Discrete Models of Diffusion and Vibration

1. Heat Conduction in an Interval

Consider the diffusion of heat energy through a long thin rod G, which we shall identify with the interval (a, b) in the *real number* line $\mathbb{R}$. Assume this rod has uniform cross section S and it is insulated along its length.

Experiment 1. A *small* section of the rod experiences an increase in the temperature $u(t)$ during the time interval $[t_1, t_2]$ when a quantity of heat Q is supplied to it. We find that this quantity is proportional to the mass and the temperature increment, that is, for some constant c we have

$$Q = c\rho Sh(u(t_2) - u(t_1)) ,$$

where h is the length, so Sh is the volume of the section, and ρ is the volume-distributed *density*. The constant c is a material property called *specific heat*, and it provides a measure of the amount of heat energy required to raise the temperature of a unit mass of the material by a degree. The temperature is then a measure of heat energy. Taking the limit as $t_2 - t_1 \to 0$, we see the *rate* at which heat is supplied to this segment is $c\rho Sh\dot{u}(t)$. The superscript dot will be used to denote a time derivative.

We observe that heat (thermal energy) flows from the hotter portions to the cooler portions of the rod by a process called *conduction*. The rate of this transfer depends on the material of the rod.

Experiment 2. The ends $x = x_1$ and $x = x_2$ of a section of the homogeneous rod of length $h = x_2 - x_1$ and cross section S are maintained at temperatures u_1 and u_2, respectively. After a period of time (which depends on the material) the temperature distribution is observed to be linear: the temperature at the position x is given by

$$u(x) = (\frac{x_2 - x}{h})u_1 + (\frac{x - x_1}{h})u_2, \quad x_1 \le x \le x_2 .$$

The quantity of heat per unit time and unit area which flows to the right is called the *flux*, and it is observed to be given by

$$q = -k\frac{u_2 - u_1}{h} \tag{1}$$

for some constant k which is called *conductivity*. The conductivity is a property of the material that is a measure of the flow rate per unit area, *i.e.*, the *flux* q induced by a given temperature gradient, $\frac{\partial u}{\partial x}$. The equation (1) defining k is known as *Fourier's law*. The minus sign arises since the heat flow is directed toward the lower temperature.

1.1. The Initial-Value Problem. We shall use the two preceding experimental observations to formulate a discrete model of the flow of heat through the rod. Partition the rod $G = (a, b)$ into N sections of equal length h, so $Nh = b-a$. Here we are assuming that the mesh size h of the partition is so small that we can approximate the temperature distribution in each section by a constant. Thus, let $u_j(t)$ be the temperature in the j-th section $[x_{j-1}, x_j]$ at the time $t \geq 0$, where the endpoints are given by $x_j = a + jh$, $0 \leq j \leq N$. In particular, $x_0 = a$ and $x_N = b$. The heat flux past x_j is given by

$$q_j(t) = -k\frac{u_{j+1}(t) - u_j(t)}{h}, \quad 1 \leq j \leq N-1\,. \tag{2a}$$

If we have a source of volume intensity $f_j(t)$ in the jth section, this contributes heat at the rate $Shf_j(t)$. The total rate of heat supplied to the jth section from left and right adjoining sections and the additional heat supply rate from internal sources is given by $S(q_{j-1}(t) - q_j(t)) + Shf_j(t)$. Setting this equal to the rate at which heat is stored, we arrive at the *conservation of energy* in the jth section in the form

$$c\rho Sh\dot{u}_j(t) = S(q_{j-1}(t) - q_j(t)) + hSf_j(t)\,, \quad 1 \leq j \leq N\,. \tag{2b}$$

In order to complete this system, it remains to determine the flux terms q_0 and q_N at the ends. The resulting ordinary differential equations are to be supplemented with the *initial conditions*

$$c\rho u_j(0) = c\rho u_j, \quad 1 \leq j \leq N, \tag{2c}$$

where the initial temperatures u_j for $1 \leq j \leq N$ are also given. This is an *initial-value problem* for the N unknown functions $u_j(t)$, $1 \leq j \leq N$.

Note that we can eliminate the interior flux terms (2a) to obtain an equivalent form of (2b), the *system of ordinary differential equations*

$$c\rho\dot{u}_j(t) = \frac{k}{h^2}(u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)) + f_j(t), \quad 2 \leq j \leq N-1. \tag{3}$$

These will remain the same for any choice of q_0 and q_N. However the first and last of the energy equations (2b) will depend on these choices.

1.2. The Boundary conditions. We consider a number of typical possibilities for determining the endpoint conditions. The first four of these will be illustrated with a condition at the right end, $x_N = b$. Note that in each of these cases another such condition will also be prescribed at the left end, $x_0 = a$. In the last example, the two conditions each involve data at both endpoints.

1. The value of the temperature could be specified at the end point. To obtain this situation, we append an additional section which is always at a known temperature, $u_b(t)$. Thus, we have

$$u_{N+1}(t) = u_b(t), \quad t > 0\,. \tag{4}$$

This is the *Dirichlet* boundary condition, or boundary condition of *first type* which describes *perfect contact* with the boundary value. It arises when the value of the end point temperature is known, usually from a direct measurement. Special cases can also correspond to fixing the boundary temperature at a prescribed value, for example, $u_b(t) = 0$

if the end is submerged in ice-water. Note that the Dirichlet condition (4) is equivalent to supplementing the flux equations (2a) with

$$q_N(t) = -k\frac{u_b(t) - u_N(t)}{h}$$

2. The rate at which heat is supplied to the rod could be specified at the end point:

$$-q_N(t)S = f_b(t), \quad t > 0\,.$$

This is the *Neumann* boundary condition, or boundary condition of *second type.* It depends on a given *source* of heat, $f_b(t)$ at the end. The homogeneous conditon with $f_b(t) = 0$ corresponds to an *insulated* end point.

3. The end of the rod is exposed to a known outside temperature $u_b(t)$, and the flux is determined by an exchange of heat through the end at a rate proportional to the difference between inside and outside temperatures:

$$-q_N(t)S + k_b(u_N(t) - u_b(t))S = f_b(t), \quad t > 0\,.$$

This is the *Robin* boundary condition, or boundary condition of *third type* which describes *partially insulated* end points. Here both $u_b(\cdot)$ and $f_b(\cdot)$ are prescribed. The first is the outside temperature and the second is a heat source concentrated on the end point. This model for heat loss determined by the difference $u_N(t) - u_b(t)$ is called *Newton's law* for cooling. It is merely a discrete form of the Fourier law and defines the *effective conductivity* k_b. For very large values $k_b \to \infty$, we obtain formally the Dirichlet boundary condition, while for very small values $k_b \to 0$ we get the Neumann condition. Thus the effective conductivity k_b is a new quantity which interpolates between the first two types.

