Continuous Models of Diffusion and Vibration

1. Heat Conduction in an Interval

We shall describe the diffusion of heat energy through a long thin rod G with uniform cross section S. As before, we identify G with the open interval (a, b), and we assume the rod is perfectly insulated along its length. Let $u(x, t)$ denote the temperature within the rod at a point $x \in G$ and at time $t > 0$. Our previous experience shows that the *heat flux* $q(x, t)$ at the point x, *i.e.*, the flow rate to the right per unit area, is proportional to the temperature gradient $\frac{\partial u}{\partial x}$. The precise statement of this experimental fact is *Fourier's law* of heat conduction,

$$q(x, t) = -k(x) \frac{\partial u}{\partial x}(x, t), \tag{1}$$

and this equation defines the *thermal conductivity* $k(x)$ of the material at the point $x \in (a, b)$. Since heat flows in the direction of decreasing temperature, we see again that the minus sign is appropriate.

The amount of heat stored in the rod within a section $[x, x + h]$ with $h > 0$ is given by

$$\int_x^{x+h} c(s)\rho(s)Su(s, t)\, ds,$$

where $c(\cdot)$ is the *specific heat* of the material, and $\rho(\cdot)$ is the volume-distributed *density*. The specific heat provides a measure of the amount of heat energy required to raise the temperature of a unit mass of the material by a degree.

Now, equating the rate at which heat is stored within the section to the rate at which heat flows into the section plus the rate at which heat is generated within this section, we arrive at the *conservation of energy* equation for the section $[x, x + h]$

$$\frac{\partial}{\partial t} \int_x^{x+h} c(s)\rho(s)Su(s, t)\, ds = S\big(q(x, t) - q(x + h, t)\big) + \int_x^{x+h} f(s, t)S\, ds$$

where $f(x, t)$ represents the rate at which heat is generated per unit volume. This heat generation term is assumed to be a known function of space and time. Dividing this by Sh and letting $h \to 0$ yields the *conservation of energy* equation

$$c(x)\rho(x) \frac{\partial u}{\partial t}(x, t) + \frac{\partial q}{\partial x}(x, t) = f(x, t). \tag{2}$$

Finally, by substituting the Fourier law (1) into the energy conservation law (2), we obtain the one–dimensional heat conduction equation

$$c(x)\rho(x)\frac{\partial u}{\partial t}(x,t) - \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}(x,t)\right) = f(x,t). \tag{3}$$

This is also known as the *diffusion equation.*

Remarks:

i. Equation (3) represents the continuous analog of the discrete heat equation developed in Chapter 1, which we list below for reference,

$$c\rho\dot{u}_j(t) - \frac{k}{h^2}(u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)) = f_j(t), \quad 1 \le j \le N.$$

ii. Given an *isotropic* material specified by a bounded domain $G \subset \mathbb{R}^n$, and letting $u(x,t)$ denote the temperature within G at a point x and at time $t > 0$, the corresponding n–dimensional heat equation has the form

$$c(x)\rho(x)\frac{\partial u}{\partial t}(x,t) - \vec{\nabla}\cdot\left(k(x)\vec{\nabla}u(x,t)\right) = f(x,t).$$

If we assume the material properties $c(\cdot), \rho(\cdot)$, and $k(\cdot)$ are constants, then equation (3) may be written in the form

$$\frac{\partial u}{\partial t}(x,t) - \alpha^2\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{1}{c\rho}f(x,t), \qquad x \in G,\ t > 0, \tag{4}$$

where $\alpha^2 \equiv \frac{k}{c\rho}$ is called the *thermal diffusivity* of the material and is a measure of the rate of change of temperature of the material. For example, if the material is made out of a substance with a high thermal conductivity and low heat capacity $c\rho$, then the material will react very quickly to transient external conditions.

1.1. Initial and Boundary conditions. Since the heat equation is second–order in space and first–order in time, one may expect that in order to have a well–posed problem, two boundary conditions and one initial condition should be specified. We shall see that this is true here.

We want to find a solution of (3) which satisfies an *initial condition* of the form

$$u(x,0) = u_0(x), \quad a < x < b,$$

where $u_0(\cdot)$ is given. Following the outline in Chapter 1, we describe some examples of appropriate *boundary conditions.* Each of these will be illustrated with a condition at the right end point, $x = b$, and we note that another such condition will be prescribed at the left end point, $x = a$.

1. Dirichlet Boundary Conditions

One can specify the value of the temperature at an end point:

$$u(b,t) = d_b(t), \quad t > 0\,,$$

where $d_b(\cdot)$ is given. This type of boundary condition describes *perfect contact* with the boundary value, and it arises when the value of the end point temperature is known (usually from a direct measurement). Such a condition arises when one sets the boundary temperature to a prescribed value, for example, $d_b(t) = 0$ when the end point is submerged in ice-water.

2. Neumann Boundary Conditions

One can specify the heat flux into the rod at an end point:

$$k\frac{\partial u}{\partial x}(b,t)S = f_b(t), \quad t > 0\,,$$

where $f_b(\cdot)$ is given. This type of boundary condition corresponds to a known heat *source* $f_b(\cdot)$ at the end. The homogeneous case $f_b(t) = 0$ occurs at an *insulated* end point.

3. Robin Boundary Conditions

The heat flux is assumed to be lost through the end at a rate proportional to the difference between the inside and outside temperatures:

$$k\frac{\partial u}{\partial x}(b,t)S + k_b(u(b,t) - d_b(t))S = f_b(t), \quad t > 0\,.$$

Such a boundary condition arises from a *partially insulated* end point, and it corresponds to Newton's law of cooling at the end point $x = b$. This is just the discrete form of Fourier's law. Here, both $d_b(\cdot)$ and $f_b(\cdot)$ are given functions. The first is the outside temperature and the second is a heat source concentrated on the end. Note that the first two boundary conditions can be formally obtained as extreme cases of the Robin boundary condition; that is, as $k_b \to \infty$, $u(b,t) \to d_b(t)$, which formally yields the Dirichlet boundary condition, and as $k_b \to 0$ we similarly obtain the Neumann boundary condition. Thus, this third type of boundary condition is an interpolation between the first two types for intermediate values of k_b.

