Contents

Eigenvalue Problems and Fourier Series

1. The Finite-Dimensional Case

Suppose that A is an $n \times n$ real symmetric matrix. It has a set of eigenvectors $\mathbf{x}_j$, $1 \leq j \leq n$, written as columns, $\mathbf{x}_j = (x_{ij})$, and these form an *orthonormal basis* for $\mathbb{R}^n$. That is, we have

$$A\mathbf{x}_j = \lambda_j \mathbf{x}_j,\ 1 \leq j \leq n,$$

with corresponding eigenvalues λ_j, and their scalar products satisfy $(\mathbf{x}_i, \mathbf{x}_j) = \delta_{ij}$. If C is the unitary matrix which diagonalizes A, its (orthonormal) columns are the eigenvectors $\mathbf{x}_j$, $1 \leq j \leq n$, and we have

$$C'AC = diag(\lambda_j) \equiv D, \quad C = \big(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\big)\,.$$

This resolution of A into its eigenvectors makes many fundamental problems involving A quite easy. This is illustrated by the following examples.

1.1. Stationary Systems. Consider the algebraic system of n equations in n unknowns,

$$\mathbf{x} \in \mathbb{R}^n : \quad A\mathbf{x} + \lambda\mathbf{x} = F \text{ in } \mathbb{R}^n \tag{1}$$

for a given number λ and column F. We first find the diagonalization of A as above. Then look for a solution in the form $\mathbf{x} = \sum_{i=1}^n u_i \mathbf{x}_i$. The unknown coefficients u_i must satisfy $\sum_{i=1}^n (\lambda_i + \lambda) u_i \mathbf{x}_i = F$, so by the orthogonality of the eigenvectors this is equivalent to the 'separated' equations

$$(\lambda_i + \lambda) u_i = (F, \mathbf{x}_i)_{\mathbb{R}^n},\ 1 \leq i \leq n.$$

Suppose that $\lambda_i + \lambda \neq 0$ for all i. Then there exists exactly one solution determined by $u_i = (\lambda_i + \lambda)^{-1} (F, \mathbf{x}_i)_{\mathbb{R}^n}$, hence,

$$\mathbf{x} = \sum_{i=1}^n (\lambda_i + \lambda)^{-1} (F, \mathbf{x}_i)_{\mathbb{R}^n} \mathbf{x}_i\,.$$

Note that we have $F = \sum_{=1}^n (F, \mathbf{x}_i)_{\mathbb{R}^n} \mathbf{x}_i$ and the solution is of the form $\mathbf{x} = (A + \lambda)^{-1} F$ suggested by the *spectral form* of the diagonalization as $A\mathbf{x} = \sum_{i=1}^n \lambda_i (\mathbf{x}, \mathbf{x}_i)_{\mathbb{R}^n} \mathbf{x}_i$.

Suppose that $\lambda_J + \lambda = 0$ for some J with $1 \leq J \leq n$. From above, it follows that there exists a solution *only if* F satisfies the *orthogonality constraint*

$$(F, \mathbf{x}_j)_{\mathbb{R}^n} = 0 \text{ for all } j \text{ such that } \lambda_j = \lambda_J\,.$$

Then there exist (many) solutions, and these are obtained as above in the form

$$\mathbf{x} = \sum_{\lambda_i \neq \lambda_J} (\lambda_i + \lambda)^{-1}(F, \mathbf{x}_i)_{\mathbb{R}^n}\mathbf{x}_i + \sum_{\lambda_i = \lambda_J} u_i \mathbf{x}_i$$

in which the coefficients u_i for which $\lambda_i = \lambda_J$, are *arbitrary.* The number of these indices i for which $\lambda_i = \lambda_J$ is the *multiplicity* of the eigenvalue λ_J. Note that the constraint on the data F is that it must be orthogonal to the whole *eigenspace* $Ker(A - \lambda_J I)$.

1.2. Systems of Ordinary Differential Equations, I. Next we consider the linear system of ordinary differential equations

$$\dot{\mathbf{u}}(t) + A\mathbf{u}(t) = \mathbf{F}(t) \text{ in } \mathbb{R}^n, \tag{2a}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \tag{2b}$$

with the matrix A given as above. We use directly the orthonormal basis $\{\mathbf{x}_j\}$ of eigenvectors of the matrix A to compute the solution. First we represent the non-homogeneous term and initial condition from (2) as

$$\mathbf{F}(t) = \sum_{j=1}^{n} f_j(t)\mathbf{x}_j\,, \quad f_j(t) = (\mathbf{F}(t), \mathbf{x}_j)_{\mathbb{R}^n}\,,$$

$$\mathbf{u}(0) = \sum_{j=1}^{n} u_j^0 \mathbf{x}_j\,, \quad u_j^0 = (\mathbf{u}(0), \mathbf{x}_j)_{\mathbb{R}^n}\,.$$

Then we look for the solution of (2) in the form

$$\mathbf{u}(t) = \sum_{j=1}^{n} u_j(t)\mathbf{x}_j\,. \tag{3}$$

A direct substitution into (2) gives us the equivalent system

$$\dot{u}_j(t) + \lambda_j u_j(t) = f_j(t) \text{ in } \mathbb{R},$$
$$u_j(0) = u_j^0\,.$$

The solution of this separated system is given by

$$u_j(t) = e^{-\lambda_j t}{u_j}^0 + \int_0^t e^{-\lambda_j(t-s)} f_j(s)\, ds\,,$$

so we obtain our solution in the form

$$\mathbf{u}(t) = \sum_{j=1}^{n} e^{-\lambda_j t}(\mathbf{u}(0), \mathbf{x}_j)\mathbf{x}_j + \int_0^t \sum_{j=1}^{n} e^{-\lambda_j(t-s)}(\mathbf{F}(s), \mathbf{x}_j)\mathbf{x}_j\,.$$

For the special case of $\mathbf{F} = \mathbf{0}$, we obtain a representation for the exponential of the matrix A:

$$e^{-tA}\mathbf{x} = \sum_{j=1}^{n} e^{-\lambda_j t}(\mathbf{x}, \mathbf{x}_j)\mathbf{x}_j.$$

When $t = 0$ this is just the orthonormal expansion of $\mathbf{x}$.

