(A).$$

Then for each $f \in L^1$ there is a unique pair $u \in L^1$, $v \in D(A)$ such that

$$u + Av = f \quad and \quad v(x) \in \alpha\big(u(x)\big), \quad a.e \quad x \in G.$$

If u_1, v_1 and u_2, v_2 are solutions corresponding to f_1, f_2 as above, then

$$\|(u_1 - u_2)^+\|_{L^1} \leq \|(f_1 - f_2)^+\|_{L^1}, \quad \|(u_1 - u_2)^-\|_{L^1} \leq \|(f_1 - f_2)^-\|_{L^1},$$

and, hence,

$$\|u_1 - u_2\|_{L^1} \leq \|f_1 - f_2\|_{L^1}.$$

If $f_1 \geq f_2$ a.e. then $u_1 \geq u_2$ a.e. on G.

EXAMPLE 9.A SCALAR CONSERVATION LAW. The first problem to be discussed here is an initial-boundary-value problem for a *scalar conservation law*: find a pair of functions, $u(\cdot, \cdot)$, $v(\cdot, \cdot)$, on $(a, b) \times (0, T)$ which satisfy

(9.2)
$$\begin{aligned}
\frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} &= f, \quad v \in \alpha(u) \quad in (a, b) \times (0, T), \\
v(a, t) &= cv(b, t) \quad for\ t \in (0, T), \\
u &= u_0 \quad on\ [a, b] \times \{0\},
\end{aligned}$$

where $a < b$ and $0 \leq c < 1$ are given.

The first order spatial operator is the L^1 realization of Example I.4.B. Set $D(A) = \{v \in W^{1,1}(a, b) : v(a) = cv(b)\}$. Define $A = \partial$ on $D(A)$. We show that it satisfies the hypotheses in Theorem II.9.2. It is easy to check that there is a solution $u \in D(A)$ of $(I + \lambda A)(u) = f$ for each $f \in L^1(a, b)$ and $\lambda > 0$. This resolvent equation is a simple problem for an ordinary differential equation, and we can solve it directly. For any such solution, we multiply the equation by $\mathrm{sgn}(u)$ and integrate to get $\|u\|_{L^1(a,b)} \leq \|f\|_{L^1(a,b)}$, so (i) holds. Multiply the identity

$$u(x) - k + \lambda Au(x) = f(x) - k$$

by $\mathrm{sgn}^+(u(x) - k)$, integrate, and set $k = \|f^+\|_{L^\infty(a,b)}$ to obtain

$$\|(u - k)^+\|_{L^1(a,b)} + (u(b) - k)^+ - (cu(b) - k)^+ \leq 0.$$

Since $k \geq 0$, $c \geq 0$ and the *positive part* function $(\cdot)^+$ is monotone, we have

$$\begin{aligned}
(u(b) - k)^+ - (cu(b) - k)^+ &\geq (u(b) - k)^+ - (cu(b) - ck)^+ \\
&\geq (1 - c)(u(b) - k)^+ \geq 0,
\end{aligned}$$

so $(u - k)^+ = 0$ and we have (ii). To check (iii), we let $Au = f$. Multiply by $\mathrm{sgn}(u)$ and integrate to get

$$|u(x)| \leq |u(a)| + \|f\|_{L^1(a,b)} , \qquad a \leq x \leq b .$$

Integrating $Au = f$ and using the boundary condition give $u(a) = \frac{c}{1-c}\|f\|_{L^1(a,b)}$, so with the above we obtain (iii) from

$$|u(x)| \leq \frac{1}{1-c}\|f\|_{L^1(a,b)} , \qquad a \leq x \leq b .$$

Theorem II.9.2 asserts that for any maximal monotone α in $\mathbb{R} \times \mathbb{R}$ with $0 \in \alpha(0)$, the operator $\mathbb{A} = A \circ \alpha$ is m-accretive in $L^1(a,b)$. Specifically, if $\lambda > 0$ and $f \in L^1(a,b)$ there is a unique pair

$$u \in L^1(a,b) \quad , \quad v \in W^{1,1}(a,b) ,$$

for which $v(a) = cv(b)$ and

$$u(x) + \lambda \partial v(x) = f(x) \quad , \quad v(x) \in \alpha(u(x)) , \quad \text{a.e.} \quad x \in (a,b) ,$$

and the mapping $f \mapsto u$ is an $L^1(a,b)$-contraction.