4. Another type of boundary condition arises if there is a *concentrated capacity* at the end point, for instance, if the end of the rod is submerged in an insulated container of well-stirred fluid, or if the end is attached to a region of very highly conductive material. Then we introduce a *new unknown* $u_{N+1}(t)$ for the temperature of the water, or the temperature of the material, respectively, and we have the *dynamic* boundary condition

$$c_0\dot{u}_{N+1}(t)S + \frac{k}{h}(u_{N+1}(t) - u_N(t))S = f_b(t), \quad t > 0\,,$$

together with the flux condition

$$q_N(t) = -\frac{k}{h}(u_{N+1}(t) - u_N(t)) \quad t > 0\,.$$

This is the boundary condition of *fourth type* and defines the *effective specific heat* c_0. This boundary condition must be supplemented with an additional initial condition,

$$c_0u_{N+1}(0) = c_0u_{N+1}\,.$$

Note that each of the first three examples led to a system of ordinary differential equations of size $N \times N$, whereas this fourth example is of size $(N + 1) \times (N + 1)$. That is, an extra differential equation was necessarly introduced to account for the additional energy storeage at the end point, and its form is independent of the mesh size h.

Each of the preceding examples was a *local* boundary condition, *i.e.*, it involves only information at the one end. For each of these, another boundary condition needs to be prescribed for the other end.

5. Here is an example of a *nonlocal boundary condition.* If the rod is bent around and the ends are joined to form a large ring, we identify the end points $x_0 = a$ and $x_N = b$, and the N-th section is now situated to the left of the 1-st section. Thus at the endpoints we match the temperature and the flux:

$$u_0(t) = u_N(t), \quad t > 0\,,$$
$$q_0 = q_N = -k\frac{u_1(t) - u_N(t)}{h}\,.$$

These are *periodic* boundary conditions. Again, since the first is only naming another variable, we have a total of two boundary constraints to supplement the initial value problem.

EXERCISE 1. *The rod above is submerged in a perfectly insulated container of well-stirred water. It is partially insulated along its length, so there is some limited heat exchange with the surrounding water along the length, $a < x < b$. The rod is in perfect contact with the water at the end points. Find an initial value problem for the discrete model of this situation. (Hint: The temperature of the water is unknown.)*

1.3. Examples: Dirichlet Boundary Conditions. We compute solutions of some particular cases in order to see if our models are giving solutions which correspond to our intuition.

Example $N = 1$. Set $\alpha = \frac{k}{h^2 c\rho}$. The system is given by

$$\dot{u}_1(t) + 2\alpha u_1(t) = F_1(t)$$
$$u_1(0) = u_1$$

where $F_1(t) \equiv \alpha(u_a(t) + u_b(t)) + \frac{1}{c\rho} f_1(t)$. The solution is given explicitly by

$$u_1(t) = e^{-2\alpha t} u_1 + \int_0^t e^{-2\alpha(t-s)} F_1(s)\, ds\,.$$

Suppose that $f_1(t) = 0$ and consider the following cases.

$u_a(t) = u_b(t) = 0$, thus, $F_1(t) = 0$. Then we get $u_1(t) = e^{-2\alpha t} u_1$, so the solution dissipates to zero. The heat has been drained out of the end points.

$u_a(t) = u_a,\ u_b(t) = u_b$. Then we get

$$u_1(t) = e^{-2\alpha t} u_1 + \frac{u_a + u_b}{2}(1 - e^{-2\alpha t})\,,$$

so the solution approaches the mean value of the endpoint temperatures. Heat has been lost or gained from the end points.

Example $N = 2$. The system of equations describing the discrete Dirichlet problem is given by

$$\dot{u}_1(t) + 2\alpha u_1(t) - \alpha u_2(t) = \frac{1}{c\rho} f_1(t) + \alpha u_a(t) \equiv F_1(t), \quad u_1(0) = u_1$$
$$\dot{u}_2(t) - \alpha u_1(t) + 2\alpha u_2(t) = \frac{1}{c\rho} f_2(t) + \alpha u_b(t) \equiv F_2(t), \quad u_2(0) = u_2\,.$$

Assume that $F_j(t) = 0$ and look for solutions in the form

$$u_1(t) = a_1 e^{-\lambda t},\ u_2(t) = a_2 e^{-\lambda t}.$$

This leads to the linear algebra problem

$$-\lambda a_1 + \alpha(2a_1 - a_2) = 0,$$
$$-\lambda a_2 + \alpha(-a_1 + 2a_2) = 0.$$

The eigenvalues are given by $\lambda = \alpha,\ 3\alpha$ with corresponding eigenvalues

$$\boldsymbol{\xi_1} \equiv \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{\xi_2} \equiv \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

A family of solutions of the homogeneous system is given by

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = c_1 e^{-\alpha t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3\alpha t} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

that is,

$$\mathbf{u}(t) = c_1 e^{-\alpha t} \boldsymbol{\xi_1} + c_2 e^{-3\alpha t} \boldsymbol{\xi_2}\,.$$

A solution satisfying the given initial conditions $u_1,\ u_2$ can be obtained by choosing constants with

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

This system is solvable because the pair $\boldsymbol{\xi_1}$, $\boldsymbol{\xi_2}$ is a *basis* for the plane, $\mathbb{R}^2$. This can be done to match any initial conditions. Note that the second term in the solution decays at a rate three times as fast as the first. This corresponds to the additional loss of heat between the two components in the second term, whereas the only loss of heat in the first term is to the end points.