1.3. Systems of Ordinary Differential Equations, II. For an initial-value problem with the system of second-order differential equations

$$\ddot{\mathbf{u}}(t) + A\mathbf{u}(t) = \mathbf{F}(t) \text{ in } \mathbb{R}^n, \tag{4a}$$

$$\mathbf{u}(0) = \mathbf{u}_0,\ \mathbf{u}'(0) = \mathbf{v}_0\,, \tag{4b}$$

a direct substitution of (3) into (4) gives us the equivalent system

$$\ddot{u}_j(t) + \lambda_j u_j(t) = f_j(t) \text{ in } \mathbb{R},$$
$$u_j(0) = u_j^0\ \ u_j'(0) = v_j^0.$$

If all eigenvalues are positive, the solution of this system is given by

$$u_j(t) = \cos(\lambda_j^{1/2}t)u_j^0 + \tfrac{1}{\lambda_j^{1/2}}\sin(\lambda_j^{1/2}t)v_j^0 + \int_0^t \tfrac{1}{\lambda_j^{1/2}}\sin(\lambda_j^{1/2}(t-s))f_j(s)\,ds\,,$$

and we obtain our solution as before from (3). Similar calculations follow if some of the eigenvalues are negative.

We shall show that the solution of boundary-value problems and of initial-boundary-value problems for the corresponding time-dependent partial differential equations to be discussed below will follow this same pattern. First we introduce two elementary applications which lead to boundary-value problems on an interval. Then we shall obtain an analagous representation of the solutions by means of eigenfuctnion expansions.

2. Heat Conduction in an Interval

We shall describe the conduction of heat energy through a long thin rod with uniform cross section S. We identify the rod with the open interval (a, b), and we assume the rod is perfectly insulated along its length. Let $u(x)$ denote the *temperature* within the rod at a point $x \in (a, b)$. The ends $x = x_1$ and $x = x_2$ of a section of the homogeneous rod of length $h = x_2 - x_1$ are maintained at temperatures u_1 and u_2, respectively. In the absence of heat sources, after a period of time (which depends on the material) the temperature distribution is observed to be linear: the temperature at the position x is given by

$$u(x) = (\frac{x_2 - x}{h})u_1 + (\frac{x - x_1}{h})u_2, \quad x_1 \le x \le x_2\,.$$

The quantity of heat per unit time and unit area which flows to the right is called the *heat flux*, and it is observed to be given by

$$q = -k\frac{u_2 - u_1}{h} \tag{5}$$

for some constant k which is called *conductivity*. The conductivity is a property of the material that is a measure of the flow rate per unit area, *i.e.*, the *flux* q induced by a given temperature gradient, $\frac{du}{dx}$. The equation (5) defining k is the discrete form of *Fourier's law*. The minus sign arises since the heat flow is directed toward the lower temperature. By letting $h \to 0$, we see that the *heat flux* $q(x)$ at the point x, *i.e.*, the flow rate to the right per unit area, is proportional to the temperature gradient $\frac{du}{dx}$. The precise statement of this experimental fact is *Fourier's law* of heat conduction,

$$q(x) = -k(x)\frac{du}{dx}(x), \tag{6}$$

and this equation defines the *thermal conductivity* $k(x)$ of the material at the point $x \in (a,b)$. Since heat flows in the direction of decreasing temperature, we see again that the minus sign is appropriate. This is a *constitutive law*.

Now, setting to zero the rate at which heat flows into the section plus the rate at which heat is generated within this section, we arrive at the *conservation of energy* equation for the section $[x, x+h]$

$$S\big(q(x) - q(x+h)\big) + \int_x^{x+h} f(s)S\,ds = 0$$

where $f(x)$ represents the rate at which heat per unit volume is delivered to the rod. This heat source term is assumed to be a known function of location. Dividing this by Sh and letting $h \to 0$ yields the *conservation law*

$$-\frac{dq}{dx}(x) + f(x) = 0. \tag{7}$$

Finally, by substituting the Fourier law (6) into the energy conservation law (7), we obtain the stationary one–dimensional heat conduction equation

$$-\frac{d}{dx}\left(k(x)\frac{du}{dx}(x)\right) = f(x). \tag{8}$$

This is also known as the *stationary diffusion equation* or *potential equation*. If we modify this to the more general situation in which the rod is *partially insulated*, then there is a distributed loss given by a discrete Fourier law, $K(x)(u(x) - U(x))$, where $K(x)$ is the conductivity and $U(x)$ is the outside temperature. This leads to the potential equation

$$-\frac{d}{dx}\left(k(x)\frac{du}{dx}(x)\right) + K(x)u(x) = f(x) + K(x)U(x). \tag{9}$$

If we assume that $k(\cdot)$ and $K(\cdot)$ are constants and rename $f(x)$, then equation (9) may be written in the form

$$-k\frac{d^2u}{dx^2}(x) + Ku = f(x), \qquad x \in (a,b). \tag{10}$$

NOTE 1. *If an* isotropic *material is specified by a bounded domain $G \subset \mathbb{R}^n$ and the temperature at $x \in G$ is denoted by $u(x)$, the corresponding n–dimensional equation has the form*

$$-\vec{\nabla}\cdot\left(k(x)\vec{\nabla}u(x)\right) = f(x).$$

Boundary conditions. Since the heat equation is second–order, one may expect that in order to have a well–posed problem, two boundary conditions should be specified. We shall see that this is true here. We describe some examples of appropriate *boundary conditions*. Each of these will be illustrated with a condition at the right end point, $x = b$, and we note that another such condition will be prescribed at the left end point, $x = a$.

Dirichlet Boundary Conditions. One can specify the value of the temperature at an end point:

$$u(b) = d_b \,,$$

where d_b is given. This type of boundary condition describes *perfect contact* with the boundary value, and it arises when the value of the end point temperature is known (usually from a direct measurement). Such a condition arises when one sets the boundary temperature to a prescribed value, for example, $d_b = 0$ when the end point is submerged in ice-water.

Neumann Boundary Conditions. One can specify the heat flux into the rod at an end point:

$$k(b)\frac{du}{dx}(b)S = f_b \,,$$

where f_b is given. This type of boundary condition corresponds to a known heat *source* f_b at the end. The homogeneous case $f_b = 0$ occurs at an *insulated* end point.

