Flow and Transport

1. The Transport Equation

We shall describe the transport of a dissolved chemical by water that is traveling with uniform velocity ν through a long thin tube G with uniform cross section S. (The very same discussion applies to the description of the transport of gas by air moving through a pipe.) We identify G with the open interval (a, b), and the velocity $\nu > 0$ is in the (rightward) positive direction of the x-axis. Since the tube is very thin, we can assume that the concentration of the chemical is constant across the cross section S at each point $x \in G$. Let $c(x, t)$ denote this concentration within the tube at a point $x \in G$ and at time $t > 0$.

The Conservation Law. The amount of chemical stored in the tube within a section $[x, x + h]$ of length $h > 0$ is given by

$$\int_x^{x+h} c(s,t) S \, ds,$$

The *flux* $q(x, t)$ at the point x is the mass flow rate of the chemical to the right per unit area, Equating the rate at which the chemical is stored within the section $[x, x + h]$ to the rate at which it flows into the section plus the rate at which the chemical is generated within this section, we arrive at

$$\frac{d}{dt} \int_x^{x+h} c(s,t) S \, ds = S\big(q(x,t) - q(x+h,t)\big) + \int_x^{x+h} F(s,t) S \, ds$$

where $F(x, t)$ represents the rate at which chemical is generated per unit volume. This *source* term is assumed to be a known function of space and time. We differentiate the integral on the left side and write the difference on the right side as an integral of a derivative to obtain

$$\int_x^{x+h} \frac{\partial c(s,t)}{\partial t} S \, ds = S \int_x^{x+h} \left(-\frac{\partial q(s,t)}{\partial x} + F(s,t) \right) ds.$$

Dividing this by Sh and letting $h \to 0$ yield the *conservation law*

$$\frac{\partial c}{\partial t} + \frac{\partial q}{\partial x} = F(x,t). \tag{1}$$

The transport equation. The *flux* $q(x,t)$ at the point x is given by

$$q(x,t) = \nu c(x,t). \tag{2}$$

This is the mass flow rate of the chemical due to *advection*, the direct transport by the moving water. Substituting this into the conservation law (1) yields the one–dimensional *transport equation*

$$\frac{\partial c}{\partial t} + \nu \frac{\partial c}{\partial x} = F(x,t). \tag{3}$$

This is also known as the *first-order wave equation*. It is just the differential form of the conservation equation (1) combined with the constitutive equation (2) of convection. It is assumed here that the velocity ν is sufficiently large that we can ignore the comparatively smaller effects of diffusion, *i.e.*, the natural motion of the chemical from areas of high concentration to those of lower concentration. This will be a major topic later.

Initial and Boundary conditions. Since the transport equation is first–order in space and time, one may expect that in order to have a well–posed problem, one boundary condition and one initial condition should be specified. We shall see that this is true here. We want to find a solution of (3) which satisfies an *initial condition* of the form

$$c(x,0) = c_0(x), \quad a < x < b,$$

where $c_0(\cdot)$ is given. That is, we need to specify the *initial state* of the model. Furthermore we shall specify the value of the concentration at the left end point where the substance is entering the tube:

$$c(a,t) = c_a(t), \quad t > 0\,,$$

where $c_a(\cdot)$ is given. This type of *boundary condition* arises when the value of the concentration at the end point is known, usually from a direct measurement. Such a condition arises when one sets the boundary concentration to a prescribed value, for example, $c_a(t) = 0$ when only pure water with no chemical is entering the left end of the tube.

The Solution: introduction of characteristic curve. Consider the homogeneous transport equation

$$c = c(x,t): \quad c_t + \nu\, c_x = 0\,.$$

If we make a change of variable to the new coordinates ξ, τ which are defined implicitly by

$$x = \xi + \tau, \quad \nu\, t = \xi - \tau,$$

and use the chain rule,

$$\frac{\partial c}{\partial \xi} = \frac{\partial c}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial c}{\partial t}\frac{\partial t}{\partial \xi}\,,$$

we find that the transport equation is equivalent to

$$c = c(\xi,\tau): \quad \frac{\partial c}{\partial \xi} = 0\,.$$

This shows that the solutions are given by

$$c(\xi,\tau) = f(\tau)\,,$$

so the general solution of the transport equation is of the form

$$c(x,t) = f(x - \nu\, t)$$

for some function $f(\cdot)$. Now let's choose this function so that $c(\cdot,\cdot)$ satisfies the initial and boundary conditions. The initial condition requires

$$c(x,0) = f(x) = c_0(x), \quad x \geq a\,,$$

and the boundary condition likewise requires

$$c(a,t) = f(a - \nu\, t) = c_a(t)\,, \quad t \geq 0\,,$$

so we must have $f(s) = c_a(\frac{a-s}{\nu})$, $s \leq a$. Thus, the solution of the initial-boundary-value problem is given by

$$c(x,t) = \begin{cases} c_0(x - \nu\, t), & x \geq \nu\, t \geq 0\,, \\ c_a(\frac{a-x+\nu\, t}{\nu}), & a \leq x \leq \nu\, t + a\,, \end{cases}$$

Note that the important fact behind these calculations is that the solution was constant along the curves where τ is constant, *i.e.*, along the curves where $x - \nu\, t$ is constant. It followed from this that the solution is a pure translation to the right with velocity ν, certainly no surprise in view of the origin of the transport equation. These special curves are the *characteristic curves* for the transport equation, and they will arise in our discussions of first–order equations.

EXERCISE 1. *Suppose that* $c_a(\cdot)$ *and* $c_0(\cdot)$ *are continuous. Show that the solution of the initial-boundary-value problem is continuous if and only if* $c_a(0^+) = c_0(a^+)$.

EXERCISE 2. *Find the solution of the initial-value problem for the* non-homogeneous *transport equation* (3) *on the region* $\{(x,t) : \ t \geq 0\}$ *with the initial condition* $c(x,0) = c_0(x)$ *on* $-\infty < x < +\infty$.