Let's review this procedure. Define the matrix and the (column) vector

$$\mathbf{A} = \alpha \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

Then the system can be written in matrix form

(5) $$\dot{\mathbf{u}}(t) + \mathbf{A}\mathbf{u}(t) = \mathbf{F}(t)\,.$$

To look for a solution $\mathbf{u}(t) = \boldsymbol{\xi} e^{-\lambda t}$ of the homogeneous equation with $\mathbf{F}(t) = \mathbf{0}$, substitute this into the differential equation to get the *eigenvalue problem*

(6) $$\mathbf{A}\boldsymbol{\xi} = \lambda \boldsymbol{\xi}\,.$$

If $\boldsymbol{\xi_1},\ \lambda_1$ and $\boldsymbol{\xi_2},\ \lambda_2$ are solutions, we obtain the corresponding solutions of the ordinary differential equation (5) in the form

$$\mathbf{u}(t) = c_1 e^{-\lambda_1 t}\boldsymbol{\xi_1} + c_2 e^{-\lambda_2 t}\boldsymbol{\xi_2}. \tag{7}$$

If the pair $\boldsymbol{\xi_1},\ \boldsymbol{\xi_2}$ is a basis, we can choose the constants to satisfy any initial condition

$$\mathbf{u}(0) = c_1\boldsymbol{\xi_1} + c_2\boldsymbol{\xi_2} + = \mathbf{u}_0\,.$$

It follows that every solution of the system can be obtained this way, so we can say the representation (7) gives the *general solution* of the homogeneous equation.

It is clear how we would like to extend this procedure to larger N's. For the case of Dirichlet boundary conditions, the matrix form of the problem is given by (5) where the unknown, matrix, and data are given by

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ \dots \\ u_{N-1}(t) \\ u_N(t) \end{bmatrix}, \quad \mathbf{A} = \alpha \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{pmatrix},$$

$$\mathbf{F}(t) = \begin{bmatrix} \frac{1}{c\rho} f_1(t) + \alpha u_a(t) \\ \frac{1}{c\rho} f_2(t) \\ \frac{1}{c\rho} f_3(t) \\ \dots \\ \frac{1}{c\rho} f_{N-1}(t) \\ \frac{1}{c\rho} f_N(t) + \alpha u_b(t) \end{bmatrix}.$$

Example $N = 3$. For our final example of the discrete Dirichlet problem, consider the case with $N = 3$ sections. The matrix is given by

$$\mathbf{A} = \alpha \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix},$$

and the matrix of the eigenvalue problem has the form

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2\alpha - \lambda & -\alpha & 0 \\ -\alpha & 2\alpha - \lambda & -\alpha \\ 0 & -\alpha & 2\alpha - \lambda \end{pmatrix},$$

so the characteristic equation is

$$det(\mathbf{A} - \lambda\mathbf{I}) = (2\alpha - \lambda)((2\alpha - \lambda)^2 - 2\alpha^2) = 0\,.$$

The characteristic values are

$$\lambda_1 = (2 - \sqrt{2})\alpha,\ \lambda_2 = 2\alpha,\ \lambda_3 = (2 + \sqrt{2})\alpha\,,$$

and the corresponding vectors are

$$\boldsymbol{\xi_1} = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \ \boldsymbol{\xi_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \ \boldsymbol{\xi_3} = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}.$$

Thus, the solution of the system (5) with $\mathbf{F}(t) = \mathbf{0}$ is given by

$$\mathbf{u}(t) = c_1 e^{-\lambda_1 t}\boldsymbol{\xi_1} + c_2 e^{-\lambda_2 t}\boldsymbol{\xi_2} + c_3 e^{-\lambda_3 t}\boldsymbol{\xi_3},$$

where the constants are chosen to satisfy the initial condition

$$c_1\boldsymbol{\xi_1} + c_2\boldsymbol{\xi_2} + c_3\boldsymbol{\xi_3} = \mathbf{u}_0 .$$

Note that the characteristic values are arranged in increasing magnitude and the corresponding eigenvectors have more and more dissipation of heat between adjacent components. This can be quantified by the number of sign changes between adjacent components of the successive eigenvalues.

1.4. Examples: Other Boundary Conditions. For all the other boundary435 conditons, the matrix is changed, but the change occurs *only* in the first or last row, depending on whether the condition was changed at $x = a$ or at $x = b$. For example, with the Dirichlet condition at the left and the Neumann condition at the right end, this leads to the system of ordinary differential equations (5) as before, but the matrix is given by

$$\mathbf{A} = \alpha \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & \dots & \dots & 0 & -1 & 1 \end{pmatrix}.$$

Example $N = 3$. For the case of Neumann conditions at *both* ends, the matrix is given by

$$\mathbf{A} = \alpha \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

and the matrix of the eigenvalue problem has the form

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} \alpha - \lambda & -\alpha & 0 \\ -\alpha & 2\alpha - \lambda & -\alpha \\ 0 & -\alpha & \alpha - \lambda \end{pmatrix},$$

so the characteristic equation is

$$det(\mathbf{A} - \lambda\mathbf{I}) = (\alpha - \lambda)(3\alpha - \lambda)\lambda = 0 .$$

The characteristic values are

$$\lambda_1 = 0, \ \lambda_2 = \alpha, \ \lambda_3 = 3\alpha ,$$

and the corresponding vectors are

$$\boldsymbol{\xi_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \boldsymbol{\xi_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \ \boldsymbol{\xi_3} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Thus, the solution of the system (5) with $\mathbf{F}(t) = \mathbf{0}$ is given by

$$\mathbf{u}(t) = c_1\boldsymbol{\xi_1} + c_2 e^{-\alpha t}\boldsymbol{\xi_2} + c_3 e^{-3\alpha t}\boldsymbol{\xi_3},$$

where the constants are chosen to satisfy the initial condition

$$c_1\boldsymbol{\xi_1} + c_2\boldsymbol{\xi_2} + c_3\boldsymbol{\xi_3} = \mathbf{u}_0 .$$

The characteristic value $\lambda = 0$ indicates the matrix is *singular*, and the 'constant' vector $\boldsymbol{\xi_1}$ corresponds to the *stationary* solution of the heat conduction problem with insulated ends. Again note that the characteristic values are arranged in increasing magnitude and the corresponding eigenvectors have more and more dissipation of heat between adjacent components. This can be quantified by the number of sign changes between adjacent components of the successive eigenvectors.

For the case of *periodic boundary conditions*, the matrix takes the form

$$\mathbf{A} = \alpha \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & -1 & 2 & -1 \\ -1 & 0 & \dots & 0 & 0 & -1 & 2 \end{pmatrix}.$$

The periodic structure of this matrix is clear.

