Linear Systems

1. Linear Algebra

1.1. Vector Spaces. The triple $\{V, +, \cdot\}$ consisting of a set V and two operations, $+$ on $V \times V$ and $\cdot$ on $\mathbb{R} \times V$, is called a *vector space* if $\{V, +\}$ is an abelian group, that is,

$$(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z}), \quad \vec{x} + \vec{y} = \vec{y} + \vec{x}, \quad \text{there is a } \vec{0} \in V : \vec{x} + \vec{0} = \vec{x},$$
$$\text{and for each } \vec{x} \in V \text{ there is a } -\vec{x} \in V \text{ with } \vec{x} + (-\vec{x}) = \vec{0},$$

and $\{V, \cdot\}$ satisfies

$$(a + b) \cdot \vec{x} = a \cdot \vec{x} + b \cdot \vec{x}, \quad (ab) \cdot \vec{x} = a \cdot (b \cdot \vec{x}),$$
$$a \cdot (\vec{x} + \vec{y}) = a \cdot \vec{x} + b \cdot \vec{y}, \quad \text{and } 1 \cdot \vec{x} = \vec{x}\,.$$

We shall usually suppress the symbol $\cdot$, and we denote $\vec{x} + (-\vec{y})$ by $\vec{x} - \vec{y}$. The elements of V are called *vectors.*
Example. The space $\mathbb{R}^N$ consists of the *column vectors*,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_N \end{bmatrix} \in \mathbb{R}^N\,,$$

and the vector space operations are defined component-wise.

Other examples are $\mathbb{R}^\infty$, $\mathbb{R}^S$, where S is a set, and these are discussed below.

A *subspace* of V is a nonempty set W which is closed under both operations, that is, $W + W \subset W$ and $\mathbb{R} \cdot W \subset W$. For any subset $S \subset V$ we define the *linear span* of S to be the set $\langle S \rangle$ of all linear combinations of elements from S, that is, $\langle S \rangle \equiv \{\sum_{j=1}^n a_j \vec{v}_j :$ all $\vec{v}_j \in S\}$. We say that S spans V if $\langle S \rangle = V$. The finite set $S = \{\vec{v}_1, \ldots \vec{v}_n\}$ of vectors is *independent* if the equality $\sum_{j=1}^n a_j \vec{v}_j = \vec{0}$ implies that necessarily all $a_j = 0$, $1 \leq j \leq n$. An arbitrary set is independent if each finite subset of it is independent. A *basis* for V is an ordered set of vectors which is independent and spans V.

THEOREM 1. *Every basis has the same number of elements.*

This number is called the *dimension* of the vector space.
Example. The space $\mathbb{R}^n$ has dimension n, and $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ is the *standard basis*, where $\vec{e}_j$ denotes the column vector with zeros in all rows except for a 1 in the j-th row. We denote with the square brackets a *column vector.*

DEFINITION 1. *If V and W are vector spaces and $T : V \to W$ is a function, then T is called* linear *if $T(a\vec{x} + b\vec{y}) = aT(\vec{x}) + bT(\vec{y})$ for all a, $b \in \mathbb{R}$ and $\vec{x}$, $\vec{y} \in V$. If also T is one-to-one and onto, then T is called an* isomorphism.

There are two subsets which arise naturally from a linear map. The *range* of T is $Rg(T) = \{T(\vec{x}) : \vec{x} \in V\}$ and the *kernel* of T is $Ker(T) = \{\vec{x} \in V : T(\vec{x}) = \vec{0}\}$.

THEOREM 2. *If $T : V \to W$ is linear, then $Rg(T)$ is a subspace of W, and $Ker(T)$ is a subspace of V. Also, $Ker(T) = \{\vec{0}\}$ if and only if the function T is one-to-one.*

EXERCISE 1. *Prove Theorem 2.*

1.1.1. *Coordinate map.* Let V be a vector space of dimension n with basis $\boldsymbol{\beta} = \{\vec{\beta}_1, \vec{\beta}_2, \ldots \vec{\beta}_n\}$. For each $\vec{x} \in V$, there is a unique vector $[x_1, x_2, \ldots, x_n] \in \mathbb{R}^n$ for which $\vec{x} = \sum_{j=1}^n x_j \vec{\beta}_j$. Define $\mathbb{C}_\beta : V \to \mathbb{R}^n$ by $\mathbb{C}_\beta(\vec{x}) = [x_1, x_2, \ldots, x_n]$. That is, $\mathbb{C}_\beta(\vec{x})$ is just the (column) vector of *coordinates* of $\vec{x}$ with respect to the basis $\boldsymbol{\beta}$. Note that $\mathbb{C}_\beta$ is a one-to-one function that maps V onto $\mathbb{R}^n$. Furthermore, it is easy to check that $\mathbb{C}_\beta$ is linear.

EXERCISE 2. *Show that $\mathbb{C}_\beta$ is linear, that is,*

$$\mathbb{C}_\beta(a\vec{x} + b\vec{y}) = a\mathbb{C}_\beta(\vec{x}) + b\mathbb{C}_\beta(\vec{y})$$

for each pair of vectors $\vec{x}$, $\vec{y}$ and numbers a, b.

EXERCISE 3. *Show that if $\boldsymbol{e} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is the standard basis for $V = \mathbb{R}^n$, then $\mathbb{C}_{\boldsymbol{e}}$ is the identity. That is,* coordinates *are just* components.

The map $\mathbb{C}_\beta : V \to \mathbb{R}^n$ defined above by the basis $\boldsymbol{\beta}$ is an isomorphism, and it is called the *coordinate map* of that basis. Note that $\mathbb{C}_\beta(\vec{\beta}_j) = \vec{e}_j,\ 1 \le j \le n$, the standard basis. Conversely, if $T : V \to \mathbb{R}^n$ is an isomorphism, then it follows that $\{T^{-1}(\vec{e}_j) : 1 \le j \le n\}$ is a basis for V.