4. Dynamic Boundary Conditions

As in the discrete case, another type of boundary condition arises when we assume the end of the rod has an *effective specific heat* given by $c_0 > 0$: this is a *concentrated capacity* at the end. For example, the end of the rod can be capped by a piece of material whose conductivity is very high, so the entire piece is essentally at the same temperature, or the end is submerged in an insulated container of well-stirred water. If we assume the heat energy is supplied to the concentrated capacity by the flux from the interior of the rod and supplemented by a given heat source located at that end point, $f_b(t)$, then we are led to the boundary condition

$$c_0 u_t(b,t) + k u_x(b,t)S = f_b(t), \qquad t > 0.$$

Implicit in this boundary condition is the assumption that we have "perfect" contact between the end of the rod and the concentrated capacity. If we permit some insulation between the end point and the concentrated capacity, then the temperature $u_c(\cdot)$ of the concentrated capacity is different from that of the endpoint of the rod, and the boundary condition is

$$\begin{cases} c_0\frac{du_c(t)}{dt} + k_b(u_c(t) - u(b,t)) = f_b(t), & t > 0, \\ k\frac{\partial u}{\partial x}(b,t) + k_b(u(b,t) - u_c(t)) = 0. \end{cases}$$

Here we have an extra condition, but we have also introduced an additional unknown, so this pair of equations should be regarded as a single constraint. Note that, as $k_b \to \infty$,

$u(b,t) \to u_c(t)$, and this partially insulated dynamic boundary condition formally yields the "perfect" contact dynamic boundary condition.

Each of the preceding boundary conditions must be supplemented with an additional boundary condition at the other endpoint, $x = a$, and the two boundary conditions need not be of the same type. These two boundary constraints together with the initial condition on a solution of (3) will comprise a well–posed problem.

5. Nonlocal Boundary Conditions

If the rod is bent around and the ends are joined to form a large ring, then at the endpoints we must match the temperature and the flux:

$$u(a,t) = u(b,t), \quad t > 0,$$
$$k\frac{\partial u}{\partial x}(a,t)S = k\frac{\partial u}{\partial x}(b,t)S\,.$$

These are called *periodic* boundary conditions. Note that again we have a total of two boundary constraints for the problem, but here the constraints depend on the solution at more than a single point.

Another example arises in the case that we submerge the entire rod into a bath of well-stirred water and assume the end points of the rod are in perfect contact with the water. Then we obtain the *dynamic and nonlocal* boundary conditions

$$u(a,t) = u(b,t) = u_c(t), \quad t > 0,$$
$$c_0\dot{u}_c(t) + k(b)\frac{\partial u}{\partial x}(b,t) - k(a)\frac{\partial u}{\partial x}(a,t) = 0,$$

where the common value of the endpoint temperatures $u_c(t)$ is unknown. Note that we have *not* introduced an additional unknown, so these two equations provide the two boundary conditions.

Exercise 1. *Assume the rod G is submerged in a perfectly insulated container of well-stirred water. The rod is partially insulated along its length, so there is some limited heat exchange with the surrounding water along the length, $a < x < b$. Also, assume the rod is in perfect contact with the water at the end points. Find an initial-boundary-value problem which models this situation.*

2. The Eigenfunction Expansion Method

We shall illustrate the method of *separation of variables* by obtaining the eigenfunction expansion of the solution of an initial-boundary-value problem with Dirichlet boundary conditions. The same method works for the other boundary conditions.

Example 1. *Suppose the rod $G = (0,\ell)$ is perfectly insulated along its length and made of an isotropic material with thermal diffusivity α^2. Assuming no internal heat* *sources or sinks,* i.e., $f(x,t) = 0$, *suppose both ends of the rod are held at a fixed temperature of zero and the initial temperature distribution is given by* $u_0(x)$. *The initial-boundary-value problem for this scenario is*

$$\frac{\partial u}{\partial t}(x,t) = \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t), \quad 0 < x < \ell, \quad t > 0, \tag{5a}$$

$$u(0,t) = 0, \; u(\ell,t) = 0, \qquad t > 0, \tag{5b}$$

$$u(x,0) = u_0(x), \quad 0 < x < \ell. \tag{5c}$$

Exercise 2. *Let* $u(x,t)$ *be a solution of the initial-boundary-value problem and show that*

$$\frac{d}{dt}\int_0^\ell u^2(x,t)\,dx \leq 0.$$

Show that this implies there is at most one *solution of the problem.*

We begin by looking for a non-null solution of the form

$$u(x,t) = X(x)T(t). \tag{6}$$

Substituting (6) into (5a) and dividing by u yields

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}, \qquad \text{for all } x \text{ and } t.$$

Note that the left side of this last equation is exclusively a function of t, while the right side of is exclusively a function of x. The only way this equation can hold for all values of x and t is for each side to equal a common constant. Denoting this constant by $-\lambda$ leads to the pair of ordinary differential equations

$$\begin{aligned} T'(t) \;&+\; \lambda\alpha^2 T(t) = 0, \quad t > 0, \\ X''(x) \;&+\; \lambda X(x) = 0, \quad 0 < x < \ell. \end{aligned}$$

The boundary conditions given in (5b) imply that $X(0) = X(\ell) = 0$.

Note that if $X(\cdot)$ and $T(\cdot)$ are solutions of these respective equations, then it follows directly that their product is a solution of (5a). The first of these ordinary differential equations has the solution $T(t) = e^{-\lambda\alpha^2 t}$. Thus, it remains to find a non-null solution of the boundary-value problem

$$\begin{cases} X''(x) + \lambda X(x) = 0, \quad 0 < x < \ell, \\ X(0) = 0, \; X(\ell) = 0. \end{cases} \tag{7}$$

This is a *"regular" Sturm-Liouville boundary-value problem* and we will see later that such problems have very special properties. Since this is a linear equation with constant coefficients, we can explicitly write down all possible solutions, and they depend on the sign of λ. First we check that for the cases of $\lambda < 0$ and $\lambda = 0$, the only solution of the boundary-value problem (7) is the null solution. For the case of $\lambda > 0$, we get the general solution of the differential equation in the form

$$X(x) = C_1 \sin(\sqrt{\lambda}x) + C_2 \cos(\sqrt{\lambda}x),$$

and then from the boundary conditions we see that necessarily

$$C_2 = 0, \quad \text{and} \quad C_1 \sin(\ell\sqrt{\lambda}) = 0,$$

respectively. Since $\sin(\ell\sqrt{\lambda}) = 0$ has solutions $\lambda_n = (n\pi/\ell)^2$, this does *not* force $C_1 = 0$. These specific values for λ are called the *eigenvalues* of the regular Sturm-Liouville problem (7), and the solutions to (7), namely, multiples of