From Theorem 8.2 it follows that there is a unique C^0 *solution* of the initial-boundary-value problem (9.2), and that it is obtained as the limit of solutions of the corresponding problem with the backward difference approximations of the time derivative. It is easy to show by examples that such partial differential equations do not admit smooth solutions, not even *continuous* solutions, even if $\alpha(\cdot)$ is a smooth invertible function and the initial data $u_0(\cdot)$ is smooth. For the case of a smooth function $\alpha(\cdot)$ the equation has the form

$$\frac{\partial u}{\partial t} + \alpha'(u)\frac{\partial u}{\partial x} = f,$$

where $\alpha'(u)$ is the *signal speed*. Thus, any solution u is constant along a trajectory $x(t)$ with the signal speed $x'(t) = \alpha'(u(x(t),t))$, and this speed depends on the value of the solution, u. In particular, if $\alpha'(u)$ is increasing and positive, then waves move rightward and get steeper until they cease to be functions. However, it is known that the solution obtained here is the *entropy* solution of the scalar conservation law.

EXAMPLE 9.B POROUS MEDIUM EQUATION. We consider the Dirichlet initial-boundary-value problem for the *porous medium equation*: find a pair of functions, $u(\cdot,\cdot)$, $v(\cdot,\cdot)$, on $G \times (0,T)$ for which

(9.3.a) $\qquad\qquad u_t - \Delta v = f, \quad v \in \alpha(u) \quad$ in $G \times (0,T)$,

(9.3.b) $\qquad\qquad\qquad\qquad\qquad v = 0 \qquad$ on $\partial G \times (0,T)$,

(9.3.c) $\qquad\qquad\qquad\qquad\qquad u = u_0 \qquad$ on $G \times \{0\}$,

where $\alpha(\cdot)$ is a maximal monotone graph as above, G is a bounded domain in $\mathbb{R}^m$, and $T > 0$ denotes the length of the time interval.

We record here a special case of Proposition II.9.1. Define $D(A) = \{v \in W_0^{1,1}(G) : Av \in L^1(G)\}$, where $Av = f \in L^1(G)$ means

$$v \in W_0^{1,1}(G) : \int_G \nabla v \cdot \nabla w = \int_G fw , \qquad w \in W_0^{1,\infty}(G) .$$

Thus, $Av = -\Delta v \in L^1(G)$ with Dirichlet boundary conditions. Let α be given as above. Then for each $\lambda > 0$ and $f \in L^1(G)$ there is a unique pair

$$u \in L^1(G) \quad , \quad v \in W_0^{1,1}(G)$$

for which the Laplacian $\Delta v \in L^1(G)$, and

$$u(x) - \lambda\Delta v(x) = f(x) \quad , \quad v(x) \in \alpha(u(x)) , \quad \text{a.e.} \quad x \in G .$$

The mapping $f \mapsto u$ is a contraction in $L^1(G)$. The operator $\mathbb{A} = A \circ \alpha$ corresponds to the stationary problem for the *porous medium equation*, and the above shows that it is m-accretive in $L^1(G)$. Thus, Theorem 8.2 gives the existence and uniqueness of the C^0 solution of (9.3) in $L^1(G)$ for each appropriate choice of initial data, u_o, and source term, f. One can construct similar operators by varying either the linear elliptic part, $-\Delta$, as in Proposition II.9.1, or the boundary conditions. It is not difficult to add perturbations, such as $\beta(u)$, especially if β is continuous and monotone, but more generality leads to some difficulties, especially if both α and β are multi-valued.

There is a wealth of information on the regularity of solutions of the porous medium equation, and it is intimately dependent on the form of the function (or graph), α. One can show by examples that the derivative $\frac{\partial u}{\partial t}$ need not even exist without further assumptions on α. In particular, it is known that if α is Lipschitz continuous, then the component v of the solution of (9.3) is continuous; if α^{-1} is Lipschitz continuous, then u is continuous and $\frac{\partial u}{\partial t} \in L^2(G)$. The first case occurs in the example of the *Stefan Problem* in which the function has the *degenerate* form $\alpha(s) = s^- + (s-1)^+$, and the *singular* case arises in problems of *partial saturation* in which $\alpha^{-1}(s) = 1 + (s-1)^-$. Even the sense in which the partial differential equation is satisfied can be an issue. If α is continuous and u is bounded, then it follows that the equation holds in the sense of generalized functions, i.e., in $\mathcal{D}(G)^*$. If α is onto, then it can be shown that the L^1 solution is also the (strong) $H^{-1}-$solution and thereby satisfies the partial differential equation in $\mathcal{D}(G)^*$.