1.1.2. *Change of Basis.* Now let $\boldsymbol{\alpha} = \{\vec{\alpha}_1, \vec{\alpha}_2, \ldots \vec{\alpha}_n\}$ and $\boldsymbol{\beta} = \{\vec{\beta}_1, \vec{\beta}_2, \ldots \vec{\beta}_n\}$ be a pair of bases for the space V. Define the *$\boldsymbol{\alpha\beta}$-transition matrix* to be $C = (c_{ij})$ where $\vec{\beta}_j = \sum_{i=1}^n c_{ij}\vec{\alpha}_i,\ 1 \le j \le n$. Then $\vec{\xi} = \mathbb{C}_\beta(\vec{x})$ if and only if $\vec{x} = \sum_{j=1}^n \xi_j \vec{\beta}_j$, so we have

$$\vec{x} = \sum_{j=1}^{n} \xi_j \sum_{i=1}^{n} c_{ij}\vec{\alpha}_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} c_{ij}\xi_j \right) \vec{\alpha}_i \,,$$

and it follows that $\mathbb{C}_\alpha(\vec{x}) = [c_{ij}\xi_j] = C\vec{\xi}$. That is, we have

$$\mathbb{C}_\alpha(\vec{x}) = C\mathbb{C}_\beta(\vec{x}) \,,$$

for which the right side is the indicated product of matrices. In particular, if $\boldsymbol{\beta}$ is the standard basis for $V = \mathbb{R}^n$, then $\mathbb{C}_\beta(\vec{x}) = [x_1, x_2, \ldots, x_n]$ and $\mathbb{C}_\alpha(\vec{\alpha}_j)$ is just the j-th column of C.

1.2. Linear Maps and Matrices. Let $T : V \to V$ be linear and let $\{\vec{\beta}_1, \vec{\beta}_2, \dots \vec{\beta}_n\} = \boldsymbol{\beta}$ be a basis for V. The *β-matrix of T* is the matrix $B = (b_{ij})$ for which $T(\vec{\beta}_j) = \sum_{i=1}^n b_{ij}\vec{\beta}_i$, $1 \le j \le n$. If $\vec{x} = \sum_{j=1}^n \xi_j \vec{\beta}_j$, then $T(\vec{x}) = \sum_{i=1}^n \left(\sum_{j=1}^n b_{ij}\xi_j\right)\vec{\beta}_i$, and so it follows that

$$\mathbb{C}_\beta T(\vec{x}) = B\mathbb{C}_\beta(\vec{x})$$

This shows that the value of the linear map T at a vector has coordinates which are obtained by multiplying the coordinates of that vector by the matrix. That is, every linear map is equivalent to matrix multiplication. Specifically, note that the coordinates of $T(\vec{\beta}_j)$ are just the j-th column of B.

Example. If $\vec{\beta}_j = \vec{e}_j$, then $T(\vec{e}_i) = i$-th column of B.

1.2.1. *Change of Basis.* Let A be the α-matrix of T in V and let B be the β-matrix of T in V. That is, we have

$$T(\vec{\alpha}_j) = \sum_{i=1}^n a_{ij}\vec{\alpha}_i \quad \text{and} \quad T(\vec{\beta}_j) = \sum_{i=1}^n b_{ij}\vec{\beta}_i,\ 1 \le j \le n.$$

Let C be the transition matrix: $\vec{\beta}_j = \sum_{i=1}^n c_{ij}\vec{\alpha}_i$, $1 \le j \le n$. Then we compute

$$T(\vec{\beta}_j) = \sum_{i=1}^n c_{ij}T(\vec{\alpha}_i) = \sum_{i,k=1}^n c_{ij}a_{ki}\vec{\alpha}_k = \sum_{k=1}^n \left(\sum_{i=1}^n a_{ki}c_{ij}\right)\vec{\alpha}_k\,,$$
$$T(\vec{\beta}_j) = \sum_{i,k=1}^n b_{ij}c_{ki}\vec{\alpha}_k = \sum_{k=1}^n \left(\sum_{i=1}^n c_{ki}b_{ij}\right)\vec{\alpha}_k\,,$$

and this shows that $AC = CB$.

We just established this equality directly from the definitions. However, we had already shown each of the identities

$$\mathbb{C}_\alpha T = A\mathbb{C}_\alpha\,, \quad \mathbb{C}_\beta T = B\mathbb{C}_\beta\,, \quad \mathbb{C}_\alpha = C\mathbb{C}_\beta\,,$$

and a direct substitution gives $CB\mathbb{C}_\beta = AC\mathbb{C}_\beta$, and this gives the same result.

A pair of matrices A, B is called *similar* if there is an invertible matrix C for which $B = C^{-1}AC$. The preceding calculation shows that this is equivalent to saying that they are representations of the same linear map on a vector space.

One of our goals is to find a basis for the space in which the matrix is particularly simple, *e.g.*, a diagonal matrix.

DEFINITION 2. *Let $T : V \to V$ be linear. A non-zero vector $\vec{v}$ for which $T(\vec{v}) = \lambda\vec{v}$ for some $\lambda \in \mathbb{R}$ is called a (real)* eigenvector *and λ is a corresponding* eigenvalue. *The subspace $Ker(T - \lambda I)$ is called the* λ-eigenspace *for T.*

A fortunate situation occurs when V has a basis $\{\vec{\beta}_1, \vec{\beta}_2, \dots \vec{\beta}_n\} = \boldsymbol{\beta}$ consisting of eigenvectors of T, that is, with

$$T(\vec{\beta}_j) = \lambda_j\vec{\beta}_j, \quad 1 \le j \le n,$$

for then the $\boldsymbol{\beta}$-matrix of T is the *diagonal matrix*

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \lambda_{n-1} & 0 \\ 0 & \dots & \dots & 0 & 0 & \lambda_n \end{pmatrix}$$

and we say that T is *diagonalizable.* This occurs when T has n distinct eigenvalues or when T is symmetric. This is the situation that we discuss below.