$$X_n(x) = \sin(\sqrt{\lambda_n}x),$$

are the corresponding *eigenfunctions*. If we combine these with the corresponding time-dependent solutions $T_n(t) = e^{-\lambda_n \alpha^2 t}$, we obtain solutions $e^{-\lambda_n \alpha^2 t} \sin(\sqrt{\lambda_n}x)$ of (5a) and (5b). From the superposition principle we obtain a large class of solutions of (5a) and (5b) in the form

$$u(x,t) = \sum_{n=1}^{N} A_n e^{-\lambda_n \alpha^2 t} X_n(x), \tag{8}$$

for any integer N. We check directly that (8) satisfies (5a) and (5b) for any choice of the coefficients $\{A_n\}$. In order to satisfy the initial condition (5c), the coefficients must be chosen to satisfy

$$u_0(x) = \sum_{n=1}^{N} A_n X_n(x). \tag{9}$$

Now this is a *severe* restriction on the initial data, but we shall find that we can go to a corresponding series with $N = +\infty$, and then there is essentially *no restriction* on the initial data! This will follow from the observation that the corresponding coefficients in (8) have exponentially decaying factors that make the series converge extremely rapidly for $t > 0$.

Let's take a preliminary look at the boundary-value problem (7). We have denoted its non-null solutions by $X_n(\cdot),\ \lambda_n,\ n \geq 1$ First we compute

$$\begin{aligned}(\lambda_m - \lambda_n) \int_0^\ell X_m(x) X_n(x)\, dx &= -\int_0^\ell \big(X_m''(x) X_n(x) - X_m(x) X_n''(x)\big)\, dx \\ &= -\int_0^\ell \frac{d}{dx}\big(X_m'(x) X_n(x) - X_m(x) X_n'(x)\big)\, dx = 0\,.\end{aligned}$$

Since $\lambda_m \neq \lambda_n$ for $m \neq n$, this shows that the eigenfunctions $X_n(\cdot)$ are orthogonal with respect to the scalar-product $(\cdot,\cdot) \equiv \int_0^\ell (\cdot,\cdot)\, dx$ on the linear space of continuous functions on the interval $[0,\ell]$. By replacing each such $X_n(\cdot)$ by the function obtained by dividing it by the corresponding *norm* $\|X_n(\cdot)\| = (X_n(\cdot), X_n(\cdot))^{\frac{1}{2}}$, we obtain an *orthonormal* set of functions in that space. That is, we have

$$(X_m(\cdot), X_n(\cdot)) = \delta_{mn} \text{ for } m,n \geq 1\,,$$

where we have scaled the eigenfunctions to get the *normalized eigenfunctions*

$$X_n(x) = \sqrt{\frac{2}{\ell}} \sin(\frac{n\pi}{\ell}x).$$

Now it is clear how to choose the coefficients A_n in (9): take the scalar product of that equation with $X_m(\cdot)$ to obtain

$$(u_0(\cdot), X_m(\cdot)) = A_m, \quad m \geq 1\,.$$

Thus we obtain

$$u(x,t) = \sum_{n=1}^{N} e^{-\lambda_n \alpha^2 t}(u_0(\cdot), X_n(\cdot))X_n(x), \tag{10}$$

when $u_0(\cdot)$ is appropriately restricted. We shall see below that we can go to the corresponding series with $N = +\infty$ as indicated with essentially no restriction on $u_0(\cdot)$.

EXERCISE 3. *Given $m, n \in \mathbb{N}$, show directly that*

(a) $\displaystyle\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = \begin{cases} 0, & m \neq n, \\ \frac{1}{2}, & m = n. \end{cases}$

(b) $\displaystyle\int_0^1 \cos(m\pi x)\cos(n\pi x)\,dx = \begin{cases} 0, & m \neq n, \\ \frac{1}{2}, & m = n. \end{cases}$

(c) $\displaystyle\int_0^1 \sin(m\pi x)\cos(n\pi x)\,dx = 0\,.$

The technique used in Example 1 is called the *method of separation of variables.* Since it depends on superposition, this technique is appropriate for solving linear initial-boundary-value problems with homogeneous boundary conditions. It can easily be modified to solve initial-boundary-value problems that contain *constant* non-homogeneous boundary conditions. In particular, for the problem

$$\begin{cases} u_t(x,t) = \alpha^2 u_{xx}(x,t), & 0 < x < \ell, t > 0, \\ u(0,t) = d_0,\ u(\ell,t) = d_\ell, & t > 0, \\ u(x,0) = u_0(x), & 0 < x < \ell, \end{cases} \tag{11}$$

define $w(x,t) = u(x,t) - \left(d_0\frac{\ell - x}{\ell} + d_\ell\frac{x}{\ell}\right)$ and transform the above problem into an equivalent initial-boundary-value problem for $w(x,t)$. Note that $w(0,t) = w(\ell,t) = 0$, and so now we have a problem with homogeneous boundary conditions as in Example 1.

EXERCISE 4. *Compute the solution of* (11) *for the case of* $u_0(\cdot) = 0$, $d_0 = 0$ *and* $d_\ell = 1$.

However, if the boundary values are given by a pair of time dependent functions, $d_0(t)$, $d_\ell(t)$, then we are led to a non-homogeneous partial differential equation. More generally, we can start with a *non-homogeneous* initial-boundary-value problem of the form

$$\begin{aligned} u_t(x,t) = \alpha^2 u_{xx}(x,t) + f(x,t), &\qquad 0 < x < \ell, \quad t > 0, \\ u(0,t) = d_0(t),\ u(\ell,t) = d_\ell(t), &\qquad t > 0, \\ u(x,0) = u_0(x), &\qquad 0 < x < \ell\,, \end{aligned}$$

and then define $w(x,t) = u(x,t) - \left(d_0(t)\frac{\ell-x}{\ell} + d_\ell(t)\frac{x}{\ell}\right)$ to transform the above problem into an equivalent initial-boundary-value problem of the form

$$\begin{aligned} w_t(x,t) &= \alpha^2 w_{xx}(x,t) + \tilde{f}(x,t), && 0 < x < \ell, \quad t > 0, \\ w(0,t) &= 0, \; w(\ell,t) = 0, && t > 0, \\ w(x,0) &= w_0(x), && 0 < x < \ell, \end{aligned}$$

where $w_0(x) = u_0(x) - \left(d_\ell(0)\frac{x}{\ell} + d_0(0)\frac{\ell-x}{\ell}\right)$ and $\tilde{f}(x,t) = f(x,t) - \left(d'_\ell(t)\frac{x}{\ell} + d'_0(t)\frac{\ell-x}{\ell}\right)$. Thus, by such a change of variable, we can always reduce the initial-boundary-value problem to the form with homogeneous boundary conditions.