We can give an explicit example for the case in which the graph α satisfies $\alpha(0) \supset [0,1]$ and $\alpha(x) = \{1\}$ for $x \in [0,1]$. Thus, neither α nor α^{-1} is continuous. It is easy to check that the pair of functions given on $t < \frac{1}{8}$ by

$$u(x,t) = H(x - \sqrt{2t}) - H(x - 1 + \sqrt{2t})$$

$$v(x,t) = \min \frac{x}{\sqrt{2t,}} 1, \frac{1-x}{\sqrt{2t}}$$

for $x \in G \equiv (0,1)$, and $u = v = 0$ on $t > \frac{1}{8}$, is the unique solution of (9.3) with $u_0 = 1$ and $f = 0$. Note that this solution u is continuous into $L^1(0,1)$ and differentiable into $H^{-1}(0,1)$.

EXAMPLE 9.C EVOLUTION EQUATIONS WITH HYSTERESIS. We consider a system consisting of a parabolic partial differential equation and an ordinary differential equation which are nonlinearly coupled by the difference of the unknowns. This system has the form

(9.4.a) $$\frac{\partial}{\partial t}\alpha(u(x,t)) - \Delta u(x,t) - \gamma(v(x,t) - u(x,t)) \ni f(x,t)\ ,$$

(9.4.b) $$\frac{\partial}{\partial t}\beta(v(x,t)) + \gamma(v(x,t) - u(x,t)) \ni g(x,t)\ ,\quad x \in G\ ,\ t \in (0,T]\ ,$$

(9.4.c) $$u(s,t) = 0\ ,\quad s \in \partial G\ ,$$

in which $u = u(x,t)$ and $v = v(x,t)$ are functions defined on a bounded domain G in Euclidean space $\mathbb{R}^m$, and $T > 0$. The component (9.4.a) contains a generalized *porous medium equation*, and we make no assumptions of strict monotonicity of $\alpha(\cdot)$. In particular, we allow the degenerate case $\alpha(\cdot) \equiv 0$, and this reduces (9.4) to a *pseudoparabolic* equation.

When each of $\alpha(\cdot)$, $\beta(\cdot)$ and $\gamma(\cdot)$ is a monotone (non-decreasing) function, the inclusion symbols, $\ni$, are replaced by the corresponding equality symbol. Such systems arise in many contexts, for example, in the diffusion of chemicals through a saturated porous medium in which (9.4.b) models adsorbtion onto immobile non-diffusive sites. In that case, u is the concentration of a chemical species in the fluid which occupies the pores and v is the concentration on the surface of the medium. These are called *first order kinetic models*, and they can be regarded as a degenerate case of corresponding *parallel flow models* which contain an additional term $-\Delta v(x,t)$ in (9.4.b). This diffusion term has been deleted because of the immobility of the concentration in the adsorbtion sites.

We set $\mathbb{B} = L^1(G) \times L^1(G)$ and define an operator A on $\mathbb{B}$ by $A([u,v]) \ni [f,g]$ if $u \in W_0^{1,1}(G)$, $v \in L^1(G)$ satisfy

$$-\Delta u = g + f,\ g \in \gamma(v - u).$$

The Dirichlet Laplace, $-\Delta$, was defined in the previous Example 9.B. The operator equation $[u,v] + A([u,v]) \ni [f,g]$ in $\mathbb{B}$ is equivalent to the system

$$u \in W_0^{1,1}(G),\quad u - \Delta u - \xi = f,$$
$$v \in L^1(G),\quad v + \xi = g,\quad \xi \in \gamma(v - u),$$

for some $\xi \in L^1(G)$. This system is equivalent to

$$u - \Delta u + (v - u) - (g - u) = f,$$
$$v - u + \xi = g - u,\quad \xi \in \gamma(v - u),$$

and likewise to

$$u - \Delta u + [(I + \gamma)^{-1} - I](g - u) = f,$$
$$v - u = (I + \gamma)^{-1}(g - u).$$

The second line defines v after u is obtained as the solution of the first line, a monotone Lipschitz perturbation of an m-accretive operator, so this system always

has a solution. It is easy to check directly that A is accretive (see below), so it follows that A is m-accretive on $\mathbb{B}$.