Robin Boundary Conditions. The heat flux is assumed to be lost through the end at a rate proportional to the difference between the inside and outside temperatures:

$$k(b)\frac{du}{dx}(b)S + k_b(u(b) - d_b)S = 0 \,.$$

Such a boundary condition arises from a *partially insulated* end point, and it corresponds to Newton's law of cooling at the end point $x = b$. This is just the discrete form of Fourier's law. Here, both d_b and f_b are given functions. The first is the outside temperature and the second is a heat source concentrated on the end. Note that the first two boundary conditions can be formally obtained as extreme cases of the Robin boundary condition; that is, as $k_b \to \infty$, $u(b) \to d_b$, which formally yields the Dirichlet boundary condition, and as $k_b \to 0$ we similarly obtain the Neumann boundary condition. Thus, this third type of boundary condition is an interpolation between the first two types for intermediate values of k_b. Finally, note that there is sign change in this condition at the left end, $x = a$, since we are accounting for the flux *into* the rod and flux is *rightward*.

Nonlocal Boundary Conditions. If the rod is perfectly insulated along its length, we submerge it into a bath of well-stirred water, and we assume the end points of the rod are in perfect contact with the water, then we obtain the boundary conditions

$$u(a) = u(b), \quad k(a)\frac{du}{dx}(a)S = k(b)\frac{du}{dx}(b)S \,,$$

where the common value of the endpoint temperatures is the (unknown) water temperature, and the second condition states that the total heat flow into the water is null. Note that we have *not* introduced an additional unknown, so these two equations provide the two boundary conditions. These are called *periodic* boundary conditions if $k(a) = k(b)$.

The same boundary conditions result if we assume the rod is bent around and the ends are joined to form a large ring. Then at the endpoints we must match the temperature and the flux. Note that again we have a total of two boundary constraints for the problem, but here the constraints depend on the solution at more than a single point.

3. Transverse Displacement of Loaded String

Next we describe the small vertical (transverse) displacement of a stretched elastic string. We identify its rest position with the horizontal interval $G = (a, b)$. Let $u(x, t)$ be the upward displacement at the location $x \in (a, b)$ and time $t \geq 0$. Denote by $T(x)$ the *tension* on the string at each $x \in (a, b)$. This is the force that is directed along the length of the string in the tangential direction. It's vertical and horizontal components are $T(x)\sin\theta_x$ and $T(x)\cos\theta_x$, respectively, where θ_x is the angle of inclination of the tangent along the string. We assume that the only displacements are vertical. This is equivalent to assuming that the horizontal forces on each small section of the string $[x, x+h]$ must be in balance, and we have $T(x+h)\cos\theta_{x+h} = T(x)\cos\theta_x = T$, a constant. The upward vertical force on that section $[x, x+h]$ due to the tension on its endpoints is given by

$$T(x)\sin\theta_{x+h} - T(x)\sin\theta_x = T(\tan\theta_{x+h} - \tan\theta_x) = T(u'(x+h) - u'(x)).$$

Denote by $f(x)$ the (upward) force per unit of length distributed along the string. Then the total force acting on the string must sum to zero, so we have

$$Tu'(x+h) - Tu'(x) + \int_x^{x+h} f(s)ds = 0, \quad [x, x+h] \subset (a, b).$$

Dividing by h and letting $h \to 0$, we obtain

$$-(Tu'(x))' = f(x), \quad a < x < b.$$

Exercise 1. *If there is an additional restoring force, such as an elastic support under the string or a distribution of springs along its length, then the force-balance equation is*

$$-(Tu'(x))' + Ku(x) = f(x), \quad a < x < b. \tag{11}$$

Boundary conditions. The boundary conditions appropriate for the force-balance equation (11) are similar to those for the heat conduction equation (10). As before, we specify some examples of boundary conditions at the end $x = b$; a boundary condition is specified at the other end $x = a$ of one of these types.

Dirichlet Boundary Conditions. The displacement u_b of the end of the string is specified by

$$u(b) = u_b.$$

Neumann Boundary Conditions. If the end of the string is attached to a slider (with negligible friction or mass) and a vertical force f_b acts on that end, then

$$Tu'(b) = f_b.$$

Robin Boundary Conditions. If the position of the end is constrained by a spring which directs it towards the position u_b, the

$$Tu'(b) + k_b(u(b) - u_b) = f_b,$$

where k_b is the spring constant.

In summary, a typical *boundary-value problem* for the loaded elastic string is to find a solution of (11) on the inteval (a, b) together with one of the boundary conditions above at each end of the interval.

4. The Eigenvalue Problem

We begin by calculating the eigenfunctions for the *periodic eigenvalue problem.* Afterward we shall relate these to the eigenfunctions of other boundary-value problems. We want to find non-null solutions of the boundary-value problem

$$\begin{cases} X''(x) + \lambda X(x) = 0, & 0 < x < \ell, \\ X(0) = X(\ell),\ X'(0) = X'(\ell). \end{cases} \tag{12}$$

This is a fundamental *Sturm-Liouville eigenvalue problem.* Since the differential equation in (12) is linear with constant coefficients, we can explicitly write down all possible solutions, and they depend on the sign of λ. First we check that for the case of $\lambda < 0$, all solutions of the equation are linear combinations of exponential functions and the only solution of the boundary-value problem (12) is the null solution. If $\lambda = 0$ the solution is any constant function, so we obtain the eigenvalue-eigenfunction pair $\lambda_0 = 0,\ X_0(x) = 1$. For the case of $\lambda > 0$, we get the general solution of the differential equation in the form

$$X(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x).$$

Then from the boundary conditions in (12) we find non-zero solutions only for $\lambda_n = (2n\pi/\ell)^2$, and in that case both constants are arbitrary. That is, for each integer $n \geq 1$ we get the eigenvalue $\lambda_n = (2n\pi/\ell)^2$ and a corresponding pair of eigenfunctions, $\cos(\frac{2n\pi}{\ell}x), \sin(\frac{2n\pi}{\ell}x)$.