1.3. Scalar Product. A *scalar product* on the vector space V is a function denoted by $\vec{x}, \vec{y} \mapsto (\vec{x}, \vec{y})$ from $V \times V$ to $\mathbb{R}$ such that

- $\vec{x} \mapsto (\vec{x}, \vec{y})$ is linear $V \to \mathbb{R}$ for every $\vec{y} \in V$,
- $(\vec{x}, \vec{y}) = (\vec{y}, \vec{x})$ for every pair $\vec{x}, \vec{y} \in V$, and
- $(\vec{x}, \vec{x}) > 0$ for every $\vec{x} \neq \vec{0}$ in V.

THEOREM 3. *Let* $(\cdot, \cdot)$ *be a scalar product on* V. *Then*

(1) $|(\vec{x}, \vec{y})|^2 \leq (\vec{x}, \vec{x})(\vec{y}, \vec{y})$ *for* $\vec{x}, \vec{y} \in V$;

(2) $\|\vec{x}\| \equiv \sqrt{(\vec{x}, \vec{x})}$ *satisfies*
- $\|\vec{x}\| > 0$ *for every* $\vec{x} \neq \vec{0}$ *in* V,
- $\|a\vec{x}\| = a\|\vec{x}\|$ *for every* $a \in \mathbb{R}$, $\vec{x} \in V$,
- $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ *for every* $\vec{x}, \vec{y} \in V$;

(3) $\|\vec{x} + \vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2 = 2\|\vec{x}\|^2 + 2\|\vec{y}\|^2$ *for every* $\vec{x}, \vec{y} \in V$.

Any function $\|\cdot\| : V \to \mathbb{R}$ which satisfies the conditions in part (2) of Theorem 3 is called a *norm.* Although we see that every scalar product gives a norm as indicated, not every norm arises from a scalar product.

Example. Let $V = \mathbb{R}^n$ and define $(\vec{x}, \vec{y}) = \vec{x} \cdot \vec{y} = \sum_{i=1}^n x_i y_i$. It is easy to check that this is a scalar product on $\mathbb{R}^n$. The corresponding norm is clearly a measure of *length* of the vector. According to the *law of cosines*, $\vec{x} \cdot \vec{y} = \|\vec{x}\|\|\vec{y}\| \cos(\theta)$ where θ is the angle between the directions of $\vec{x}$ and $\vec{y}$. Thus the fraction $\frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|\|\vec{y}\|}$ is a measure of the *angle* between $\vec{x}$ and $\vec{y}$.

Example. Set $V = \{\vec{x} = (x_1, x_2, x_3, \dots) \in \mathbb{R}^\infty : \sum_{i=1}^\infty x_i^2 < \infty\}$. Note that from the inequality $|x_i + y_i|^2 \leq 2(|x_i|^2 + |y_i|^2)$ it follows that V is a subspace of $\mathbb{R}^\infty$, hence, a vector space. Then from the preceding example, we get

$$\left(\sum_{i=1}^n |x_i y_i| \right)^2 \leq \sum_{i=1}^n |x_i|^2 \sum_{i=1}^n |y_i|^2 \leq \sum_{i=1}^\infty |x_i|^2 \sum_{i=1}^n |y_i|^2$$

for each $n \geq 1$, so the series $\sum_{i=1}^\infty x_i y_i$ is (absolutely) summable, and we can define the scalar product $(\vec{x}, \vec{y}) = \sum_{i=1}^\infty x_i y_i$ on $\mathbb{R}^\infty$.

A pair of vectors $\vec{x}, \vec{y} \in V$ is said to be *orthogonal* if $(\vec{x}, \vec{y}) = 0$, and this is denoted by $\vec{x} \perp \vec{y}$. A vector $\vec{x} \in V$ is *normal* if $\|\vec{x}\| = 1$. A set $S \subset V$ is *orthonormal* if each pair from S is orthogonal and each vector in S is normal.

EXERCISE 4. *Show that every orthonormal set is necessarily linearly independent.*

1.3.1. *Orthonormal Bases.* Let $\boldsymbol{\beta} = \{\vec{\beta}_1, \vec{\beta}_2, \ldots \vec{\beta}_n\}$ be an orthonormal basis for the n-dimensional space V, and consider the coordinate map $\mathbb{C}_\beta : V \to \mathbb{R}^n$. If $\mathbb{C}_\beta(\vec{x}) = [x_1, x_2, \ldots, x_n]$, then we have $\vec{x} = \sum_{i=1}^n x_i \vec{\beta}_i$ with coefficients easily computed from the scalar product by $x_i = (\vec{x}, \vec{\beta}_i)$. In particular, we have the representation

$$\vec{x} = \sum_{i=1}^{n} (\vec{x}, \vec{\beta}_i) \vec{\beta}_i \,.$$

If $\vec{y}$ is another vector in V, we find from a similar calculation that

$$(\vec{x}, \vec{y})_V = \sum_{i=1}^{n} x_i y_i = (\mathbb{C}_\beta(\vec{x}), \mathbb{C}_\beta(\vec{y}))_{\mathbb{R}^n}.$$

Here we have put subscripts on the scalar products involved in order to emphasize the two spaces involved. In particular, it follows that the coordinate map preserves the scalar product structure on the space V.