EXAMPLE 2. *Find the* eigenfunction expansion *of the solution of the initial-boundary-value problem*

$$u_t(x,t) = \alpha^2 u_{xx}(x,t) + f(x,t), \qquad 0 < x < \ell, \quad t > 0, \tag{14a}$$

$$u(0,t) = 0, \; u(\ell,t) = 0, \qquad t > 0, \tag{14b}$$

$$u(x,0) = u_0(x), \qquad 0 < x < \ell, \tag{14c}$$

with non-homogeneous partial differential equation and *homogeneous boundary conditions.* Recall that for $f(x,t) = 0$, the solution to this case was given by equation (8). For the problem with a non-homogeneous partial differential equation (14a), we look for the solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x). \tag{15}$$

If we assume that both $f(x,t)$ and $u_0(x)$ also have eigenfunction expansions given by

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) X_n(x) \quad \text{and} \quad u_0(x) = \sum_{n=1}^{\infty} u_0^n X_n(x),$$

respectively, where

$$f_n(t) \equiv \int_0^\ell f(\zeta,t) X_n(\zeta)\, d\zeta \quad \text{and} \quad u_0^n \equiv \int_0^\ell u_0(\zeta) X_n(\zeta)\, d\zeta,$$

then substituting each of these expansions into equation (14a) yields

$$\sum_{n=1}^{\infty} \left[\dot{u}_n(t) + \lambda_n \alpha^2 u_n(t)\right] X_n(x) = \sum_{n=1}^{\infty} f_n(t) X_n(x).$$

By equating the coefficients of the series given in this last equation, we are led to the initial-value problems

$$\dot{u}_n(t) + \lambda_n \alpha^2 u_n(t) = f_n(t), \qquad t > 0 \tag{16a}$$

$$u_n(0) = u_0^n. \tag{16b}$$

The solution to (16) is

$$u_n(t) = e^{-\lambda_n \alpha^2 t} u_0^n + \int_0^t e^{-\lambda_n \alpha^2 (t-\tau)} f_n(\tau)\, d\tau$$

Now, if we use this in (15), we find that the solution to our non-homogeneous initial-boundary-value problem (14) is

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\lambda_n \alpha^2 t} u_0^n X_n(x) + \sum_{n=1}^{\infty} \int_0^t e^{-\lambda_n \alpha^2 (t-\tau)} f_n(\tau) X_n(x)\, d\tau, \tag{17}$$

with the coefficients u_0^n and $f_n(\cdot)$ computed as above.

This formula has the tyical structure of the solution of an initial-value problem. That is, if we use the first term in this representation to define a family of operators $E(t),\ t \geq 0$, on the space of functions on the interval $[0, \ell]$ by

$$[E(t)u_0](x) = \sum_{n=1}^{\infty} e^{-\lambda_n \alpha^2 t} \big(u_0, X_n\big) X_n(x)\,, \tag{18}$$

then the solution (17) takes the form

$$u(\cdot, t) = E(t)u_0 + \int_0^t E(t-\tau) f(\tau)\, d\tau\,. \tag{19}$$

In particular, the operator $E(t)$ in this specific case is an *integral operator*

$$\begin{aligned}
[E(t)u_0](x) &= \sum_{n=1}^{\infty} e^{-\lambda_n \alpha^2 t} \int_0^\ell (u_0(s)\ \sin(\frac{n\pi}{\ell} s))\, ds \frac{2}{\ell} \sin(\frac{n\pi}{\ell} x) \\
&= \int_0^\ell \frac{2}{\ell} (\sum_{n=1}^{\infty} e^{-\lambda_n \alpha^2 t} \sin(\frac{n\pi}{\ell} s) \sin(\frac{n\pi}{\ell} x)\big) u_0(s)\, ds \\
&= \int_0^\ell G(x,s,t) u_0(s)\, ds
\end{aligned}$$

for which the kernel

$$G(x,s,t) = \frac{2}{\ell} (\sum_{n=1}^{\infty} e^{-\lambda_n \alpha^2 t} \sin(\frac{n\pi}{\ell} s) \sin(\frac{n\pi}{\ell} x)\big)$$

is the *Green's function* for the problem.

Exercise 5. *Compute the solution of* (5a) *with initial condition* $u(x,0) = 0$ *and the boundary conditions* $u(0,t) = 0$ *and* $u(\ell, t) = t$.

Example 3. *For the situation of Example 1, suppose the left end of the rod is insulated while the right end has a heat loss given by* $-u_x(\ell, t) = k_\ell\, u(\ell, t)$ *where* $k_\ell \geq 0$. *The initial-boundary-value problem for this situation is given by*

$$u_t(x,t) = \alpha^2 u_{xx}(x,t), \qquad 0 < x < \ell, \quad t > 0, \tag{20a}$$

$$u_x(0,t) = 0,\ \ k_\ell\, u(\ell,t) + u_x(\ell,t) = 0, \qquad t > 0, \tag{20b}$$

$$u(x,0) = u_0(x), \qquad 0 < x < \ell. \tag{20c}$$

We seek a solution in the form

$$u(x,t) = X(x)T(t),$$

where the boundary conditions imply $X'(0) = 0$ and $k_\ell X(\ell) + X'(\ell) = 0$. The method of separation of variables leads us to the time-dependent problem

$$T'(t) + \lambda\alpha^2 T(t) = 0, \quad t > 0,$$

and the boundary-value problem

$$X''(x) + \lambda X(x) = 0, \qquad 0 < x < \ell, \tag{21a}$$

$$X'(0) = 0,\ k_\ell X(\ell) + X'(\ell) = 0. \tag{21b}$$

For values of $\lambda < 0$, we find that there are no non-null solutions. For the case of $\lambda = 0$, only if $k_\ell = 0$ do we get a non-zero solution, and this is $X_0(x) = (\frac{1}{\ell})^{\frac{1}{2}}$ with $\lambda_0 = 0$. But for $\lambda > 0$, the general solution to (21a) is

$$X(x) = C_1 \sin(\lambda^{\frac{1}{2}} x) + C_2 \cos(\lambda^{\frac{1}{2}} x),$$

and the boundary conditions (21b) imply that

$$C_1 = 0, \quad \text{and} \quad k_\ell C_2 \cos(\ell\lambda^{\frac{1}{2}}) - C_2 \lambda^{\frac{1}{2}} \sin(\ell\lambda^{\frac{1}{2}}) = 0.$$