We denote these eigenfunctions and eigenvalues of the boundary-value problem (12) by $X_n(\cdot),\ \lambda_n,\ n \geq 0$. Then we compute

$$\begin{aligned}(\lambda_m - \lambda_n) \int_0^\ell X_m(x) X_n(x)\, dx &= -\int_0^\ell \big(X_m''(x) X_n(x) - X_m(x) X_n''(x)\big)\, dx \\ &= -\int_0^\ell \frac{d}{dx}\big(X_m'(x) X_n(x) - X_m(x) X_n'(x)\big)\, dx = 0\,.\end{aligned}$$

The boundary conditions in (12) have been used to obtain the last equality. If $\lambda_m \neq \lambda_n$, this calculation shows that the corresponding eigenfunctions $X_m(\cdot), X_n(\cdot)$ are orthogonal with respect to the scalar-product $(\cdot,\cdot) \equiv \int_0^\ell (\cdot,\cdot)\, dx$ on the linear space of functions on the interval $(0, \ell)$. For each $n \geq 1$, a direct calculation shows that the two eigenfunctions $\cos(\frac{2n\pi}{\ell}x), \sin(\frac{2n\pi}{\ell}x)$ are also orthogonal, so all pairs of eigenfunctions are orthogonal. By replacing each such $X_n(\cdot)$ by the function obtained by dividing it by the corresponding *norm* $\|X_n(\cdot)\| = (X_n(\cdot), X_n(\cdot))^{\frac{1}{2}}$, we obtain an *orthonormal* set of functions in that space. That is, we have

$$(X_m(\cdot), X_n(\cdot)) = \delta_{mn} \text{ for } m, n \geq 0\,, \tag{13}$$

where we have scaled the eigenfunctions to get the *orthonormal eigenfunctions* on $(0, \ell)$

$$X_0(x) = \frac{1}{\sqrt{\ell}}, \quad X_{2k}(x) = \sqrt{\frac{2}{\ell}}\cos(\frac{2k\pi}{\ell}x),\ k \geq 1, \tag{14a}$$

$$X_{2k-1}(x) = \sqrt{\frac{2}{\ell}}\sin(\frac{2k\pi}{\ell}x),\ k \geq 1, \tag{14b}$$

with corresponding eigenvalues $\lambda_0 = 0$ and $\lambda_{2k} = \lambda_{2k-1} = (\frac{2k\pi}{\ell})^2$ for $k \geq 1$. All but the first of the eigenvalues have multiplicity 2. This normalization will greatly simplify many calculations to follow. In particular, when we can write a function $f(x),\ x \in (0, \ell)$, as the sum of a series of the normalized eigenfunctions, say

$$f(x) \cong \sum_{n=0}^{\infty} c_n X_n(x),\ x \in (0, \ell),$$

then taking the scalar product with each of the eigenfunctions $X_m(x)$ shows that the coefficients are determined by $(f, X_m) \equiv \int_0^\ell f(x)X_m(x)dx = c_m,\ m \geq 0$. That is,

$$f(x) \cong \sum_{n=0}^{\infty}(f, X_m)\, X_n(x). \tag{15}$$

This is the general *Fourier expansion* formula. We postpone the sense in which the series converges to $f(\cdot)$, so we have denoted this by the symbol $\cong$ in (15). (See Theorem 10.3.1 in Text.)

Since the eigenfunctions X_n are periodic with period ℓ, any function satisfying (15) on the interval $(0, \ell)$ can be extended to $\mathbb{R}$ as an ℓ-periodic function which satisfies (15) on all of $\mathbb{R}$. Such a function is determined by its values on any interval of length ℓ. In particular, if $\ell = 2L$, it suffices to know the values of f on the interval $(-L, L)$. Then the orthonormal eigenfunctions on $(-L, L)$ are

$$X_0(x) = \frac{1}{\sqrt{2L}}, \quad X_{2k}(x) = \frac{1}{\sqrt{L}}\cos(\frac{k\pi}{L}x),\ k \geq 1, \tag{16a}$$

$$X_{2k-1}(x) = \frac{1}{\sqrt{L}}\sin(\frac{k\pi}{L}x),\ k \geq 1, \tag{16b}$$

with corresponding eigenvalues $\lambda_0 = 0$ and $\lambda_{2k} = \lambda_{2k-1} = (\frac{k\pi}{L})^2$ for $k \geq 1$. These satisfy the *periodic eigenvalue problem*

$$\begin{cases} X''(x) + \lambda X(x) = 0, & -L < x < L, \\ X(-L) = X(L),\ X'(-L) = X'(L). \end{cases} \tag{17}$$

It is often convenient to write the Fourier expansion (15) directly in terms of the sines and cosines (without the normalization)

$$f(x) \cong \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\tfrac{n\pi}{L}x) + \sum_{n=1}^{\infty} a_n \sin(\tfrac{n\pi}{L}x), \tag{18}$$

and then the coefficients are given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(s)\cos(\tfrac{n\pi}{L}s)\,ds,\ n \geq 0, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(s)\sin(\tfrac{n\pi}{L}s)\,ds,\ n \geq 1. \tag{19}$$

NOTE 2. *This form will be useful to take advantage of symmetry properties of the solutions of various boundary-value problems. Compare with the Euler-Fourier formulas of Section 10.2 in Text.*

EXERCISE 2. *Given $m, n \in \mathbb{N}$, show directly that*

(a) $\displaystyle\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = \begin{cases} 0, & m \neq n, \\ \frac{1}{2}, & m = n. \end{cases}$

(b) $\displaystyle\int_0^1 \cos(m\pi x)\cos(n\pi x)\,dx = \begin{cases} 0, & m \neq n, \\ \frac{1}{2}, & m = n. \end{cases}$

(c) $\displaystyle\int_0^1 \sin(m\pi x)\cos(n\pi x)\,dx = 0\,.$

Symmetry and Boundary Conditions. Suppose that a *continuous* function f is *odd*: $f(-x) = -f(x)$. Then necessarily it satisfies $f(0) = 0$. Similarly, if f is odd with respect to L, i.e., $f(L-x) = -f(L+x)$, then continuity at L requires $f(L) = 0$. Likewise, if f is even and the derivative f' is continuous at 0, then $f'(0) = 0$, and if f is even with respect to L then continuity of f' at L requires $f'(L) = 0$.

Suppose the function f is $2L$-periodic and continuously differentiable. Then $f(-L+x) = f(L+x)$ so $f'(-L+x) = f'(L+x)$ and we have the boundary conditions $f(-L) = f(L)$ and $f'(-L) = f'(L)$. If f is odd, then $f(0) = f(-L) = f(L) = 0$ and then f is determined by its values on $(0, L)$ and the boundary conditions $f(0) = f(L) = 0$. If f is even, then f' is odd so we obtain $f'(0) = f'(-L) = f'(L) = 0$ and f is determined by its values on $(0, L)$ and the boundary conditions $f'(0) = f'(L) = 0$.