Now we consider the effect of a change of orthonormal bases. Let $\boldsymbol{\alpha} = \{\vec{\alpha}_1, \vec{\alpha}_2, \ldots \vec{\alpha}_n\}$ and $\boldsymbol{\beta} = \{\vec{\beta}_1, \vec{\beta}_2, \ldots \vec{\beta}_n\}$ be a pair of orthonormal bases for the space V. If C is the transition matrix defined by $\vec{\beta}_j = \sum_{i=1}^n c_{ij} \vec{\alpha}_i$, $1 \le j \le n$, then we easily compute the entries directly by $c_{ij} = (\vec{\beta}_j, \vec{\alpha}_i)$. Furthermore, we have

$$(\vec{\beta}_j, \vec{\beta}_\ell) = (\sum c_{ij} \vec{\alpha}_i, \sum c_{k\ell} \vec{\alpha}_k) = \sum \sum c_{ij} c_{k\ell} (\vec{\alpha}_i, \vec{\alpha}_k) \,.$$

From the orthogonality of the bases, we find that

$$\delta_{j\ell} = \sum_i c'_{ji} c_{i\ell} \,,$$

so $I = C'C$, where $C' = (c_{ji})$ denotes the *transpose* of the matrix $C = (c_{ij})$. A matrix C is called *unitary* if $C^{-1} = C'$ In particular, we see that for such a matrix it is a trivial matter to compute the *inverse* matrix .

Let V and W be scalar product spaces. A linear map $T : V \to W$ is called *unitary* if $\|T\vec{x}\| = 1$ for every $\vec{x} \in V$ for which $\|\vec{x}\| = 1$. This is equivalent to requiring that $\|T(\vec{x})\|_W = \|\vec{x}\|_V$ for each $\vec{x} \in V$.

LEMMA 1. *The map T is unitary if and only if $(T\vec{x}, T\vec{y})_W = (\vec{x}, \vec{y})_V$ for all $\vec{x}$, $\vec{y} \in V$.*

This follows directly from the identity $2(T\vec{x}, T\vec{y})_W = \|T(\vec{x} + \vec{y})\|_W^2 - \|T\vec{x}\|_W^2 - \|T\vec{y}\|_W^2 = 2(\vec{x}, \vec{y})_V$. It shows that the *distance preserving* linear maps are the same as the *scalar product preserving* maps.

COROLLARY 1. *The unitary linear T maps each orthonormal basis onto an orthonormal basis.*

Now let $T : V \to V$ be linear and let $\boldsymbol{\alpha} = \{\vec{\alpha}_1, \vec{\alpha}_2, \ldots \vec{\alpha}_n\}$ be an orthonormal basis for V. The α matrix of T is determined by $T(\alpha_j) = \sum_{i=1}^n a_{ij} \alpha_i$ so we have $a_{ij} = (T(\alpha_j), \alpha_i)_V$. The coordinate map satisfies $\mathbb{C}_\alpha(T(\alpha_j)) = j$th column of A. Thus, T is unitary if and only if the columns of A are orthogonal, that is, $\sum_{i=1}^{n} a_{ij}a_{ik} = \delta_{jk}$ and $A'A = I$. This shows that the α-matrix is unitary if and only if the operator T is unitary.

1.3.2. *Orientation.* Let A be the α-matrix of T in V and let B be the β-matrix of T in V. that is, we have

$$T(\vec{\alpha}_j) = \sum_{i=1}^{n} a_{ij}\vec{\alpha}_i \text{ and } T(\vec{\beta}_j) = \sum_{i=1}^{n} b_{ij}\vec{\beta}_i,\ 1 \le j \le n,$$

and let C be the transition matrix: $\vec{\beta}_j = \sum_{i=1}^{n} c_{ij}\vec{\alpha}_i,\ 1 \le j \le n$. Then $B = C^{-1}AC$. If the bases $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are orthonormal, then the matrix C is unitary, and we have $B = C'AC$. Also, from the calculaton $a_{ij} = (\vec{\alpha}_i, T\vec{\alpha}_j)_V$ it follows that the matrix A is symmetric, that is, $A' = A$, if and only if the operator T is *symmetric*: $(T\vec{x}, \vec{y})_V = (\vec{x}, T\vec{y})_V$ for all $\vec{x}, \vec{y} \in V$.

EXERCISE 5. *Let T be a symmetric operator on V which is also non-negative: $(T\vec{x}, \vec{x}) \ge 0$ for all $\vec{x} \in V$. Show that*

$$|(T\vec{x}, \vec{y})|^2 \le (T\vec{x}, \vec{x})(T\vec{y}, \vec{y}), \quad \vec{x},\ \vec{y} \in V.$$

Show further that if $(T\vec{x}, \vec{x}) = 0$, then $T\vec{x} = \vec{0}$.

1.4. Eigenvectors and Diagonalization. Let V be a linear space of dimension n, $T : V \to V$ a linear map, $\boldsymbol{\alpha} = \{\vec{\alpha}_1,\ \vec{\alpha}_2,\ \dots\ \vec{\alpha}_n\}$ a basis for V, and let A be the α-matrix for T. Thus, we have $\mathbb{C}_\alpha T\vec{x} = A\mathbb{C}_\alpha(\vec{x})$ for each $\vec{x} \in V$. Recall that a vector $\vec{x} \ne \vec{0}$ is an eigenvector of T if $T\vec{x} = \lambda\vec{x}$, and the $\lambda \in \mathbb{R}$ is the corresponding eigenvalue. Similarly, a (column) vector $[X] \ne [0]$ is an eigenvector of the matrix A if $AX = \lambda X$.