Since we are only interested in non-null solutions, the latter equation is equivalent to solving $\lambda^{\frac{1}{2}} \tan(\ell\lambda^{\frac{1}{2}}) = k_\ell$. That is, we have $\tan(\ell\lambda^{\frac{1}{2}}) = \frac{k_\ell}{\lambda^{\frac{1}{2}}}$. The tangent function is π-periodic, so we obtain a sequence λ_n, $n \geq 0$, of solutions to this equation, and the corresponding eigenfunctions are given by

$$X_n(x) = \left(\frac{2}{\ell}\right)^{\frac{1}{2}} \cos(\lambda_n^{\frac{1}{2}} x), \quad n \geq 1.$$

Note that the eigenvalues belong to intervals determined by $\ell(\lambda_n)^{\frac{1}{2}} \in [n\pi,\ n\pi + \frac{1}{2}\pi]$, $n \geq 0$, and that for small $\frac{k_\ell}{(\lambda_n)^{\frac{1}{2}}}$ we have

$$\lambda_n \approx \left(\frac{n\pi}{\ell}\right)^2,$$

so the eigenvalues are asymptotically close to those of the preceding example. Combining these results with the time-dependent solutions $T_n(t) = e^{-\lambda_n \alpha^2 t}$ and using the orthogonality of the eigenfunctions, we find solutions of (20a) in the form

$$u(x,t) = \sum_{n=0}^{\infty} (u_0(\cdot), X_n(\cdot)) e^{-\lambda_n \alpha^2 t} X_n(x),$$

and it is understood that the sum starts at $n = 1$ if $k_\ell > 0$.

Exercise 6. *Compute the solution of* (5a) *with initial condition* $u(\cdot, 0) = u_0(\cdot)$ *and the boundary conditions* $u(0,t) = 0$ *and* $u_x(\ell, t) = 0$.

Exercise 7. *Consider the problem*

$$u''(x) + u'(x) = f(x)$$

$$u'(0) = u(0) = \frac{1}{2}[u'(\ell) + u(\ell)],$$

where $f(x)$ *is a given function.*

(a) *Is the solution unique?*

(b) *Does a solution necessarily exist, or is there a condition that* $f(x)$ *must satisfy for existence?*

EXERCISE 8. *Let the rod* G *be defined over the interval* $(0,1)$*, and suppose its lateral surface is perfectly insulated along its length. Furthermore, let's assume the material properties of the rod are constant, its thermal diffusivity is* α^2*, and there are no internal heat sources or sinks. Assuming both ends of the rod are insulated and the initial temperature distribution in the rod is given by* $u_0(x) = x$*, find the temperature distribution* $u(x,t)$ *within rod* G*.*

EXERCISE 9. *Given the same setup as in the previous example, find the temperature distribution* $u(x,t)$ *within rod* G*, under the following conditions:*

i. *The thermal diffusivity* α^2 *of the rod is some known constant.*
ii. *The left end of the rod is held at the fixed constant temperature* $u(0,t) = T_L$*, while the right end is held at the fixed constant temperature* $u(1,t) = T_R$*.*
iii. *The initial temperature distribution within the rod is given by* $u_0(x)$*.*

3. Longitudinal Vibrations

We recall the discussion of the longitudinal vibrations in a long narrow cylindrical rod of cross section area S. The rod is located along the $x-$axis, and we identify it with the interval $[a,b]$ in $\mathbb{R}$. The rod is assumed to stretch or contract in the horizontal direction, and we assume that the vertical plane cross-sections of the rod move only horizontally. Denote by $u(x,t)$ the *displacement* in the positive direction from the point $x \in [a,b]$ at the time $t > 0$. The corresponding *displacement rate* or *velocity* is denoted by $v(x,t) \equiv u_t(x,t)$.

Let $\sigma(x,t)$ denote the local *stress*, the force per unit area with which the part of the rod to the right of the point x acts on the part to the left of x. Since force is positive to the right, the stress is positive in conditions of *tension*. For a section of the rod, $x_1 < x < x_2$, the total (rightward) force acting on that section due to the remainder of the rod is given by

$$\big(\sigma(x_2,t) - \sigma(x_1,t)\big)S\,.$$

If the density of the rod at x is given by $\rho > 0$, the momentum of this section is just

$$\int_{x_1}^{x_2} \rho\, u_t(x,t)\, S\, dx\ .$$

If we let $F(x,t)$ denote any external applied force per unit of volume in the positive $x-$direction, then we obtain from Newton's second law that

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho\, u_t(x,t)\, S\, dx = \big(\sigma(x_2,t) - \sigma(x_1,t)\big)S + \int_{x_1}^{x_2} F(x,t)\, S\, dx$$

for any such $x_1 < x_2$. For a sufficiently smooth displacement $u(x,t)$, we obtain the *conservation of momentum* equation

$$\rho\, u_{tt}(x,t) - \sigma_x(x,t) = F(x,t)\ , \qquad a < x < b,\ t > 0\ . \tag{22}$$

The *stress* $\sigma(x,t)$ is determined by the type of material of which the rod is composed and the amount by which the neighboring region is stretched or compressed, i.e., on the *elongation* or *strain*, $\varepsilon(x,t)$. In order to define this, first note that a section $[x, x+h]$ of the rod is deformed by the displacement to the new position $[x+u(x), (x+h)+u(x+h)]$. The *elongation* is the limiting increment of the change in the length due to the deformation as given by

$$\lim_{h\to 0} \frac{[u(x+h)+(x+h)]-[u(x)+x]-h}{h} = \frac{d\,u(x)}{dx}\,,$$

so the strain is given by $\varepsilon(x,t) \equiv u_x(x,t)$.