Example $N = 3$**.** The matrix is given by

$$\mathbf{A} = \alpha \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix},$$

and the matrix of the eigenvalue problem has the form

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2\alpha - \lambda & -\alpha & -\alpha \\ -\alpha & 2\alpha - \lambda & -\alpha \\ -\alpha & -\alpha & 2\alpha - \lambda \end{pmatrix},$$

so the characteristic equation is

$$det(\lambda\mathbf{I} - \mathbf{A}) = \lambda(\lambda - 3\alpha)^2 = 0.$$

The characteristic values are

$$\lambda_1 = 0, \ \lambda_2 = 3\alpha, \ \lambda_3 = 3\alpha,$$

and the corresponding vectors are

$$\boldsymbol{\xi_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \boldsymbol{\xi_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \ \boldsymbol{\xi_3} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Thus, the solution of the system (5) with $\mathbf{F}(t) = \mathbf{0}$ is given by

$$\mathbf{u}(t) = c_1\boldsymbol{\xi_1} + c_2 e^{-3\alpha t}\boldsymbol{\xi_2} + c_3 e^{-3\alpha t}\boldsymbol{\xi_3}\,,$$

where the constants are chosen to satisfy the initial condition

$$c_1\boldsymbol{\xi_1} + c_2\boldsymbol{\xi_2} + c_3\boldsymbol{\xi_3} = \mathbf{u}_0\,.$$

The characteristic value $\lambda = 0$ indicates the matrix is *singular*, and the 'constant' vector $\boldsymbol{\xi_1}$ corresponds to the *stationary* periodic solution of the heat conduction problem. Again note that the higher characteristic values have more dissipation of heat between adjacent components. This can be quantified by the number of sign changes between adjacent components of the successive eigenvectors.

EXERCISE 2. *Write out the matrix* $\mathbf{A}$ *and forcing term* $\mathbf{F}(t)$ *that arise in the system* (5) *for each of the boundary conditions of Section 1.2 at both ends.*

EXERCISE 3. *Write out the matrix* $\mathbf{A}$ *and forcing term* $\mathbf{F}(t)$ *corresponding to the boundary conditions of Exercise 1.*

EXERCISE 4. *Find the general solution of the homogeneous system* (5) *for* $N = 2$ *and with Dirichlet condition at* $x = a$ *and Neumann condition at* $x = b$*.*

EXERCISE 5. *Find the general solution of the homogeneous system* (5) *for* $N = 2$ *and with Neumann conditions at both* $x = a$ *and* $x = b$*.*

2. Vibrations in an Interval

We begin with a very simple classical example of a vibrating system. A particle of *mass* $m > 0$ is suspended by a spring in a *rest position* at which the force of gravity is balanced by the restoring force of the spring. Let $u(t)$ denote the displacement (upward) of the particle, at any time $t \geq 0$ from this rest position which corresponds to $u = 0$.

Experiment 1. If the particle is moved below this position, *i.e.*, if $u(t) < 0$, then the spring exerts an additional restoring force (upward) whose magnitude is proportional to $u(t)$. The same happens when the particle is moved above the rest position. This is known as *Hooke's law*: the restoring force (upward) is given by

$$F_s(t) = -ku(t)\,,$$

and the constant $k > 0$ measures the *stiffness* of the spring, *i.e.*, its resistance to compression or dilation. This property describes the *elasticity* of the spring. Note that it is *rate independent*, and the minus sign denotes that the direction of force is opposite to that of the displacement.

Experiment 2. At higher velocities, there is an additional force generated by the passage through the medium, a *friction* force which we note is always increasing with increasing velocity. This force is always directed opposite to the direction of motion, and when this force is not too large, it is proportional to the velocity. We find that it is given by

$$F_f(t) = -r\dot{u}(t)\,.$$

Here the velocity is given by the time derivative of displacement, $\dot{u}(t) \equiv \frac{du(t)}{dt}$.

Newton's law states that the sum of all forces on a particle is equal to the product of mass and acceleration of the particle. This is the *balance of momentum*. For the mass-spring system described above, this takes the form

$$m\ddot{u}(t) + r\dot{u}(t) + ku(t) = F(t)\,, \tag{8a}$$

where $F(t)$ denotes all additional (upward) forces acting on the particle. The appropriate *initial data* consists of the initial displacement and the initial velocity,

$$u(0) = u_0\,, \qquad \dot{u}(0) = v_0\,. \tag{8b}$$

Let's look at some solutions of the problem (8) with $F(t) = 0$. Thus, try $u(t) = e^{\lambda t}$ in (8a) to get the quadratic equation

$$m\lambda^2 + r\lambda + k = 0\,.$$

The roots are obtained from the equivalent form

$$(\lambda + \frac{r}{2m})^2 = \left(\frac{r}{2m}\right)^2 - \frac{k}{m}\,.$$

In the case of *no friction*, $r = 0$, we have the purely imaginary roots $\lambda = \pm\sqrt{\frac{k}{m}}i$, and the *periodic* solution is given by

$$u(t) = c_1 \cos(\sqrt{\frac{k}{m}}\,t) + c_2 \sin(\sqrt{\frac{k}{m}}\,t)\,.$$

If $0 < r^2 < 4mk$, then $\lambda = -\frac{r}{2m} \pm i\sqrt{\frac{k}{m} - \frac{r^2}{4m^2}}$ is the complex conjugate pair of roots, and the corresponding *weakly damped* solution is

$$u(t) = e^{-\frac{r}{2m}t}\left(c_1 \cos(\frac{\sqrt{4mk - r^2}}{2m}\,t) + c_2 \sin(\frac{\sqrt{4mk - r^2}}{2m}\,t)\right)\,.$$

If $r^2 = 4mk$, then $\lambda = -\frac{r}{2m}$ is the double root, and the *critically damped* solution is

$$u(t) = e^{-\frac{r}{2m}t}(c_1 + c_2\,t)\,.$$

Finally, if $r^2 > 4mk$, then both roots are negative, $\lambda = -\frac{r}{2m} \pm \frac{\sqrt{r^2 - 4mk}}{2m}$, and the *strongly damped* solution is

$$u(t) = e^{-\frac{r}{2m}t}(c_1 e^{\frac{\sqrt{r^2-4mk}}{2m}\,t} + c_2 e^{-\frac{\sqrt{r^2-4mk}}{2m}\,t})\,.$$

In each case, the constants c_1, c_2 are determined by the initial data (8b).