Assume for simplicity that the monotone functions $\alpha(\cdot)$ and $\beta(\cdot)$ are Lipschitz continuous. Then the corresponding operators on $L^1(G)$ are continuous and accretive, so it follows that the sum is m-accretive. Thus, for each $\varepsilon > 0$ and each $[f, g] \in \mathbb{B}$ there is a unique solution of the system

(9.5.a) $$\varepsilon u_\varepsilon + \alpha(u_\varepsilon) - \Delta u_\varepsilon - \xi_\varepsilon = f,$$

(9.5.b) $$\varepsilon v_\varepsilon + \beta(v_\varepsilon) + \xi_\varepsilon = g, \quad \xi_\varepsilon \in \gamma(v_\varepsilon - u_\varepsilon).$$

Now define the composite operator $\mathbb{A}$ in $\mathbb{B}$ by $\mathbb{A}([U, V]) \ni [f, g]$ if there exist

$$u \in W_0^{1,1}(G), \ v \in L^1(G): \quad U = \alpha(u), \ V = \beta(v)$$
$$-\Delta u = g + f, \ g \in \gamma(v - u),$$

that is, $A([u, v]) \ni [f, g]$ in $\mathbb{B}$ and $U = \alpha(u), \ V = \beta(v)$ in $L^1(G)$. The equation

$$[U, V] + \mathbb{A}([U, V]) \ni [f, g]$$

is just the system

$$U = \alpha(u), \quad \alpha(u) - \Delta u - \xi = f,$$
$$V = \beta(v), \quad \beta(v) + \xi = g, \quad \xi \in \gamma(v - u).$$

In order to show that $\mathbb{A}$ is accretive on $\mathbb{B}$, let $[f_1, g_1], \ [f_2, g_2] \in \mathbb{B}$ and let $[U_1, V_1], [U_2, V_2]$ be corresponding solutions. Subtract the respective equations to get

$$(U_1 - U_2) - \Delta(u_1 - u_2) - (\xi_1 - \xi_2) = f_1 - f_2,$$
$$(V_1 - V_2) + (\xi_1 - \xi_2) = g_1 - g_2,$$
$$\xi_1 \in \gamma(v_1 - u_1), \quad \xi_2 \in \gamma(v_2 - u_2).$$

Multiply the first equation by $\mathrm{sgn}_0(U_1 - U_2 + u_1 - u_2)$, the second by $\mathrm{sgn}_0(V_1 - V_2 + v_1 - v_2)$, integrate over G and add. Note that $\mathrm{sgn}_0(U_1 - U_2 + u_1 - u_2) \in \mathrm{sgn}(U_1 - U_2)$ and $\mathrm{sgn}_0(U_1 - U_2 + u_1 - u_2) \in \mathrm{sgn}(u_1 - u_2)$, and similar inclusions hold for the second component. Using Theorem 8.2 and the monotonicity of $\alpha(\cdot)$, $\beta(\cdot)$ and $\gamma(\cdot)$, we obtain

$$\|U_1 - U_2\|_{L^1} + \|V_1 - V_2\|_{L^1} \leq \|f_1 - f_2\|_{L^1} + \|g_1 - g_2\|_{L^1}.$$

The same procedure holds with $\mathbb{A}$ replaced by $\varepsilon\mathbb{A}$ for any $\varepsilon > 0$, so $\mathbb{A}$ is accretive in $\mathbb{B}$.

The sum $I + \mathbb{A}$ will be *onto* $\mathbb{B}$ if we can show that for each pair $[f, g] \in \mathbb{B}$ there is a unique solution of the system

(9.6)
$$\alpha(u) - \Delta u - \xi = f,$$
$$\beta(v) + \xi = g, \quad \xi \in \gamma(v - u).$$

To this end, for each $\varepsilon > 0$ let $u_\varepsilon, \ v_\varepsilon$ be the solution of (9.5). As above, we obtain the estimates

$$\varepsilon\|u_\varepsilon\|_{L^1} + \varepsilon\|v_\varepsilon\|_{L^1} + \|U_\varepsilon\|_{L^1} + \|V_\varepsilon\|_{L^1} \leq \|f\|_{L^1} + \|g\|_{L^1}.$$