The relation between the stress and strain is a *constitutive law*, usually determined by experiment, and it depends on the type of material. In the simplest case, with small displacements, we find by experiment that $\sigma(x,t)$ is proportional to $\varepsilon(x,t)$, i.e., that there is a constant k called *Young's modulus* for which

$$\sigma(x,t) = k\,\varepsilon(x,t)\ .$$

The constant k is a property of the material, and in this case we say the material is purely *elastic*. A rate-dependent component of the stress-strain relationship arises when the force generated by the elongation depends not only on the magnitude of the strain but also on the speed at which it is changed, i.e., on the *strain rate* $\varepsilon_t(x,t) = v_x(x,t)$. The simplest such case is that of a *visco-elastic* material defined by the linear constitutive equation

$$\sigma(x,t) = k\,\varepsilon(x,t) + \mu\,\varepsilon_t(x,t)\ ,$$

in which the material constant μ is the *viscosity* or internal friction of the material. Finally, if we include the effect of the transverse motions of the rod that result from the elongations under conditions of constant volume or mass, we will get an additional term to represent the transverse inertia. If the constant P denotes *Poisson's ratio*, and r is the average radius of that cross section, then the corresponding *stress-strain* relationship is given as before by

$$\sigma(x,t) = k\,\varepsilon(x,t) + \mu\,\varepsilon_t(x,t) + \rho\, r^2\, P\,\varepsilon_{tt}(x,t)\,.$$

In terms of displacement, the total stress is

$$\sigma(x,t) = k u_x(x,t) + \mu u_{xt}(x,t) + \rho\, r^2\, P u_{xtt}(x,t)\,. \tag{23}$$

The partial differential equation for the longitudinal vibrations of the rod is obtained by substituting (23) into (22).

3.1. Initial and Boundary conditions. Since the momentum equation is second–order in time, one may expect that in order to have a well–posed problem, two initial conditions should be specified. Thus, we shall specify the *initial conditions*

$$u(x,0) = u_0(x), \quad u_t(x,t) = v_0(x), \qquad a < x < b\,,$$

where $u_0(\cdot)$ and $v_0(\cdot)$ are the initial displacement and the initial velocity, respectively.

We list a number of typical possibilities for determining the two boundary conditions. Each of these is illustrated as before with a condition at the right end, $x = b$, and we note that another such condition will also be prescribed at the left end, $x = a$.

1. The displacement could be specified at the end point:

$$u(b,t) = d_b(t), \quad t > 0\,.$$

This is the *Dirichlet* boundary condition, or boundary condition of *first type.* It could be obtained from observation of the endpoint position, or it could be imposed directly on the endpoint. The homogeneous case $d_b(t) = 0$ corresponds to a *clamped end.*

2. The horizontal force on the rod could be specified at the end point:

$$\sigma(b,t) = f_b(t), \quad t > 0\,.$$

For the purely elastic case, $\sigma = ku_x$, this is the *Neumann* boundary condition, or boundary condition of *second type.* The homogeneous condition with $f_b(t) = 0$ corresponds to a *free end.*

3. The force on the end is determined by an elastic constraint, a restoring force proportional to the displacement:

$$\sigma(b,t) + k_0(u(b,t) - d_b(t)) = f_b(t), \quad t > 0\,.$$

For the purely elastic case this is the *Robin* boundary condition, or boundary condition of *third type.* Here both $d_b(\cdot)$ and $f_b(\cdot)$ are prescribed. The first is a prescribed displacement of the spring reference, and the second is a horizontal force concentrated on the right end point. For $k_0 \to \infty$, we obtain formally the Dirichlet boundary condition, while for $k_0 \to 0$ we get the Neumann condition. Thus the *effective tension* k_0 interpolates between the first two types.

4. Another type of boundary condition arises if there is a *concentrated mass* at the end point. Then $u(b,t)$ is the displacement of this mass, and we have the *dynamic* boundary condition

$$\rho_0 u_{tt}(b,t) + \sigma(b,t) = f_b(t), \quad t > 0\,,$$

which is the boundary condition of *fourth type* for the elastic case.

4. The Eigenfunction Expansion, II

We shall apply the method of *separation of variables* to the initial-boundary-value problem for longitudinal vibrations with Dirichlet boundary conditions.

EXAMPLE 4. *Suppose the rod* $(0,\ell)$ *is perfectly elastic and set* $\alpha^2 = \frac{k}{\rho}$. *Assume there are no internal forces,* i.e., $f(x,t) = 0$, *and that both ends of the rod are fixed. The initial displacement and velocity are given by* $u_0(x)$ *and* $v_0(x)$, *respectively. The initial-boundary-value problem for this scenario is*

$$u_{tt}(x,t) = \alpha^2 u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0, \tag{24a}$$

$$u(0,t) = 0,\ u(\ell,t) = 0, \qquad t > 0, \tag{24b}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = v_0(x), \quad 0 < x < \ell. \tag{24c}$$

We look for non-null solutions of the form $u(x,t) = X(x)T(t)$ and find as before that

$$\frac{T''(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}, \qquad 0 < x < \ell,\ t > 0\,.$$

Each side must be equal a common constant, denoted by $-\lambda$, and this leads to the pair of ordinary differential equations

$$\begin{aligned} T''(t) &+ \lambda\,\alpha^2\,T(t) = 0, \quad t > 0, \\ X''(x) &+ \lambda\,X(x) = 0, \quad 0 < x < \ell. \end{aligned}$$

The boundary conditions given in (24b) imply that $X(0) = X(\ell) = 0$.

Note that if $X(\cdot)$ and $T(\cdot)$ are solutions of these respective equations, then it follows directly that their product is a solution (24a). We have already found the non-null solutions of the *boundary-value problem*

$$\left\{ \begin{array}{l} X''(x) + \lambda X(x) = 0, \quad 0 < x < \ell, \\ X(0) = 0,\; X(\ell) = 0. \end{array} \right. \tag{25}$$

The solutions are the *normalized eigenfunctions*

$$X_n(x) = \sqrt{\frac{2}{\ell}}\sin(\frac{n\pi}{\ell}x)$$

corresponding to the eigenvalues $\lambda_n = (n\pi/\ell)^2$. If we combine these with the corresponding time-dependent solutions $\cos(\alpha\sqrt{\lambda_n}t)$ and $\sin(\alpha\sqrt{\lambda_n}t)$ of the first differential equation and take linear combinations, we obtain a large class of solutions of the partial differential equation (24a) and boundary conditions (24b) in the form of a series

$$u(x,t) = \sum_{n=1}^{\infty}\big(A_n\cos(\alpha\sqrt{\lambda_n}t) + B_n\sin(\alpha\sqrt{\lambda_n}t)\big)X_n(x),$$

where the sequences $\{A_n\}$ and $\{B_n\}$ are to be determined. From the initial conditions (24c), it follows that these coefficients must satisfy

$$\sum_{n=1}^{\infty} A_n X_n(x) = u_0(x),\; \sum_{n=1}^{\infty} B_n\alpha\sqrt{\lambda_n}X_n(x) = v_0(x), \quad 0 < x < \ell,$$

so we obtain

$$A_m = (u_0(\cdot), X_m(\cdot)),\; B_m = \frac{(v_0(\cdot), X_m(\cdot))}{\alpha\sqrt{\lambda_m}}, \quad m \ge 1\,.$$