These observations lead to very useful special cases of the Fourier expansion (15). Assume the function f is given on the interval $(0, L)$. Extend it to $(-L, 0)$ as an odd function, and then extend it $2L$-periodically to $\mathbb{R}$. The Fourier expansion (15) simplifies considerably. All even coefficients vanish, $a_n = 0$, and the remaining odd coefficients can be written over $(0, L)$ as

$$b_n = \frac{2}{L}\int_0^L f(s)\sin(\frac{n\pi}{L}s)ds, \quad n \geq 1, \tag{20}$$

so we obtain the *Fourier sine series*

$$f(x) \cong \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{L}x).$$

Similarly, if we extend f to $(-L, 0)$ as an even function and then extend it $2L$-periodically to $\mathbb{R}$, all odd coefficienents vanish, $b_n = 0$, and the remaining even coefficients can be written over $(0, L)$ as

$$a_n = \frac{2}{L}\int_0^L f(s)\cos(\frac{n\pi}{L}s)ds, \quad n \geq 1, \tag{21}$$

and we have the *Fourier cosine series*

$$f(x) \cong \frac{1}{L}\int_0^L f(x)dx + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi}{L}x).$$

Stationary System. We can use the eigenfunctions (13) to determine the solution of the corresponding non-homogeneous boundary-value problem

$$-X''(x) + \lambda X(x) = F(x),\ 0 < x < \ell, \tag{22a}$$

$$X(0) = X(\ell),\ X'(0) = X'(\ell), \tag{22b}$$

where $\lambda \in \mathbb{R}$ and the function $F(\cdot)$ are given. As for the algebraic case (1), we begin by looking for a solution $X(\cdot)$ given by a series of the eigenfunctions (14) of the corresponding homogeneous boundary-value problem (12),

$$X(x) = \sum_{n=1}^{\infty} c_n X_n(x). \tag{23}$$

Taking the scalar product of the ordinary differential equation (22a) with the eigenfunction $X_m(x)$, we integrate by parts and use the boundary conditions (22b) to obtain

$$(X'', X_m) = (X, X_m'') = -\lambda_m(X, X_m).$$

This shows that the coefficient $\{c_m\}$ must satisfy

$$(\lambda_m + \lambda)c_m = (F, X_m) \equiv \int_0^\ell F(x)X_m(x)dx, \quad m \geq 0.$$

From this it follows that if $\lambda \neq -\lambda_m$ for all $m \geq 0$, then (22) has exactly one solution given by

$$X(x) = \sum_{n=0}^{\infty} \frac{(F, X_n)}{\lambda_n + \lambda} X_n(x). \tag{24}$$

If for some $N \geq 0$ we have $\lambda = -\lambda_N$, then there are solutions of (22) only if $F(\cdot)$ satisfies the orthogonality constraint $(F, X_n) = 0$ for all n such that $\lambda_n = \lambda_N$, and the solutions are given by

$$X(x) = \sum_{\lambda_n \neq \lambda_N} \frac{(F, X_n)}{\lambda_n - \lambda_N} X_n(x) + \sum_{\lambda_n = \lambda_N} c_n X_n(x) \tag{25}$$

with coefficients c_n for $\lambda_n = \lambda_N$ that are arbitrary. As in the algebraic case, the number of indices n for which $\lambda_n = \lambda_N$ is the *multiplicity* of the eigenvalue λ_N and the constraint on the data F is that it must be orthogonal to all eigenfunctions of (12) with $\lambda = \lambda_N$.

5. Dynamic Heat Conduction

We shall describe the diffusion of heat energy through a long thin rod G with uniform cross section S. As before, we identify G with the open interval (a, b), and we assume the rod is perfectly insulated along its length. Let $u(x, t)$ denote the temperature within the rod at a point $x \in G$ and at time $t > 0$. Our previous experience shows that the *heat flux* $q(x, t)$ at the point x, *i.e.*, the flow rate to the right per unit area, is proportional to the temperature gradient $\frac{\partial u}{\partial x}$. The precise statement of this experimental fact is *Fourier's law* of heat conduction,

$$q(x,t) = -k(x)\frac{\partial u}{\partial x}(x,t), \tag{26}$$

and this equation defines the *thermal conductivity* $k(x)$ of the material at the point $x \in (a, b)$. Since heat flows in the direction of decreasing temperature, we see again that the minus sign is appropriate.

The amount of heat stored in the rod within a section $[x, x + h]$ with $h > 0$ is given by

$$\int_x^{x+h} c(s)\rho(s)Su(s,t)\,ds,$$

where $c(\cdot)$ is the *specific heat* of the material, and $\rho(\cdot)$ is the volume-distributed *density*. The specific heat provides a measure of the amount of heat energy required to raise the temperature of a unit mass of the material by a degree.

Now, equating the rate at which heat is stored within the section to the rate at which heat flows into the section plus the rate at which heat is generated within this section, we arrive at the *conservation of energy* equation for the section $[x, x + h]$

$$\frac{d}{dt}\int_x^{x+h} c(s)\rho(s)Su(s,t)\,ds = S\big(q(x,t) - q(x+h,t)\big) + \int_x^{x+h} f(s,t)S\,ds$$

where $f(x, t)$ represents the rate at which heat is generated per unit volume. This heat generation term is assumed to be a known function of space and time. Dividing this by Sh and letting $h \to 0$ yields the *conservation of energy* equation

$$c(x)\rho(x)\frac{\partial u}{\partial t}(x,t) + \frac{\partial q}{\partial x}(x,t) = f(x,t). \tag{27}$$

Finally, by substituting the Fourier law (26) into the energy conservation law (27), we obtain the one–dimensional heat conduction equation

$$c(x)\rho(x)\frac{\partial u}{\partial t}(x,t) - \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}(x,t)\right) = f(x,t). \tag{28}$$

This is also known as the *diffusion equation*.