LEMMA 2. *The vector $\vec{x} \ne \vec{0}$ is an eigenvector of T if and only if the column vector $[X] = \mathbb{C}_\alpha(\vec{x}) \ne [0]$ is an eigenvector of the matrix A.*

LEMMA 3. *Let $\vec{x}_j$ be eigenvectors of the* symmetric *T with corresponding eigenvalues λ_j for $j = 1, 2$. If $\lambda_1 \ne \lambda_2$, then $\vec{x}_1 \perp \vec{x}_2$.*

PROOF. From the calculation $\lambda_1(\vec{x}_1, \vec{x}_2) = (T\vec{x}_1, \vec{x}_2) = (\vec{x}_1, T\vec{x}_2) = \lambda_2(\vec{x}_1, \vec{x}_2)$, we have $(\lambda_1 - \lambda_2)(\vec{x}_1, \vec{x}_2) = 0$. □

Now we want to show that there *exist* eigenvectors of a *symmetric* T. The real-valued function $\vec{x} \mapsto (T\vec{x}, \vec{x})$ is continuous on V, and the sphere $S = \{\vec{x} \in V : \|\vec{x}\| = 1\}$ is a closed and bounded set. Therefore, this function has a maximum value somewhere on S. That is, there is an $\vec{x}_1 \in S$ with

$$(T\vec{x}, \vec{x}) \le (T\vec{x}_1, \vec{x}_1) \equiv \lambda_1, \quad \vec{x} \in S\,.$$

Now for any non-zero $\vec{x} \in V$ we have $\|\vec{x}\|^{-1}\vec{x} \in S$, so it follows that $\|\vec{x}\|^{-2}(T\vec{x}, \vec{x}) \le \lambda_1$, that is, $\lambda_1\|\vec{x}\|^2 - (T\vec{x}, \vec{x}) \ge 0$. This shows that we have

$$((\lambda_1 I - T)\vec{x}, \vec{x}) \ge 0,\ \vec{x} \in V, \quad ((\lambda_1 I - T)\vec{x}_1, \vec{x}_1) = 0\,.$$

It follows from Exercise 5 that $(\lambda_1 I - T)\vec{x}_1 = 0$, that is,

$$T\vec{x}_1 = \lambda_1\vec{x}_1\,.$$

This proves the following result.

LEMMA 4. *The symmetric linear map $T : V \to V$ has an eigenvector $\vec{x}_1$, and the corresponding eigenvalue λ_1 is the maximimum of $(T\vec{x}, \vec{x})$ over the unit sphere. This maximum is attained at $\vec{x}_1$.*

This is the first and most important step for the diagonalization theorem below. Another useful result is the following.

LEMMA 5. *Let $T : V \to V$ be a symmetric and linear map on the scalar-product space V. If W is a subspace of V, and if $T(W) \subset W$, then $T(W^\perp) \subset W^\perp$. That is, if a subspace W is* invariant *under T, then the orthogonal complement $W^\perp$ is also invariant under T.*

PROOF. If $w \in W,\ v \in W^\perp$, then $(Tv, w) = (v, Tw)$, so it follows that $Tv \in W^\perp$. □

Let $\vec{x_0} \in V$ and let W be a subspace of V. The real-valued function $w \mapsto \|\vec{x_0} - \vec{w}\|^2$ is quadratic and therefore has a minimum value on W, and this is achieved at some $\vec{w_0} \in W$. Thus, For each $\vec{w} \in W$, the map $t \mapsto \|(\vec{x_0} - \vec{w_0}) + t\,\vec{w}\|^2$ is minimal at $t = 0$, and there the derivative is zero, so we have

$$\frac{d}{dt}\|(\vec{x_0} - \vec{w_0}) + t\vec{w}\|^2|_{t=0} = 2(\vec{x_0} - \vec{w_0}, \vec{w}) = 0.$$

This proves the following.

LEMMA 6. *If W is a subspace of the finite-dimensional scalar-product space V, then for each $\vec{x_0} \in V$ there is a unique pair $\vec{w_0} \in W$ and $\vec{w_1} \in W^\perp$ for which $\vec{x_0} = \vec{w_0} + \vec{w_1}$.*

PROOF. We choose $\vec{w_0}$ as above and $\vec{x_0} = \vec{w_0} + (\vec{x_0} - \vec{w_0})$. □

THEOREM 4. *Let $T : V \to V$ be linear and symmetric, and $dim(V) = n$. Then there is an orthonormal basis $\boldsymbol{\beta} = \{\vec{\beta}_1, \vec{\beta}_2, \dots \vec{\beta}_n\}$ of eigenvectors of T, that is, $T(\vec{\beta}_j) = \lambda_j\vec{\beta}_j,\ 1 \le j \le n$, so the $\boldsymbol{\beta}$-matrix of T is the diagonal matrix of eigenvalues.*

PROOF. Lemma 4 shows there is an eigenpair $\vec{\beta}_1,\ \lambda_1$. We normalize $\vec{\beta}_1$ and define its linear span by $W_1 =< \vec{\beta}_1 >$. then W_1 is invariant under T, so by Lemma 5 so also is $W_1^\perp$. Now apply Lemma 4 to $T : W_1^\perp \to W_1^\perp$ to obtain an eigenpair $\vec{\beta}_2,\ \lambda_2$ with a normal $\vec{\beta}_2 \in W_1^\perp$. Then define the linear span $W_2 =< \vec{\beta}_1, \vec{\beta}_2 >$ and note that $T : W_2^\perp \to W_2^\perp$, so we can apply Lemma 4 again. Repeat this for n steps to obtain an orthonormal basis for V. □

COROLLARY 2. *Let A be an $n \times n$ symmetric matrix. Then there is a unitary matrix C for which $C'AC$ is the diagonal matrix of eigenvalues.*

1.4.1. *Computations.* Suppose that A is an $n \times n$ symmetric matrix, that C is unitary, and that $B = C'AC$ is diagonal. Partition C by columns and write $C = \big([X_1], [X_2], \dots, [X_n]\big)$. Then we have

$$AC = \big(A[X_1], A[X_2], \dots, A[X_n]\big),\ CB = \big(\lambda_1[X_1], \lambda_2[X_2], \dots, \lambda_n[X_n]\big)\ ,$$

so it follows that $AX_j = \lambda_j X_j$ for $j = 1, 2, \dots, n$, and X_j is the j-th eigenvector. These eigenvectors are an orthonormal basis for $\mathbb{R}^n$. This gives a method for constructing the orthogonal transition matrix which diagonalizes the symmetric matrix A.