In summary, the solution of the initial-boundary-value problem (24) is given by the series

$$u(x,t) = \sum_{n=1}^{\infty}\big(\cos(\alpha\sqrt{\lambda_n}t)(u_0(\cdot), X_n(\cdot)) + \sin(\alpha\sqrt{\lambda_n}t)\frac{(v_0(\cdot), X_n(\cdot))}{\alpha\sqrt{\lambda_n}}\big)X_n(x).$$

Denote the second term in the preceding formula by

$$[S(t)v_0](x) = \sum_{n=1}^{\infty}\big(\sin(\alpha\sqrt{\lambda_n}t)\frac{(v_0(\cdot), X_n(\cdot))}{\alpha\sqrt{\lambda_n}}\big)X_n(x).$$

This defines the operator $S(t)$ on the space of functions on $[0,\ell]$. We can use this operator to represent the solution by

$$u(\cdot,t) = S'(t)u_0 + S(t)v_0\,. \tag{26}$$

Example 5. *Suppose the rod $(0,\ell)$ is perfectly elastic and set $\alpha^2 = \frac{k}{\rho}$. Assume that both ends of the rod are fixed, the initial displacement and velocity are both null, and that there are distributed forces $f(x,t)$ along its length. The initial-boundary-value problem for this case is*

$$u_{tt}(x,t) = \alpha^2 u_{xx}(x,t) + f(x,t), \quad 0 < x < \ell, \quad t > 0, \tag{27a}$$

$$u(0,t) = 0, \; u(\ell,t) = 0, \qquad t > 0, \tag{27b}$$

$$u(x,0) = 0, \; u_t(x,0) = 0, \quad 0 < x < \ell. \tag{27c}$$

For this problem with a non-homogeneous partial differential equation (27a), we look for the solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x). \tag{28}$$

If we assume that $f(x,t)$ has the eigenfunction expansion

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) X_n(x)\,,$$

then the coefficients are given by

$$f_n(t) \equiv \int_0^\ell f(s,t) X_n(s)\, ds\,,$$

and substituting this expansion into equation (27a) yields

$$\sum_{n=1}^{\infty} \left[\ddot{u}_n(t) + \lambda_n \alpha^2 u_n(t)\right] X_n(x) = \sum_{n=1}^{\infty} f_n(t) X_n(x)\,.$$

By equating the coefficients of the series given in this last equation, we are led to the sequence of initial-value problems

$$\ddot{u}_n(t) + \lambda_n \alpha^2 u_n(t) = f_n(t), \qquad t > 0 \tag{29a}$$

$$u_n(0) = 0, \; \dot{u}_n(0) = 0\,. \tag{29b}$$

The solution to (29) is

$$u_n(t) = \int_0^t \frac{\ell}{n\pi\alpha} \sin(\frac{n\pi\alpha}{\ell}(t-\tau)) f_n(\tau)\, d\tau$$

Now, if we use this in (28), we find that the solution to our non-homogeneous initial-boundary-value problem (14) is

$$u(x,t) = \int_0^t \sum_{n=1}^{\infty} \frac{\ell}{n\pi\alpha} \sin(\frac{n\pi\alpha}{\ell}(t-\tau)) f_n(\tau) X_n(x)\, d\tau. \tag{30}$$

Note that we can use the operator $S(t)$ to represent this formula as

$$u(\cdot,t) = \int_0^t S(t-\tau) f(\tau)\, d\tau. \tag{31}$$

As before, each $S(t)$ is an integral operator of the form

$$\begin{aligned}[S(t)v_0](x) &= \sum_{n=1}^{\infty} \frac{\ell}{n\pi\alpha} \sin(\frac{n\pi\alpha}{\ell}t) \int_0^{\ell} (v_0(s)\ \sin(\frac{n\pi}{\ell}s))\, ds \frac{2}{\ell} \sin(\frac{n\pi}{\ell}x) \\ &= \int_0^{\ell} \frac{2}{\ell} (\sum_{n=1}^{\infty} \frac{\ell}{n\pi\alpha} \sin(\frac{n\pi\alpha}{\ell}t) \sin(\frac{n\pi}{\ell}s) \sin(\frac{n\pi}{\ell}x)\big) v_0(s)\, ds \\ &= \int_0^{\ell} H(x,s,t) v_0(s)\, ds\end{aligned}$$

for which the kernel

$$H(x,s,t) = \frac{2}{\ell}\big(\sum_{n=1}^{\infty} \frac{\ell}{n\pi\alpha} \sin(\frac{n\pi\alpha}{\ell}t) \sin(\frac{n\pi}{\ell}s) \sin(\frac{n\pi}{\ell}x)\big)$$

is the *Green's function* for the problem.

EXAMPLE 6. *Suppose the left end of the elastic rod is free while the right end has an elastic constraint given by $u(\ell,t) + u_x(\ell,t) = 0$. The initial-boundary-value problem for this situation is*

$$u_{tt}(x,t) = \alpha^2 u_{xx}(x,t), \qquad 0 < x < \ell, \qquad t > 0, \tag{32a}$$

$$u_x(0,t) = 0, \quad u(\ell,t) + u_x(\ell,t) = 0, \qquad t > 0, \tag{32b}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), \qquad 0 < x < \ell. \tag{32c}$$

We seek a solution in the form

$$u(x,t) = X(x)T(t),$$

where the boundary conditions imply that $X_x(0) = 0$ and $X(\ell) + X_x(\ell) = 0$. The method of separation of variables leads us to the boundary-value problem

$$\begin{aligned} X_{xx}(x) + \lambda X(x) = 0, &\qquad 0 < x < \ell, \\ X'(0) = 0,\ X(\ell) + X'(\ell) = 0. &\end{aligned}$$

We obtain a sequence of eigenvalues λ_n and corresponding eigenfunctions given by

$$X_n(x) = \big(\frac{2}{\ell}\big)^{\frac{1}{2}} \cos(\lambda_n^{\frac{1}{2}} x). \tag{33}$$

Note that for large λ_n we have

$$\lambda_n \approx (\frac{n\pi}{\ell})^2,$$

so the eigenvalues are asymptotically close to those of the preceding example. Combining these results with the time-dependent solutions and using the orthogonality of the eigenfunctions, we find solutions of the initial-boundary-value problem (32) in the form

$$u(x,t) = \sum_{n=1}^{\infty} \big(\cos(\alpha\sqrt{\lambda_n}t)(u_0(\cdot), X_n(\cdot)) + \sin(\alpha\sqrt{\lambda_n}t) \frac{(v_0(\cdot), X_n(\cdot))}{\alpha\sqrt{\lambda_n}}\big) X_n(x),$$

with the eigenfunctions given by (33). Once again, this can be represented in the form (26) for an appropriate family of operators $\{S(t) : t \geq 0\}$.