From (9.5.b) we get a bound on $\|\xi_\varepsilon\|_{L^1}$ and then from (9.5.a) a bound on $\|\Delta u_\varepsilon\|_{L^1}$ and on $\|u_\varepsilon\|_{L^1}$. We shall assume additionally that β^{-1} is a continuous function and that

$$|s| \le C(|\beta(s)| + 1), \quad s \in \mathbb{R},$$

for some constant $0 < C$. Then Nemytskii's Theorem II.3.2 shows that β^{-1} gives a continuous operator on L^1, and this gives a bound on $\|v_\varepsilon\|_{L^1}$. Thus, u_ε, v_ε is a solution of (9.6) with right side given by

$$f_\varepsilon \equiv f - \varepsilon u_\varepsilon \to f, \quad g_\varepsilon \equiv g - \varepsilon v_\varepsilon \to g.$$

Since $\mathbb{A}$ is accretive, we obtain, as before, Cauchy-type estimates which imply that

$$U_\varepsilon \to U, \quad V_\varepsilon \to V, \quad u_\varepsilon \to u, \quad \Delta u_\varepsilon \to \Delta u, \quad \xi_\varepsilon \to \xi$$

in L^1, by the continuity of β^{-1} that $v_\varepsilon \to v$, and then that $U = \alpha(u)$, $V = \beta(v)$, $\xi \in \gamma(v - u)$. Thus, we obtain a solution of (9.6), and it follows that $\mathbb{A}$ is m-accretive. From Theorem 8.2 it follows that there is a unique C^0 *solution* of the initial-boundary-value problem (9.4) for each appropriate choice of initial data, $\alpha(u_0)$, $\beta(v_0)$, and source terms, f, g.

Here we have permited $\gamma(\cdot)$ to be multi-valued. This generalization includes a very elegant treatment of parabolic problems with variational inequalities. For example, if we define

$$\gamma(s) = \begin{cases} 0, & s < 0, \\ [0, \infty), & s = 0, \end{cases}$$

then the solution of (9.4.b) with $g = 0$ satisfies the *Signorini conditions*

$$u(t) \ge v(t), \quad \frac{\partial}{\partial t} \beta(v(t)) \le 0, \quad (u(t) - v(t)) \frac{\partial}{\partial t} \beta(v(t)) = 0,$$

in which the exchange term is a *unilateral constraint*. In particular we obtain such equations with *hysteresis* nonlinearities. These appear the form

$$\frac{\partial}{\partial t} (\alpha(u) + \mathcal{H}(u)) - \Delta u = f$$

in which $\mathcal{H}$ denotes a rate-independent *hysteresis* functional, that is, its value depends not only on the current value of the input, u, but also on the *history* of the input, and it does not depend on the rate of the input. For a more interesting example, a *simple play* hysteresis functional, we choose $\gamma \equiv \operatorname{sgn}^{-1}$. Then for a given input, $u(t)$, the output is the solution $w(t) \equiv \mathcal{H}(u)(t)$ of the equation

$$\frac{\partial}{\partial t} \beta(v(t)) + \gamma(v(t) - u(t)) \ni 0,$$

and it is given by $w(t) = \beta(v(t))$, where $|v(t) - u(t)| \le 1$ and

$$\begin{cases} w'(t) \ge 0 & \text{if } v(t) = u(t) - 1, \\ w'(t) = 0 & \text{if } |v(t) - u(t)| < 1, \\ w'(t) \le 0 & \text{if } v(t) = u(t) + 1. \end{cases}$$

By replacing the Dirichlet operator $-\Delta$ in (9.4) by an appropriate first-order operator such as in Example 9.A, we can as well include scalar conservation laws with unilateral constraints or hysteresis functionals.

EXAMPLE 9.D DYNAMIC BOUNDARY CONDITIONS. A problem with the same formal structure is the initial-boundary-value problem

(9.7.a)
$$\frac{\partial}{\partial t}\alpha(u) - \Delta u \ni f , \qquad x \in G ,$$

(9.7.b)
$$\frac{\partial}{\partial t}\beta(v) + \frac{\partial u}{\partial \nu} \ni g \text{ and}$$

(9.7.c)
$$\frac{\partial u}{\partial \nu} \in \gamma(v - u) , \qquad s \in \Gamma ,$$

for $t > 0$ with initial values specified at $t = 0$ for $\alpha(u)$ and $\beta(v)$. At each $t > 0$, $u(t)$ is a function on the bounded domain G in $\mathbb{R}^m$ with smooth boundary $\Gamma = \partial G$, and $v(t)$ is a function on Γ. Each of $\alpha(\cdot), \beta(\cdot), \gamma(\cdot)$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$. Thus, the system (9.7) consists of a generalized porous medium equation in the interior of G subject to a nonlinear dynamic constraint on the boundary Γ.