Example. Let $A = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$. Then $|A - \lambda I| = 0$ for $\lambda_1 = 3$ with $X_1 = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and $\lambda_2 = -1$ with $X_2 = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$. The transition matrix is $C = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ and we check that $C'AC = \left(\begin{smallmatrix} 3 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Example. Let $A = \left(\begin{smallmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{smallmatrix}\right)$. Then $|A - \lambda I| = (4 - \lambda)(\lambda - 1)^2$. We get $\lambda_1 = 4$ with $X_1 = \frac{1}{\sqrt{3}}\left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right]$ and $\lambda_2 = \lambda_3 = 1$ with the two orthogonal eigenvectors $X_2 = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 \\ -1 \\ 0 \end{smallmatrix}\right]$ and $X_3 = \frac{1}{\sqrt{6}}\left[\begin{smallmatrix} 1 \\ 1 \\ -2 \end{smallmatrix}\right]$. The transition matrix is

$$C = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{pmatrix},$$

and we compute $C'AC = \left(\begin{smallmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Note that there are many choices for the eigenvectors X_2, X_3. For example, we could also have used the more 'symmetric' pair $X_2 = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 \\ 0 \\ -1 \end{smallmatrix}\right]$ and $X_3 = \frac{1}{\sqrt{6}}\left[\begin{smallmatrix} 1 \\ -2 \\ 1 \end{smallmatrix}\right]$.

2. Applications

In the following, A is an $n \times n$ real symmetric matix. Let C be the unitary matrix which diagonalizes A, partitioned into the (orthonormal) columns X_j, $1 \leq j \leq n$ of eigenvectors. Thus, we have

$$C'AC = diag(\lambda_j) \equiv D, \quad AX_j = \lambda_j X_j, \quad C = \big(X_1, X_2, \ldots, X_n\big)\,.$$

2.1. Stationary Systems. Consider an algebraic system of n equations in n unknowns,

$$AX + \lambda X = F \text{ in } \mathbb{R}^n$$

for a given number λ and column F. We first find the diagonalization of A as above. Then look for a solution in the form $X = \sum_{i=1}^n u_i X_i$. The unknown coefficients u_i must satisfy $\sum_{i=1}^n (\lambda_i + \lambda) u_i X_i = F$, so by the orthogonality of the eigenvectors this is equivalent to the 'separated' equations

$$(\lambda_i + \lambda) u_i = (F, X_i)_{\mathbb{R}^n},\ 1 \leq i \leq n.$$

Suppose that $\lambda_i + \lambda \neq 0$ for all i. Then there exists exactly one solution determined by $u_i = (\lambda_i + \lambda)^{-1}(F, X_i)_{\mathbb{R}^n}$, hence,

$$X = \sum_{i=1}^n (\lambda_i + \lambda)^{-1}(F, X_i)_{\mathbb{R}^n} X_i\,.$$

Note that we have $F = \sum_{=1}^n (F, X_i)_{\mathbb{R}^n} X_i$ and the solution is of the form $X = (A + \lambda)^{-1} F$ suggested by the *spectral form* of the diagonalization as $AX = \sum_{i=1}^n \lambda_i (X, X_i)_{\mathbb{R}^n} X_i$.

Suppose that $\lambda_J + \lambda = 0$ for some J with $1 \leq J \leq n$. From above, it follows that there exists a solution *only if* F satisfies the *orthogonality constraint*

$$(F, X_j)_{\mathbb{R}^n} = 0 \text{ for all } j \text{ such that } \lambda_j = \lambda_J\,.$$

Then there exist (many) solutions, and these are obtained as above in the form

$$X = \sum_{\lambda_i \neq \lambda_J} (\lambda_i + \lambda)^{-1} (F, X_i)_{\mathbb{R}^n} X_i + \sum_{\lambda_i = \lambda_J} u_i X_i$$

in which the coefficients u_i, $\lambda_i = \lambda_J$, are *arbitrary.* The number of these indices i for which $\lambda_i = \lambda_J$ is the *multiplicity* of the eigenvalue λ_J. Note that the constraint on the data F is that it must be orthogonal to the whole *eigenspace* $Ker(A - \lambda_J I)$.

2.2. Systems of Ordinary Differential Equations. Here we are concerned with the system of ordinary differential equations

$$\dot{\mathbf{u}}(t) + A\mathbf{u}(t) = \mathbf{F}(t) \text{ in } \mathbb{R}^n, \tag{1a}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \tag{1b}$$

with the matrix A given as above. Make a change of variable,

$$\mathbf{u}(t) = C\mathbf{v}(t)\,. \tag{2}$$

Since C is invertible with $C' = C^{-1}$, this defines the function $\mathbf{v}(\cdot)$, which is characterized by

$$\dot{\mathbf{v}}(t) + D\mathbf{v}(t) = C'\mathbf{F}(t) \text{ in } \mathbb{R}^n, \tag{3a}$$

$$\mathbf{v}(0) = C'\mathbf{u}_0 \tag{3b}$$

with the diagonal matrix $D = diag(\lambda_j)$ made up from the eigenvalues of A.