EXERCISE 10. *Suppose the rod* $(0, \ell)$ *is elastic and that we account for the inertia of lateral extension. Set* $\alpha^2 = \frac{k}{\rho}$ *and* $\beta^2 = r^2 P$*, where* P *is Poisson's ratio and* r *is the average radius of a cross section as above. Assume there are no internal forces,* i.e., $f(x, t) = 0$*, and that both ends of the rod are fixed. The initial displacement and velocity are given by* $u_0(x)$ *and* $v_0(x)$*, respectively. The initial-boundary-value problem is*

$$u_{tt}(x,t) = \alpha^2 u_{xx}(x,t) + \beta^2 u_{xxtt}(x,t), \quad 0 < x < \ell, \quad t > 0, \tag{34a}$$

$$u(0,t) = 0, \ u(\ell,t) = 0, \qquad t > 0, \tag{34b}$$

$$u(x,0) = u_0(x), \ u_t(x,0) = v_0(x), \quad 0 < x < \ell. \tag{34c}$$

Find the solution by separation of variables. Find the family of operators $\{S(t) : t \geq 0\}$ *for which this can be represented in the form* (26).

5. Duhamel Formulae: Variation of Parameters

We would like to investigate further the structure of the solutions of both the first and second order equations that we found above. The point is that the forms we found above are indeed quite general, and we find in each case that the formula for a solution of the general non-homogeneous problem can be written down immediately, once we know the formula for a *basic* problem.

5.1. First Order Equations. Suppose that we know the *basic initial-value problem* for the first-order evolution equation

$$\dot{u}(t) + Au(t) = 0, \qquad u(0) = \varphi,$$

is well-posed, that is, that there exists exactly one solution of this problem for each choice of the initial function φ. This defines the *semigroup* of operators $\{E_A(t)\}$ by

$$u(t) \equiv E_A(t)\varphi, \quad t \geq 0\,.$$

In particular cases, this permits us to write a formula for $E_A(\cdot)$ as an integral operator with an explicit kernel. Now suppose that we have a solution $u(\cdot)$ of the more general problem with a nonhomogeneous equation,

$$\dot{u}(t) + Au(t) = f(t), \qquad u(0) = \varphi.$$

From the formal computation

$$\frac{d}{d\tau} E_A(t-\tau)u(\tau) = E_A(t-\tau)\{\dot{u}(\tau) + Au(\tau)\} = E_A(t-\tau)f(\tau)\,,$$

and an integration in time, we obtain

$$u(t) = E_A(t)\varphi + \int_0^t E_A(t-\tau)f(\tau)\,d\tau\,.$$

This is just the form (19). In particular, we can use our formula for the integral operators $E_A(\cdot)$ to obtain an explicit representation and verify directly that this formula gives the solution of the non-homogeneous problem.

5.2. Second Order Equation. Now for the second-order evolution equation, we suppose that the *basic initial-value problem*

$$\ddot{w}(t) + Aw(t) = 0, \qquad w(0) = 0, \quad \dot{w}(0) = \psi,$$

is well-posed. This defines the operators $\{S_A(t)\}$ by

$$w(t) \equiv S_A(t)\psi\,.$$

Note first that the derivative of these operators can be used to represent the solution of the corresponding problem

$$\ddot{w}(t) + Aw(t) = 0, \qquad w(0) = \psi, \quad \dot{w}(0) = 0,$$

by the formula

$$w(t) = S'_A(t)\psi\,.$$

Now consider the general initial-value problem with non-homogeneous data in the form

$$\ddot{w}(t) + Aw(t) = f(t), \qquad w(0) = \varphi, \quad \dot{w}(0) = \psi.$$

We make the formal computation

$$\frac{d}{d\tau}\{S_A(t-\tau)\dot{w}(\tau) + S'_A(t-\tau)w(\tau)\} = \\ S_A(t-\tau)\{\ddot{w}(\tau) + Aw(\tau)\} = S_A(t-\tau)f(\tau),$$

and then an integration in time yields the representation

$$w(t) = S'_A(t)\varphi + S_A(t)\psi + \int_0^t S_A(t-\tau)f(\tau)\,d\tau.$$

This is just the combination of the two formulae (26) and (31). Again, we find that if we find the formula for the integral operators $S_A(\cdot)$ which gives the explicit representation of the solution to the *basic* initial-value problem, then the corresponding formula for the solution of the general non-homogeneous problem can be written immediately.

EXERCISE 11. *Suppose the rod $(0,\ell)$ is perfectly elastic and set $\alpha^2 = \frac{k}{\rho}$. Assume that the left end of the rod is fixed, the right end is free, and the initial displacement is null. The initial-boundary-value problem for this situation is*

$$\begin{aligned} u_{tt}(x,t) &= \alpha^2 u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0, \\ u(0,t) &= 0,\ u_x(\ell,t) = 0, \qquad\qquad t > 0, \\ u(x,0) &= 0,\ u_t(x,0) = v_0(x), \quad 0 < x < \ell. \end{aligned}$$

Find the kernel $H(x,s,t)$ for which the solution is given by the integral operator

$$u(x,t) = \int_0^\ell H(x,s,t)v_0(s)\,ds\,.$$

Find a corresponding formula for the solution of the non-homogeneous problem with given initial displacement, velocity, and distributed forces $f(x,t)$ along its length:

$$\begin{aligned} u_{tt}(x,t) &= \alpha^2 u_{xx}(x,t) + f(x,t), \quad 0 < x < \ell, \quad t > 0, \\ u(0,t) = 0,\ u_x(\ell,t) &= 0, \qquad\qquad\qquad\qquad\quad t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) &= v_0(x), \quad 0 < x < \ell. \end{aligned}$$