If we assume the material properties $c(\cdot), \rho(\cdot)$, and $k(\cdot)$ are constants, then equation (28) may be written in the form

$$\frac{\partial u}{\partial t}(x,t) - \alpha^2\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{1}{c\rho}f(x,t), \qquad x \in G,\ t > 0, \tag{29}$$

where $\alpha^2 \equiv \frac{k}{c\rho}$ is called the *thermal diffusivity* of the material and is a measure of the rate of change of temperature of the material. For example, if the material is made out of a substance with a high thermal conductivity and low heat capacity $c\rho$, then the material will react very quickly to transient external conditions.

Initial and Boundary conditions. Since the heat equation is second–order in space and first–order in time, one may expect that in order to have a well–posed problem, two boundary conditions and one initial condition should be specified. We shall see that this is true here.

We want to find a solution of (28) which satisfies an *initial condition* of the form

$$u(x, 0) = u_0(x), \quad a < x < b,$$

where $u_0(\cdot)$ is given. Following the outline in Chapter 1, we describe some examples of appropriate *boundary conditions*. Each of these will be illustrated with a condition at the right end point, $x = b$, and we note that another such condition will be prescribed at the left end point, $x = a$.

Dirichlet Boundary Conditions. One can specify the value of the temperature at an end point:

$$u(b, t) = d_b(t), \quad t > 0\,,$$

where $d_b(\cdot)$ is given. This type of boundary condition describes *perfect contact* with the boundary value, and it arises when the value of the end point temperature is known (usually from a direct measurement). Such a condition arises when one sets the boundary temperature to a prescribed value, for example, $d_b(t) = 0$ when the end point is submerged in ice-water.

Neumann Boundary Conditions. One can specify the heat flux into the rod at an end point:

$$k\frac{\partial u}{\partial x}(b, t)S = f_b(t), \quad t > 0\,,$$

where $f_b(\cdot)$ is given. This type of boundary condition corresponds to a known heat *source* $f_b(\cdot)$ at the end. The homogeneous case $f_b(t) = 0$ occurs at an *insulated* end point.

Robin Boundary Conditions. The heat flux is assumed to be lost through the end at a rate proportional to the difference between the inside and outside temperatures:

$$k\frac{\partial u}{\partial x}(b, t)S + k_b(u(b, t) - d_b(t))S = f_b(t), \quad t > 0\,.$$

Such a boundary condition arises from a *partially insulated* end point, and it corresponds to Newton's law of cooling at the end point $x = b$. This is just the discrete form of Fourier's law. Here, both $d_b(\cdot)$ and $f_b(\cdot)$ are given functions. The first is the outside temperature and the second is a heat source concentrated on the end. Note that the first two boundary conditions can be formally obtained as extreme cases of the Robin boundary condition; that is, as $k_b \to \infty$, $u(b, t) \to d_b(t)$, which formally yields the Dirichlet boundary condition, and as $k_b \to 0$ we similarly obtain the Neumann boundary condition. Thus, this third type of boundary condition is an interpolation between the first two types for intermediate values of k_b.

Dynamic Boundary Conditions. As in the discrete case, another type of boundary condition arises when we assume the end of the rod has an *effective specific heat* given by $c_0 > 0$: this is a *concentrated capacity* at the end. For example, the end of the rod can be capped by a piece of material whose conductivity is very high, so the entire piece is essentally at the same temperature, or the end is submerged in an insulated container of well-stirred water. If we assume the heat energy is supplied to the concentrated capacity by the flux from the interior of the rod and supplemented by a given heat source located at that end point, $f_b(t)$, then we are led to the boundary condition

$$c_0 u_t(b,t) + k u_x(b,t) S = f_b(t), \qquad t > 0.$$

Implicit in this boundary condition is the assumption that we have "perfect" contact between the end of the rod and the concentrated capacity. If we permit some insulation between the end point and the concentrated capacity, then the temperature $u_c(\cdot)$ of the concentrated capacity is different from that of the endpoint of the rod, and the boundary condition is

$$\begin{cases} c_0 \frac{du_c(t)}{dt} + k_b(u_c(t) - u(b,t)) = f_b(t), & t > 0, \\ k\frac{\partial u}{\partial x}(b,t) + k_b(u(b,t) - u_c(t)) = 0. \end{cases}$$

Here we have an extra condition, but we have also introduced an additional unknown, so this pair of equations should be regarded as a single constraint. Note that, as $k_b \to \infty$, $u(b,t) \to u_c(t)$, and this partially insulated dynamic boundary condition formally yields the "perfect" contact dynamic boundary condition.

Each of the preceding boundary conditions must be supplemented with an additional boundary condition at the other endpoint, $x = a$, and the two boundary conditions need not be of the same type. These two boundary constraints together with the initial condition on a solution of (28) will comprise a well–posed problem.

Nonlocal Boundary Conditions. If the rod is bent around and the ends are joined to form a large ring, then at the endpoints we must match the temperature and the flux:

$$u(a,t) = u(b,t), \quad t > 0,$$
$$k\frac{\partial u}{\partial x}(a,t)S = k\frac{\partial u}{\partial x}(b,t)S\,.$$

These are called *periodic* boundary conditions. Note that again we have a total of two boundary constraints for the problem, but here the constraints depend on the solution at more than a single point.

Another example arises in the case that we submerge the entire rod into a bath of well-stirred water and assume the end points of the rod are in perfect contact with the water. Then we obtain the *dynamic and nonlocal* boundary conditions

$$u(a,t) = u(b,t) = u_c(t), \quad t > 0,$$
$$c_0 \frac{du_c(t)}{dt} + k(b)\frac{\partial u}{\partial x}(b,t) - k(a)\frac{\partial u}{\partial x}(a,t) = 0,$$

where the common value of the endpoint temperatures $u_c(t)$ is unknown. Note that we have *not* introduced an additional unknown, so these two equations provide the two boundary conditions.

Exercise 3. *Assume the rod G is submerged in a perfectly insulated container of well-stirred water. The rod is partially insulated along its length, so there is some limited heat exchange with the surrounding water along the length, $a < x < b$. Also, assume the rod is in perfect contact with the water at the end points. Find an initial-boundary-value problem which models this situation.*

6. The Porous Medium Equation

We shall describe the diffusion of fluid through the long thin tube G with uniform cross section S. As before, we identify G with the open interval (a, b), but here we assume that the tube is packed with a distribution of particles, such as sand or gravel, which impede the flow of the fluid. The fluid is constrained to flow in the complementary region of open channels and pores *not* occupied by the particles. The porous medium is assumed to be *rigid*, so its structure is not deforming. It is also assumed to remain *fully saturated*: all the pore space is occupied by the fluid. This is often the case for fluids in the subsurface, for example, in the groundwater region or below a lake or sea. The one-dimensional model can be appropriate for depth in a wide region for which horizontal variations are insignificant.