A remarkable variety of boundary conditions are included in (9.7). For example, if $\beta \equiv 0$ we have an explicit *Neumann boundary condition*, and if $\gamma \equiv 0$ it is homogeneous. If $\beta^{-1} = 0$, then $v \equiv 0$ and we have a nonlinear Robin constraint, and if $\gamma^{-1} = 0$ we get $v = u$ on Γ, and this satisfies the nonlinear *dynamic boundary condition*

$$\frac{\partial}{\partial t}\beta(u) + \frac{\partial u}{\partial \nu} \ni g \quad \text{in } L^1(\Gamma).$$

If $\beta^{-1} = 0$ and $\gamma^{-1} = 0$ we have the homogeneous *Dirichlet boundary condition* $u = 0$. Additionally we have seen in the previous Example 9.C that (9.7.b) and (9.7.c) can represent *boundary hysteresis*. That is, the map $u \mapsto \beta(v) \mapsto \frac{\partial u}{\partial \nu}$ is a hysteresis functional. Adsorption in porous media may be governed by conditions on the surfaces of the solid material that are of hysteresis type. If one assumes that the process is governed by certain thresholds, the adsorption rate shows a hysteresis phenomenon of the kind discussed above.

We can show that the problem (9.7) can be realized in the form (9.1) on the Banach space $L^1(G) \times L^1(\Gamma)$. In order to illustrate the types of estimates that are involved for the problem (9.7), we shall do this for simplicity in the special case of single valued functions, $\alpha(\cdot)$, $\beta(\cdot)$, $\gamma(\cdot)$. The operator $\mathbb{A}$ is constructed so that the *resolvent equation*, $(I + \varepsilon\mathbb{A})([U, V]) \ni [f, g]$ with $\varepsilon > 0$, takes the form

$$U = \alpha(u), \quad U - \varepsilon\Delta u \ni f \quad \text{in } L^1(G) ,$$

$$V = \beta(v), \quad V + \varepsilon\frac{\partial u}{\partial \nu} \ni g, \quad \frac{\partial u}{\partial \nu} \in \gamma(v - u), \quad \text{in } L^1(\Gamma) .$$

In order to show how one obtains the essential estimates that are needed, multiply the respective equations by appropriate functions φ on G and ψ on Γ and integrate to obtain

$$\int_G \left(\alpha(u)\varphi + \varepsilon\vec{\nabla}u \cdot \vec{\nabla}\varphi\right) dx + \int_\Gamma \left(\beta(v)\psi + \varepsilon\gamma(v - u)(\psi - \varphi)\right) ds = \int_G f\varphi \, dx + \int_\Gamma g\psi \, ds .$$

This suggests the appropriate variational formulation and leads to the essential a-priori estimates. For example, if we choose $\varphi = \text{sgn}(u)$, $\psi = \text{sgn}(v)$ and can

obtain simultaneously $\varphi = \mathrm{sgn}(\alpha(u))$, $\psi = \mathrm{sgn}(\beta(v))$ as before, then we obtain the stability estimate

$$\|\alpha(u)\|_{L^1(G)} + \|\beta(v)\|_{L^1(\Gamma)} \leq \|f\|_{L^1(G)} + \|g\|_{L^1(\Gamma)} .$$

By estimating similarly the *differences* of solutions, we establish that the *resolvent* map $[f, g] \mapsto [u, v] \mapsto [\alpha(u), \beta(v)]$ is a contraction, and this is the *accretiveness* of the operator $\mathbb{A}$. Under quite general conditions on the monotone graphs $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$, we find that $\mathbb{A}$ is m-accretive as desired. Our examples above show that it is worthwhile to retain as much generality as possible. As before, the Dirichlet operator $-\Delta$ in (9.4) can be replaced by a first-order operator as in Example 9.A in order to include scalar conservation laws with unilateral constraints or hysteresis functionals.