EXERCISE 6. *Sketch some representative solutions for each of the cases above.*

EXERCISE 7. *Let $u_r(\cdot)$ be the solution in the weakly damped case $0 < r^2 < 4mk$ with initial data $u(0) = u_0$ and $\dot{u}(0) = v_0$. Find the limit of $u_r(\cdot)$ as $r \to 2\sqrt{mk}$.*

2.1. Transverse Vibrations. We shall formulate a discrete model of the small transverse motions of a stretched elastic string. We identity its rest position with the horizontal interval $G = (a, b)$. Partition this interval into N sections of equal lengths h, so $Nh = b-a$ and the right end point of the j-th section is located at $x_j = a+jh$, $1 \le j \le N$. Here we are assuming that the mesh size h of the partition is so small that we can approximate the vertical displacement of each section by a constant. Thus, let $u_j(t)$ be the upward displacement of the j-th section at the time $t \ge 0$. The displacements at the endpoints are $u_0(t)$ and $u_{N+1}(t)$. Let ρ denote the linear density, so the mass of each section is ρh. Denote by T the (uniform) *tension* on the string. This is a force that is directed along the length of the string. The vertical component of this force is generated by the relative displacement of u_{j+1} with respect to u_j, and it is given by $T \sin\alpha$, where α is the angle of the line segment with slope $\tan\alpha = \frac{u_{j+1}-u_j}{h}$. As long as this angle is small, we can approximate the vertical component of force of u_{j+1} on u_j by $T\sin\alpha \approx T\tan\alpha$. Thus, the vertical force exerted on the jth segment by the $j+1$st segment is given by

$$\tau_j = T\frac{u_{j+1}-u_j}{h}, \quad 1 \le j \le N-1\,. \tag{9a}$$

The *conservation of momentum* of the jth segment due to the vertical forces from the two neighbors together with an additional distributed vertical force of mass density $f_j(t)$ lead to the equation

$$\rho h \ddot{u}_j(t) = \tau_j - \tau_{j-1} + \rho h f_j(t), \quad 1 \le j \le N\,. \tag{9b}$$

These are to be supplemented with the *initial conditions*

$$u_j(0) = u_j, \quad \rho\dot{u}_j(0) = \rho v_j, \quad 1 \le j \le N, \tag{9c}$$

where the initial displacements u_j and velocities v_j are also given for $1 \le j \le N$. This is an *initial-value problem* for the N unknown functions $u_j(t)$, $1 \le j \le N$. The endpoint displacements $u_0(t)$, $u_{N+1}(t)$ for $t > 0$ are yet to be determined.

Note that substitution of (9a) into (9b) yields the *system of ordinary differential equations*

$$\rho\ddot{u}_j(t) = \frac{T}{h^2}(u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)) + \rho f_j(t), \quad 1 \le j \le N. \tag{10}$$

It is useful to note that this system is formally equivalent to the representation of a mass-spring system of N particles which are connected in a line by springs and suspended.

2.2. The Boundary conditions. We consider a number of typical possibilities for determining the endpoint conditions. We illustrate each of these as before with a condition at the right end, $x_N = b$, and note that another such condition will also be prescribed at the left end, $x_0 = a$.

1. The displacement could be specified at the end point:

$$u_{N+1}(t) = u_b(t), \quad t > 0\,.$$

This is the *Dirichlet* boundary condition, or boundary condition of *first type.*

2. The vertical force on the string could be specified at the end point:

$$\tau_N = T\frac{u_{N+1}(t) - u_N(t)}{h} = f_b(t), \quad t > 0\,.$$

This is the *Neumann* boundary condition, or boundary condition of *second type.* It depends on a given vertical force $f_b(t)$ at the end. The homogeneous condition with $f_b(t) = 0$ corresponds to a *free end.*

3. The force on the end is determined by an elastic constraint, a restoring force proportional to the displacement:

$$\tau_N + T_0(u_N(t) - u_b(t)) = f_b(t), \quad t > 0\,.$$

This is the *Robin* boundary condition, or boundary condition of *third type.* Here both $u_b(\cdot)$ and $f_b(\cdot)$ are prescribed. The first is the prescribed displacement and the second is a vertical force concentrated on the end point. For $T_0 \to \infty$, we obtain formally the Dirichlet boundary condition, while for $T_0 \to 0$ we get the Neumann condition. Thus the *effective tension* T_0 is a new quantity which interpolates between the first two types.

4. Another type of boundary condition arises if there is a *concentrated mass* at the end point. Then $u_{N+1}(t)$ is the displacement of this mass, an additional unknown, and we have the *dynamic* boundary condition:

$$\rho_0 \ddot{u}_{N+1}(t) + \tau_N = f_b(t), \quad t > 0\,,$$

where the endpoint force has been obtained from

$$\tau_N = T\frac{u_{N+1} - u_N}{h}\,.$$

This is the boundary condition of *fourth type*, and it must be supplemented with corresponding initial conditions,

$$\rho_0 u_{N+1}(0) = \rho_0 u_{N+1}\,, \qquad \rho_0 \dot{u}_{N+1}(0) = \rho_0 v_{N+1}$$

2.3. Examples: Dirichlet Boundary Conditions. We compute solutions of some particular cases in order to see if our models are giving solutions which correspond to intuition.

Example $N = 1$. Set $\alpha = \frac{T}{\rho h^2}$. The system is given by

$$\ddot{u}_1(t) + 2\alpha u_1(t) = F_1(t)$$
$$u_1(0) = u_1,\ \dot{u}_1(0) = v_1$$

where $F_1(t) \equiv \alpha(u_0(t)+u_2(t))+f_1(t)$. This is just the mass-spring system with no friction. Thus, when the endpoint displacements and outside forces are zero, so $F_1(t) = 0$, the solution is given by

$$u(t) = u_1\cos(\sqrt{2\alpha}\,t) + \frac{v_1}{\sqrt{2\alpha}}\sin(\sqrt{2\alpha}\,t)\,.$$

Example $N = 2$. The system of equations describing the approximate Dirichlet problem is given by

$$\ddot{u}_1(t) + 2\alpha u_1(t) - \alpha u_2(t) = f_1(t) + \alpha u_0(t) \equiv F_1(t), \quad u_1(0) = u_1\,,\ \dot{u}_1(0) = v_1,$$
$$\ddot{u}_2(t) - \alpha u_1(t) + 2\alpha u_2(t) = f_2(t) + \alpha u_3(t) \equiv F_2(t), \quad u_2(0) = u_2\,,\ \dot{u}_2(0) = v_2\,.$$

Assume that each $F_j(t) = 0$ and look for solutions in the form

$$u_1(t) = a_1 e^{\mu t},\ u_2(t) = a_2 e^{\mu t}.$$

This leads to the linear algebra problem

$$\mu^2 a_1 + \alpha(2a_1 - a_2) = 0,$$
$$\mu^2 a_2 + \alpha(-a_1 + 2a_2) = 0$$

This is again the eigenvalue problem (6) with $\lambda = -\mu^2$. The eigenvalues are given by $\lambda_1 = \alpha,\ \lambda_2 = 3\alpha$ with corresponding eigenvalues

$$\boldsymbol{\xi_1} \equiv \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{\xi_2} \equiv \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