EXERCISE 6. *Carry out the details of the indicated change of variable.*

Now the system is completely separated into n separate ordinary differential equations, so we can solve it directly. The components must satisfy

$$\dot{v}_j(t) + \lambda_j v_j(t) = X_j \cdot \mathbf{F}(t) \text{ in } \mathbb{R},$$
$$v_j(0) = X_j \cdot \mathbf{u}_0$$

for $1 \leq j \leq n$. The solution is obtained directly as

$$v_j(t) = e^{-\lambda_j t} X_j \cdot \mathbf{u}_0 + \int_0^t e^{-\lambda_j (t-s)} X_j \cdot \mathbf{F}(s)\, ds\,.$$

We write this as

$$\mathbf{v}(t) = E(t)C'\mathbf{u}(0) + \int_0^t E(t-s)C'\mathbf{F}(s)\, ds\,,$$

with the diagonal matrix $E(t) \equiv diag(e^{-\lambda_j t})$. Finally, from the change of variable (2), we get the solution of the system (1) in the form

$$\mathbf{u}(t) = S(t)\mathbf{u}(0) + \int_0^t S(t-s)\mathbf{F}(s)\, ds\,, \tag{4}$$

in terms of the family of matrices $S(t) \equiv CE(t)C'$. Note that these formally play the role of the exponential matrix, $S(t) \cong e^{-At}$. The representation (4) is know as the *variation of parameters* formula for the solution of (1)

Next we take a somewhat different approach which uses directly the orthonormal basis $\{X_j\}$ of eigenvectors of the matrix A. These are the columns of C. First we represent the non-homogeneous term and initial condition from (1) as

$$\mathbf{F}(t) = \sum_{j=1}^{n} f_j(t) X_j\,, \quad f_j(t) = \mathbf{F}(t) \cdot X_j\,,$$

$$\mathbf{u}(0) = \sum_{j=1}^{n} u_j^0 X_j\,, \quad u_j^0 = \mathbf{u}(0) \cdot X_j\,.$$

Then we look for the solution of (1) in the form

$$\mathbf{u}(t) = \sum_{j=1}^{n} u_j(t) X_j\,.$$

A direct substitution into (1) gives us the equivalent system

$$\dot{u}_j(t) + \lambda_j u_j(t) = f_j(t) \text{ in } \mathbb{R},$$
$$u_j(0) = u_j^0\,.$$

This is just the system (3) obtained above. The solution of this separated system is given as before by

$$u_j(t) = e^{-\lambda_j t} {u_j}^0 + \int_0^t e^{-\lambda_j (t-s)} f_j(s)\, ds\,,$$

so we obtain our solution in the form

$$\mathbf{u}(t) = \sum_{j=1}^{n} e^{-\lambda_j t} (\mathbf{u}(0) \cdot X_j) X_j + \int_0^t \sum_{j=1}^{n} e^{-\lambda_j (t-s)} (\mathbf{F}(s) \cdot X_j) X_j\,.$$

In particular, we compare with the representation (4) to see

$$S(t)\mathbf{v} = \sum_{j=1}^{n} e^{-\lambda_j t} (\mathbf{v} \cdot X_j) X_j\,, \text{ where } \mathbf{v} = \sum_{j=1}^{n} (\mathbf{v} \cdot X_j) X_j\,.$$

EXERCISE 7. *Show that*

$$C\mathbf{v}C' = \sum_{j=1}^{n} (\mathbf{v} \cdot X_j) X_j \textit{ for any } \mathbf{v} \in \mathbb{R}^n\,.$$

EXERCISE 8. *How would you make a function $f(A)$ of the symmetric matrix A? Explain your answer. Illustrate this by constructing* $\cos(A)$ *and* $\sin(tA)$.

2.3. Implicit Differential Equations. For many problems the system of ordinary differential equations occurs most naturally in the form

$$B\dot{\mathbf{u}}(t) + A\mathbf{u}(t) = \mathbf{F}(t) \text{ in } \mathbb{R}^n, \tag{5a}$$

$$B\mathbf{u}(0) = B\mathbf{u}_0, \tag{5b}$$

in which both A and B are symmetric matrices. The matrix B differs from the identity, for instance, whenever the model is built with non-equal lengths of successive increments in the partition of the rod. This also occurs if the properties of the material, such as density or specific heat, vary from point to point, for then they change with the index j.

We shall illustrate the approach with the special case in which $B = diag(\mu_j)$ is the indicated *diagonal* matrix with every element $\mu_j > 0,\ 1 \le j \le n$. (This already covers many important situations.) Using a basis $\{X_j\}$ for the space, we look for the solution of (5) in the form

$$\mathbf{u}(t) = \sum_{j=1}^{n} u_j(t) X_j \,. \tag{6}$$

This does not simplify the computations substantially, unless the $\{X_j\}$ are solutions of the *generalized eigenvalue problem*

$$AX = \lambda BX \,. \tag{7}$$

In order to solve this, we denote by $B^{\frac{1}{2}}$ the corresponding diagonal matrix

$$B^{\frac{1}{2}} = diag(\mu_j^{\frac{1}{2}}) \,.$$

The matrix $B^{-\frac{1}{2}} A B^{-\frac{1}{2}}$ is symmetric, so this composite matrix can be diagonalized as before: there is an orthonormal basis of vectors $\{Y_j\}$ for which

$$B^{-\frac{1}{2}} A B^{-\frac{1}{2}} Y_j = \lambda_j Y_j, \quad 1 \le j \le n.$$

It follows that the family of vectors $X_j = B^{-\frac{1}{2}} Y_j$ satisfies

$$AX_j = \lambda_j B X_j, \quad 1 \le j \le n,$$

so they comprise the solution to the generalized eigenvalue problem (7). Note that the $\{Y_j\}$ are an orthonormal basis of $\mathbb{R}^n$, but we know only that $\{X_j\}$ is a *basis*. However, we observe $(Y_i, Y_j) = (B^{\frac{1}{2}} X_i, B^{\frac{1}{2}} X_j)$, and this suggests that the scalar-products $(u, v)_B \equiv (B^{\frac{1}{2}} u, B^{\frac{1}{2}} v)$ and $(u, v)_{B'} \equiv (B^{-\frac{1}{2}} u, B^{-\frac{1}{2}} v)$ could play a substantial role for the generalized initial value problem (5).