The conservation law. Let $\rho(x, t)$ denote the (average) *density* of the fluid within the tube at a point $x \in G$ and at time $t > 0$. The mass of fluid stored in the section $[x, x + h] \subset (a, b)$ of length $h > 0$ is given by

$$\int_x^{x+h} \phi(s)\rho(s, t)S\, ds,$$

where $\phi(x)$ is the *porosity* of the porous medium at x, *i.e.,* the volume fraction of the medium occupied by the fluid. We assume this is non-zero, so $0 < \phi(x) \leq 1$. The (averaged) fluid *velocity* $v(x, t)$ at the point x is the flow rate to the right per unit area measured in volume of fluid per unit area per time. Equating the rate at which fluid is stored within the section to the rate at which fluid flows into the section plus the fluid source rate within this section, we arrive at the *fluid conservation* equation for the section $[x, x + h]$,

$$\frac{\partial}{\partial t}\int_x^{x+h} \phi(s)\rho(s, t)S\, ds = S\big(\rho(x, t)v(x, t) - \rho(x + h, t)v(x + h, t)\big) + \int_x^{x+h} \rho(x, t)f(s, t)S\, ds\,,$$

where $f(x, t)$ represents the rate at which fluid volume is inserted per unit volume of themedium. This source term is assumed to be a known function of space and time. Dividing by Sh and letting $h \to 0$ yields the *fluid conservation* equation

$$\phi(x)\frac{\partial\rho(x, t)}{\partial t} + \frac{\partial(\rho(x, t)v(x, t))}{\partial x} = \rho(x, t)f(x, t). \tag{30}$$

Darcy's law. Let $p(x,t)$ denote the *pressure* of the fluid in the pores. This is measured in force per unit area. A fundamental experimental observation is that the fluid flow rate through the section $[x, x+h]$ of the porous media is proportional to the pressure difference at its ends per unit length, that is,

$$v = \frac{k(x)}{\mu\, h}\big(p(x,t) - p(x+h,t)\big). \tag{31}$$

Thus, letting $h \to 0$ we get the fluid velocity $v(x,t)$ at the point x proportional to the pressure gradient,

$$v(x,t) = -\frac{k(x)}{\mu}\frac{\partial p(x,t)}{\partial x}. \tag{32}$$

The constant μ is the *viscosity* of the fluid, a measure of its resistance to *shear*, and the equation (32) defines the *permeability* $k(x)$ of the porous medium at the point $x \in (a,b)$. It is a measure of the conductivity of the medium, *i.e.,* the inverse of resistance of the medium to internal flow. Since fluid flows in the direction of decreasing pressure, the minus sign is appropriate. In fact, if we write this in the form

$$\frac{\mu}{k(x)} v(x,t) = -\frac{\partial p(x,t)}{\partial x},$$

it is a balance of forces on the fluid as it flows through the medium, and the coefficient $\mu/k(x)$ is the *resistance* to flow. The porous medium is characterized by its porosity and the permeability, and they are related, usually by a power law. By substituting Darcy's law (32) into the fluid conservation law (30), we obtain the one–dimensional *porous medium equation*

$$\phi(x)\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial x}\left(\rho\,\frac{k(x)}{\mu}\frac{\partial p}{\partial x}\right) = \rho(x,t) f(x,t). \tag{33}$$

Note that the product $\rho\frac{\partial p}{\partial x}$ makes (33) nonlinear.

It remains to specify the *state equation*, the relation between density ρ and pressure p for the particular fluid. If the *compressibility*, $c = \frac{1}{\rho}\frac{d\rho}{dp}$, of the fluid is constant, then we have $\rho = \rho_0 e^{c(p-p_0)}$. In this case the chain rule shows that $\rho p_x = \frac{1}{c}\rho_x$, so we obtain the linear *diffusion equation* for fluid density

$$\phi(x)\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial x}\left(\frac{k(x)}{c\,\mu}\frac{\partial \rho}{\partial x}\right) = \rho(x,t) f(x,t). \tag{34}$$

If the compressibility is small, the fluid is called *slightly compressible* and the density is nearly constant, $\rho(x) \approx \rho_0$; then (33) is further simplified to the *pressure equation*

$$c\,\phi(x)\frac{\partial p}{\partial t} - \frac{\partial}{\partial x}\left(\frac{k(x)}{\mu}\frac{\partial p}{\partial x}\right) = f(x,t). \tag{35}$$

If the fluid is *incompressible*, *i.e.*, if $c = 0$, then we obtain the equation

$$-\frac{\partial}{\partial x}\left(\frac{k(x)}{\mu}\frac{\partial p}{\partial x}\right) = f(x,t). \tag{36}$$

Of course, either of these must be supplemented with appropriate initial and boundary conditions to get a well posed problem which determines the density and pressure along the length of the tube, and then these determine the Darcy velocity (32).

Initial and Boundary Conditions. We record boundary and initial conditions for the case of the pressure equation (35). The initial value of the pressure is given as

$$p(x,0) = p_0(x),\ a < x < b.$$

As before, we describe various boundary conditions for the right end $x = b$. One of these should be given, and we note that any one of these types will be prescribed independently at the left end $x = a$.

Dirichlet Boundary Conditions. The value of the pressure at an end point is specified or measured,

$$p(b,t) = p_b(t), \quad t > 0\,,$$

so $p_b(\cdot)$ is given. This corresponds to the *drained* condition in which fluid flows freely into or out of the boundary of the rod in order to maintain that pressure balance.