By taking real parts of the complex exponential solutions, we find the general (real valued) solution given by

$$\mathbf{u}(t) = (c_1 \cos(\sqrt{\alpha} t) + d_1 \sin(\sqrt{\alpha} t))\boldsymbol{\xi_1} + (c_2 \cos(\sqrt{3\alpha} t) + d_2 \sin(\sqrt{3\alpha} t))\boldsymbol{\xi_2}.$$

The specific solution satisfying the initial conditions is obtained by choosing constants with

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad d_1\sqrt{\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + d_2\sqrt{3\alpha} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

This system is solvable because the pair $\boldsymbol{\xi_1},\ \boldsymbol{\xi_2}$ is a *basis* for the plane, $\mathbb{R}^2$. Note that the second term in the solution oscillates at a rate $\sqrt{3}$ times as fast as the first. This corresponds to the additional internal stress between the two components in the second term, whereas the only stress in the first term is from the connections to the end points.

Let's review this procedure. Define the matrix and the (column) vector

$$\mathbf{A} = \alpha \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

We can write the resulting system in matrix form

$$\ddot{\mathbf{u}}(t) + \mathbf{A}\mathbf{u}(t) = \mathbf{F}(t), \tag{11a}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0. \tag{11b}$$

To look for a solution of the homogeneous equation with $\mathbf{F}(t) = \mathbf{0}$ in the form $\mathbf{u}(t) = \boldsymbol{\xi} e^{\mu t}$, substitute this into the differential equation to get the *eigenvalue problem*

$$\mathbf{A}\boldsymbol{\xi} = \lambda\boldsymbol{\xi},$$

where $\lambda \equiv -\mu^2$. If $\boldsymbol{\xi_1},\ \lambda_1$ and $\boldsymbol{\xi_2},\ \lambda_2$ are solutions of the eigenvalue problem, we obtain the corresponding real-valued solutions of the ordinary differential equation system

$$\mathbf{u}(t) = (c_1 \cos(\sqrt{\lambda_1}\, t) + d_1 \sin(\sqrt{\lambda_1}\, t))\boldsymbol{\xi_1} + (c_2 \cos(\sqrt{\lambda_2}\, t) + d_2 \sin(\sqrt{\lambda_2}\, t))\boldsymbol{\xi_2}.$$

The pair $\boldsymbol{\xi_1},\ \boldsymbol{\xi_2}$ is a basis, so this gives the *general solution* of the homogeneous equation.

It is clear how to extend this procedure to larger N's. For the case of Dirichlet boundary conditions, the matrix form of the problem is given by (11) where the unknown, matrix, and data are given by

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ \dots \\ u_{N-1}(t) \\ u_N(t) \end{bmatrix}, \quad \mathbf{A} = \alpha \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{pmatrix}, \quad \mathbf{F}(t) = \begin{bmatrix} f_1(t) + \alpha u_a(t) \\ f_2(t) \\ f_3(t) \\ \dots \\ f_{N-1}(t) \\ f_N(t) + \alpha u_b(t) \end{bmatrix}.$$

Example $N = 3$. For our final example of the approximate Dirichlet problem, consider the case with $N = 3$ sections. The matrix is given by

$$\mathbf{A} = \alpha \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

The characteristic values are

$$\lambda_1 = (2 - \sqrt{2})\alpha, \ \lambda_2 = 2\alpha, \ \lambda_3 = (2 + \sqrt{2})\alpha,$$

and the corresponding vectors are

$$\boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \ \boldsymbol{\xi}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \ \boldsymbol{\xi}_3 = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}.$$

Thus, the solution of the system (11) with $\mathbf{F}(t) = \mathbf{0}$ is given by

$$\begin{aligned} \mathbf{u}(t) = {} & (c_1 \cos(\sqrt{\lambda_1}\, t) + d_1 \sin(\sqrt{\lambda_1}\, t))\boldsymbol{\xi}_1 \\ & + (c_2 \cos(\sqrt{\lambda_2}\, t) + d_2 \sin(\sqrt{\lambda_2}\, t))\boldsymbol{\xi}_2 + (c_3 \cos(\sqrt{\lambda_3}\, t) + d_3 \sin(\sqrt{\lambda_3}\, t))\boldsymbol{\xi}_3 , \end{aligned}$$

where the constants are chosen to satisfy the initial condition

$$\begin{aligned} c_1\boldsymbol{\xi}_1 + c_2\boldsymbol{\xi}_2 + c_3\boldsymbol{\xi}_3 &= \mathbf{u}_0 , \\ d_1\sqrt{\lambda_1}\boldsymbol{\xi}_1 + d_2\sqrt{\lambda_2}\boldsymbol{\xi}_2 + d_3\sqrt{\lambda_3}\boldsymbol{\xi}_3 &= \mathbf{v}_0 . \end{aligned}$$

Note that the characteristic values are arranged in increasing magnitude and the corresponding eigenvectors have higher and higher rates of oscillation. This can be quantified by the number of sign changes between adjacent components of the successive eigenvalues.

EXERCISE 8. *Find the general solution of the homogeneous system* (11) *for* $N = 2$ *and with Neumann conditions at both* $x = a$ *and* $x = b$. *Explain the reason for the occurence of a "new" type of solution in the model.*

2.4. The Rotating String. We constructed above a discrete model of the small displacements of a stretched elastic string. These displacements take place in a vertical plane. Now we consider the situation in which the plane containing the displacements is rotated at a fixed angular speed ω about the rest axis of the string. This introduces the new *centrifugal force* directed outward (= upward) on the j-th segment and given by $h\rho\omega^2 u_j(t)$ in (10). If there are no additional outside forces acting on the string, the system (11) takes the form

$$\ddot{\mathbf{u}}(t) + \mathbf{A}\mathbf{u}(t) = \omega^2 \mathbf{u}(t)\,,$$
$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0.$$

Finally, we look for *stationary* solutions: are there any values of the angular velocity which permit the presence of *standing waves*? Any such ω must satisfy the system

$$\mathbf{A}\boldsymbol{\xi} = \omega^2 \boldsymbol{\xi}$$

for a corresponding non-zero displacement $\mathbf{u}(t) = \boldsymbol{\xi}$. But this is precisely the *eigenvalue problem* (6) with $\lambda = \omega^2$! This gives the interpretation of the solutions $\boldsymbol{\xi}$, λ of the eigenvalue problem for the matrix $\mathbf{A}$ as profiles $\boldsymbol{\xi}$ of standing waves in a rotating planar system with angular speed $\sqrt{\lambda}$.