LEMMA 7. *The basis $\{X_j\}$ is orthonormal with respect to the scalar-product $(u, v)_B$, and the basis $\{BX_j\}$ is orthonormal with respect to the scalar-product $(u, v)_{B'}$.*

In order to substitute (6) into (5), we first compute

$$Au(t) = \sum_{j=1}^{n} u_j(t) \lambda_j B X_j \,,$$

$$Bu(t) = \sum_{j=1}^{n} u_j(t) B X_j \,,$$

From the orthonormal expansion

$$F(t) = \sum_{j=1}^{n} (F(t), BX_j)_{B'} B X_j$$

we find the coefficients in (6) must satisfy

$$\dot{u}_j(t) + \lambda_j u_j(t) = (F(t), BX_j)_{B'} = X_j \cdot F(t) \text{ in } \mathbb{R},$$
$$u_j(0) = (X_j, \mathbf{u}_0)_B = X_j \cdot B\mathbf{u}_0\,.$$

The solution is easily obtained as before by

$$u_j(t) = e^{-\lambda_j t} X_j \cdot B\mathbf{u}_0 + \int_0^t e^{-\lambda_j(t-s)} X_j \cdot \mathbf{F}(s)\, ds\,.$$

Now we try another approach. The change of variable

$$\mathbf{v}(t) = B^{\frac{1}{2}}\mathbf{u}(t) \tag{8}$$

converts (5) to the form

$$\dot{\mathbf{v}}(t) + B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\mathbf{v}(t) = B^{-\frac{1}{2}}\mathbf{F}(t)\,, \tag{9a}$$

$$\mathbf{v}(0) = B^{\frac{1}{2}}\mathbf{u}_0. \tag{9b}$$

We try the expansion

$$\mathbf{v}(t) = \sum_{j=1}^{n} v_j(t) Y_j \tag{10}$$

in terms of the original orthonormal system $\{Y_j\}$. This leads directly to the system

$$\dot{v}_j(t) + \lambda_j v_j(t) = (B^{-\frac{1}{2}}F(t), Y_j) = F(t) \cdot X_j \text{ in } \mathbb{R},$$
$$v_j(0) = B^{\frac{1}{2}}\mathbf{u}_0 \cdot Y_j = B\mathbf{u}_0 \cdot X_j\,.$$

Thus, we have the same coefficient functions as before.

Finally, we indicate that everything above can be extended to the more general case of a *symmetric and non-singular* matrix B. Everything depends on the construction of the non-singular symmetric square root matrix, $B^{\frac{1}{2}}$. Let C be the unitary matrix which diagonalizes B, so we have

$$C'BC = diag(\mu_j) \equiv D.$$

Since B is non-singular, we have each $\mu_j \neq 0$. We easily take the square root of the diagonal matrix D, and then we define $B^{\frac{1}{2}} \equiv CD^{\frac{1}{2}}C'$. It is easy to check that this defines a symmetric square root of B as desired. For example, we calculate the square

$$B^{\frac{1}{2}}B^{\frac{1}{2}} = CD^{\frac{1}{2}}C'CD^{\frac{1}{2}}C' = CD^{\frac{1}{2}}D^{\frac{1}{2}}C' = CDC' = B\,,$$

and the transpose is given by

$$(B^{\frac{1}{2}})' = (CD^{\frac{1}{2}}C')' = C(D^{\frac{1}{2}})'C' = B^{\frac{1}{2}}\,.$$

2.4. Second Order Implicit Equations. We briefly indicate the procedures for the systems that arise from vibration problems. We consider also the effects of resistance and inertia. We shall assume that both A and B are symmetric matrices and that B is non-singular. Thus we can solve the generalized eigenvalue problem (7) as above.

Standard wave equation for vibration ...

$$B\ddot{\mathbf{u}}(t) + A\mathbf{u}(t) = \mathbf{F}(t) \text{ in } \mathbb{R}^n, \tag{11a}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad B\dot{\mathbf{u}}(0) = B\mathbf{u}_1. \tag{11b}$$

Damped wave equation ...

$$B\ddot{\mathbf{u}}(t) + \varepsilon B\dot{\mathbf{u}}(t) + A\mathbf{u}(t) = \mathbf{F}(t) \text{ in } \mathbb{R}^n, \tag{12a}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad B\dot{\mathbf{u}}(0) = B\mathbf{u}_1. \tag{12b}$$

Viscous wave equation ...

$$B\ddot{\mathbf{u}}(t) + \varepsilon A\dot{\mathbf{u}}(t) + A\mathbf{u}(t) = \mathbf{F}(t) \text{ in } \mathbb{R}^n, \tag{13a}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad B\dot{\mathbf{u}}(0) = B\mathbf{u}_1. \tag{13b}$$

Inertial wave equation ...

$$B\ddot{\mathbf{u}}(t) + \varepsilon A\ddot{\mathbf{u}}(t) + A\mathbf{u}(t) = \mathbf{F}(t) \text{ in } \mathbb{R}^n, \tag{14a}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad B\dot{\mathbf{u}}(0) = B\mathbf{u}_1. \tag{14b}$$