2.5. Friction. Suppose there is an additional force generated by the passage through the medium, a *friction* force on the j-th section given by $-rh\dot{u}_j(t)$. These terms must be added to the system (10), and we obtain

$$\rho\ddot{u}_j(t) + r\dot{u}_j(t) + \frac{T}{h^2}(-u_{j+1}(t) + 2u_j(t) - u_{j-1}(t)) = \rho f_j(t), \quad 1 \le j \le N. \tag{12}$$

The matrix form is given by

$$\ddot{\mathbf{u}}(t) + \frac{r}{\rho}\dot{\mathbf{u}}(t) + \mathbf{A}\mathbf{u}(t) = \mathbf{F}(t)\,, \tag{13}$$

where $\mathbf{A}$ depends on the chosen boundary conditions as before.

The model (13) can be reduced to the form (11a) by anticipating the *damping* induced by friction. To see this, note that the change of variable $\mathbf{w}(t) = e^{\frac{r}{2\rho}t}\mathbf{u}(t)$ introduces a function $\mathbf{w}(\cdot)$ which satisfies the system

$$\ddot{\mathbf{w}}(t) + (\mathbf{A} - \frac{r^2}{4\rho^2}\mathbf{I})\mathbf{w}(t) = e^{\frac{r}{2\rho}t}\mathbf{F}(t)\,.$$

Since each eigenvalue λ of $\mathbf{A}$ corresponds to the eigenvalue $\lambda - \frac{r^2}{4\rho^2}$ of $\mathbf{A} - \frac{r^2}{4\rho^2}\mathbf{I}$, and with the same eigenvector, we can immediately write out the solution of (13). Specifically, the effect of the friction term in (13) is to add a damping factor of $e^{-\frac{r}{2\rho}t}$ and to slow the rate of oscillations.

2.6. Remarks on Mechanics. Problems from mechanics of deformation of a structure concern three types of quantities: stress, strain, and displacement. *Displacement* describes the motion of a point with reference to a fixed coordinate system outside the structure during the process of deformation. *Stress* is the internal force per unit of area on the structure. In the mass-spring system, stress occured only in its *total* form, a single force independent of position, while in the model of transverse vibrations there was a stress at the interface between adjacent sections resulting from their relative displacements. *Strain* is the local deformation, for example, the increase in length of a line element divided by its original length (tensile strain). (Much more sophisticated forms of strain quantities are needed to describe the deformation of a body in $\mathbb{R}^3$.) These quantities are connected by three kinds of equations. Relations between the stresses are *momentum balance.* These are *equilibrium* conditions in stationary problems (like the rotating string model) and *equations of motion* in dynamic problems. These are expressed by some form of Newton's law. The material involved is determined by the assumed *constitutive equations* which relate stress and strain. Hooke's law is the simplest example. Equations which relate strains to displacement are called *kinematic relations.*

2.7. Longitudinal Vibrations. We shall consider small longitudinal displacements of a long thin rod with uniform cross section S. As before, we identify points on the rod with the interval $G = (a, b)$ in the *real number* line $\mathbb{R}$.

Experiment. The force required to compress or dilate a section $x_1 \le x \le x_2$ of the rod is proportional to the relative change of its length from the original length $\ell = x_2 - x_1$. If $u(x,t)$ denotes the *displacement* (rightward) of the point of the rod originally at x, this *elongation* of this section of the rod at time $t \ge 0$ is given by

$$\frac{(x_2 + u(x_2,t)) - (x_1 + u(x_1,t)) - \ell}{\ell} = \frac{u(x_2,t) - u(x_1,t)}{\ell}.$$

It is a *compression* or *dilation*, depending on its sign, and it is called the *strain* of the section. This results in a rightward force acting at x_1 given by

$$kS\frac{u(x_2,t) - u(x_1,t)}{\ell},$$

and this defines the number k, the *stiffness* of the material. The limit as $\ell \to 0$ defines the *strain* at the point x by the kinematic relation $\varepsilon(x,t) \equiv \frac{\partial u}{\partial x}$, and the corresponding (internal outward) force per unit area at x is the *stress* $\sigma(x)$. The resulting *stress-strain* relationship

$$\sigma(x,t) = k\varepsilon(x,t)$$

is *Hooke's law* for a linearly elastic model. (This is the constitutive relation.)

In order to construct our discrete model of the longitudinal motion of the rod, we partition the interval (a, b) into N sections of equal lengths h, so $Nh = b - a$, and the right end point of the j-th section is located at $x_j = a + jh$, $1 \le j \le N$. The (discrete) strain of the j^{th} section is given by $\varepsilon_j = \frac{u_j - u_{j-1}}{h}$, and the (rightward) stress at x_{j-1} due to the relative displacements within the j^{th} section is given by Hooke's law as

$$\sigma_j(t) = k\varepsilon_j = k\frac{u_j(t) - u_{j-1}(t)}{h}.$$

Likewise, the corresponding stress acting on x_j from the $(j+1)^{\text{st}}$ section is $\sigma_{j+1}(t)$. Let ρ denote the volume density, so the mass of each section is ρhS. The mesh size h of the partition is assumed to be so small that we can assume the mass of the j^{th} section is concentrated at x_j. If we have an additional distributed rightward force of mass density $f_j(t)$ acting on the j^{th} section, this contributes a total longitudinal force of $\rho hSf_j(t)$. From the balance of momentum for the j-section, we obtain

$$\rho hS\ddot{u}_j(t) = S(\sigma_{j+1}(t) - \sigma_j(t)) + \rho hSf_j(t) \qquad 1 \le j \le N.$$

After dividing by hS, we find these satisfy the system

$$\rho\ddot{u}_j(t) + \frac{k}{h^2}(-u_{j+1}(t) + 2u_j(t) - u_{j-1}(t)) = \rho f_j(t), \quad 2 \le j \le N-1\,, \tag{14}$$

and the corresponding equations at the ends will depend on the end conditions. The initial displacements u_j and velocities v_j for $1 \le j \le N$ are to be prescribed. Thus we see that this one system (10) or (14) is formally equivalent to the representation of a discrete model of either transverse vibrations, longitudinal vibrations, or a mass-spring system of N particles connected by springs. Furthermore, each of the examples of boundary conditions listed in Section 2.2 has a meaning here. One need only replace *vertical* by *horizontal* to get the corresponding description.

EXERCISE 9. *Complete the system* (14) *for the case that the left end is prescribed at the position $u_a(t)$ and at the right end there is applied a force $f_b(t)$. Find the matrix* $\mathbf{A}$ *and the forcing term* $\mathbf{F}(t)$ *which permit this system to be written in the form* (11).

2.8. Viscosity. We consider a model for the *internal* friction that results from the relative motions of the interior of the rod.

Experiment. There is an additional force required to compress or dilate a small section $x_1 \le x \le x_2$ of the rod that is proportional to the *elongation rate*. Such a force is given by

$$RS\frac{\dot{u}(x_2,t) - \dot{u}(x_1,t)}{\ell}$$

for some constant $R > 0$ and is known as *viscous friction*. The resulting *stress-strain* relationship is given in the limit $\ell \to 0$ by the kinematic relation

$$\sigma(x,t) = k\,\varepsilon(x,t) + R\,\dot{\varepsilon}(x,t)$$

in which the constant R is a material property called *viscosity*. The discrete form of this constitutive relation is

$$\sigma_j(t) = k\frac{u_j(t) - u_{j-1}(t)}{h} + R\frac{\dot{u}_j(t) - \dot{u}_{j-1}(t)}{h}\,.$$

When this is used in the momentum balance equations for the j^{th}-section of our partition of (a,b), we obtain

$$\rho\ddot{u}_j(t) + \frac{R}{h^2}(-\dot{u}_{j+1}(t) + 2\dot{u}_j(t) - \dot{u}_{j-1}(t)) + \frac{k}{h^2}(-u_{j+1}(t) + 2u_j(t) - u_{j-1}(t)) = \rho f_j(t), \quad 1 \le j \le N. \tag{15}$$

We shall see that the viscous terms in (15) have a substantially stronger effect than the corresponding friction terms in (13).

Suppose we have Dirichlet boundary conditions at both ends. With the matrix $\mathbf{A}$ as given in Section 2.3 with $\alpha = \frac{k}{\rho h^2}$, the system takes the form

$$\ddot{\mathbf{u}}(t) + \frac{R}{k}\mathbf{A}\dot{\mathbf{u}}(t) + \mathbf{A}\mathbf{u}(t) = \mathbf{F}(t)\,, \tag{16}$$

To solve the homogeneous equation with $\mathbf{F}(t) = \mathbf{0}$, we look for a solution in the form $\mathbf{u}(t) = \boldsymbol{\xi} e^{-\mu t}$. Substitute this into (16) to get

$$(\mu^2 - \frac{R}{k}\mu\mathbf{A} + \mathbf{A})\boldsymbol{\xi} = \mathbf{0}/,.$$

This is just the *eigenvalue problem*

$$\mathbf{A}\boldsymbol{\xi} = \lambda\boldsymbol{\xi}\,,$$

with $\lambda \equiv \frac{\mu^2}{\frac{R}{k}\mu - 1}$. Let $\boldsymbol{\xi}$, λ be a solution of this eigenvalue problem. Then the corresponding μ is determined by

$$(\mu - \frac{R\lambda}{2k})^2 = (\frac{R\lambda}{2k})^2 - \lambda = \frac{(R^2\lambda - 4k^2)\lambda}{4k^2}\,,$$

so the solution of (16) depends on the sign of $R^2\lambda - 4k^2$. For the case of the lower values of λ for which $0 < R^2\lambda < 4k^2$, we obtain the corresponding solutions

$$\mathbf{u}(t) = e^{-\frac{R\lambda}{2k}t}\left(c\,\cos(\frac{\sqrt{(4k^2 - R^2\lambda)\lambda}}{2k}\,t) + d\,\sin(\frac{\sqrt{(4k^2 - R^2\lambda)\lambda}}{2k}\,t)\right)\boldsymbol{\xi}\,.$$

From the remaining higher values of λ with $R^2\lambda > 4k^2$, the two real roots are negative, and we get the exponential solutions

$$\mathbf{u}(t) = e^{-\frac{R\lambda}{2k}\,t}\left(c\,e^{\frac{\sqrt{(R^2\lambda - 4k^2)\lambda}}{2k}\,t} + d\,e^{-\frac{\sqrt{(R^2\lambda - 4k^2)\lambda}}{2k}\,t}\right)\boldsymbol{\xi}\,.$$

2.9. Transverse inertia. Finally, we consider the effect of the transverse motions of the rod that result from the elongations under conditions of constant volume or mass. If we denote the transverse *radial motion* $r_j(t)$ of a particle of the rod at a unit distance from x_j along a perpendicular to the center line, we find that it is given by $r_j(t) = P\,\varepsilon_j(t)$. The constant P depends on the material and is known as *Poisson's ratio*. The corresponding transverse acceleration of a point at a distance r from the centerline is $\ddot{U}_j(t) = r\,P\,\ddot{\varepsilon}_j(t)$. By comparing the vertical work and the horizontal work, we find that the corresponding horizontal force must be $\frac{\overline{r}}{h}$ times the vertical force, where $\overline{r}$ is the average radius of that cross section, so the corresponding contribution to the longitudinal stress on that section of the rod is $\rho S\,h\frac{\overline{r}}{h}\,\overline{r}\,P\,\ddot{\varepsilon}_j(t)$. That is, the total stress of the j^{th} section of the rod on $x_j(t)$ is given by

$$\sigma_j(t) = k\frac{u_j(t) - u_{j-1}(t)}{h} + R\frac{\dot{u}_j(t) - \dot{u}_{j-1}(t)}{h} + \rho\,\overline{r}^2\,P\frac{\ddot{u}_j(t) - \ddot{u}_{j-1}(t)}{h}\,.$$

The corresponding *stress-strain* relationship is given in the limit $h \to 0$ by

$$\sigma(x,t) = k\,\varepsilon(x,t) + R\,\dot{\varepsilon}(x,t) + \rho\,\overline{r}^2\,P\,\ddot{\varepsilon}(x,t)\,.$$

EXERCISE 10. *Include the effect of* transverse inertia *in the model analogous to* (16). *Show how it affects the solution for the case of* $R = 0$, *specifically, how it limits the frequency of the response, independent of the size of the eigenvalues.*