Neumann Boundary Condition. The fluid source $v_b(t)$ is known at the *sealed* or *imperveous* boundary $x = b$, so we have

$$\frac{k(b)}{\mu}\frac{\partial p(b,t)}{\partial x} = v_b(t),\ t > 0.$$

Robin Boundary Condition. When the boundary is *permeable* to an outside pressure $p_b(t)$ with a flow rate satisfying the discrete Darcy law (31), and there is a source $v_b(t)$ at the boundary, we obtain

$$\frac{k(b)}{\mu}\frac{\partial p(b,t)}{\partial x} + \frac{K_b}{\mu}\big(p(b,t) - p_b(t)\big) = v_b(t),\ t > 0.$$

Dynamic Boundary Condition. There may be a very high porosity concentrated at the boundary, so that a substantial concentration of fluid can accumulate there. This occurs because the 'packing' of the grains of the medium is less efficient there. In that case, we will have

$$c\,\phi_b\frac{\partial p(b,t)}{\partial t} + \frac{k(b)}{\mu}\frac{\partial p(b,t)}{\partial x} = v_b(t),\ t > 0.$$

7. The Eigenfunction Expansion, I

We shall illustrate the method of *separation of variables* by obtaining the eigenfunction expansion of the solution of an initial-boundary-value problem with Dirichlet boundary conditions. The same method works for the other boundary conditions.

Example 1. *Suppose the rod $G = (0,\ell)$ is perfectly insulated along its length and made of an isotropic material with thermal diffusivity α^2. Assuming no internal heat* *sources or sinks,* i.e., $f(x,t)=0$, *suppose both ends of the rod are held at a fixed temperature of zero and the initial temperature distribution is given by* $u_0(x)$. *The initial-boundary-value problem for this scenario is*

$$\frac{\partial u}{\partial t}(x,t)=\alpha^2\frac{\partial^2 u}{\partial x^2}(x,t), \quad 0<x<\ell, \quad t>0, \tag{37a}$$

$$u(0,t)=0,\ u(\ell,t)=0, \qquad t>0, \tag{37b}$$

$$u(x,0)=u_0(x), \quad 0<x<\ell. \tag{37c}$$

Exercise 4. *Let* $u(x,t)$ *be a solution of the initial-boundary-value problem and show that*

$$\frac{d}{dt}\int_0^\ell u^2(x,t)\,dx \le 0.$$

Show that this implies there is at most one *solution of the problem.*

We begin by looking for a non-null solution of the form

$$u(x,t)=X(x)T(t). \tag{38}$$

Substituting (38) into (37a) and dividing by u yields

$$\frac{T'(t)}{\alpha^2 T(t)}=\frac{X''(x)}{X(x)}, \qquad \text{for all } x \text{ and } t.$$

Note that the left side of this last equation is exclusively a function of t, while the right side of is exclusively a function of x. The only way this equation can hold for all values of x and t is for each side to equal a common constant. Denoting this constant by $-\lambda$ leads to the pair of ordinary differential equations

$$\begin{aligned} T'(t) &+ \lambda\alpha^2 T(t)=0, \quad t>0,\\ X''(x) &+ \lambda X(x)=0, \quad 0<x<\ell. \end{aligned}$$

The boundary conditions given in (37b) imply that $X(0)=X(\ell)=0$.

Note that if $X(\cdot)$ and $T(\cdot)$ are solutions of these respective equations, then it follows directly that their product is a solution of (37a). The first of these ordinary differential equations has the solution $T(t)=e^{-\lambda\alpha^2 t}$. Thus, it remains to find a non-null solution of the boundary-value problem

$$\begin{cases} X''(x)+\lambda X(x)=0, \quad 0<x<\ell,\\ X(0)=0,\ X(\ell)=0. \end{cases} \tag{39}$$

This is a *"regular" Sturm-Liouville boundary-value problem* and we will see later that such problems have very special properties. Since this is a linear equation with constant coefficients, we can explicitly write down all possible solutions, and they depend on the sign of λ. First we check that for the cases of $\lambda<0$ and $\lambda=0$, the only solution of the boundary-value problem (39) is the null solution. For the case of $\lambda>0$, we get the general solution of the differential equation in the form

$$X(x)=C_1\sin(\sqrt{\lambda}x)+C_2\cos(\sqrt{\lambda}x),$$

and then from the boundary conditions we see that necessarily

$$C_2 = 0, \quad \text{and} \quad C_1 \sin(\ell\sqrt{\lambda}) = 0,$$

respectively. Since $\sin(\ell\sqrt{\lambda}) = 0$ has solutions $\lambda_n = (n\pi/\ell)^2$, this does *not* force $C_1 = 0$. These specific values for λ are called the *eigenvalues* of the regular Sturm-Liouville problem (39), and the solutions to (39), namely, multiples of

$$X_n(x) = \sin(\sqrt{\lambda_n}x),$$

are the corresponding *eigenfunctions*. If we combine these with the corresponding time-dependent solutions $T_n(t) = e^{-\lambda_n\alpha^2 t}$, we obtain solutions $e^{-\lambda_n\alpha^2 t}\sin(\sqrt{\lambda_n}x)$ of (37a) and (37b). From the superposition principle we obtain a large class of solutions of (37a) and (37b) in the form

$$u(x,t) = \sum_{n=1}^{N} A_n e^{-\lambda_n\alpha^2 t} X_n(x), \tag{40}$$

for any integer N. We check directly that (40) satisfies (37a) and (37b) for any choice of the coefficients $\{A_n\}$. In order to satisfy the initial condition (37c), the coefficients must be chosen to satisfy

$$u_0(x) = \sum_{n=1}^{N} A_n X_n(x). \tag{41}$$

Now this is a *severe* restriction on the initial data, but we shall find that we can go to a corresponding series with $N = +\infty$, and then there is essentially *no restriction* on the initial data! This will follow from the observation that the corresponding coefficients in (40) have exponentially decaying factors that make the series converge extremely rapidly for $t > 0$.

Let's take a preliminary look at the boundary-value problem (39). We have denoted its non-null solutions by $X_n(\cdot),\ \lambda_n,\ n \geq 1$ First we compute

$$\begin{aligned}(\lambda_m - \lambda_n)\int_0^\ell X_m(x)X_n(x)\,dx &= -\int_0^\ell \big(X_m''(x)X_n(x) - X_m(x)X_n''(x)\big)\,dx \\ &= -\int_0^\ell \frac{d}{dx}\big(X_m'(x)X_n(x) - X_m(x)X_n'(x)\big)\,dx = 0\,.\end{aligned}$$

Since $\lambda_m \neq \lambda_n$ for $m \neq n$, this shows that the eigenfunctions $X_n(\cdot)$ are orthogonal with respect to the scalar-product $(\cdot,\cdot) \equiv \int_0^\ell (\cdot,\cdot)\,dx$ on the linear space of continuous functions on the interval $[0,\ell]$. By replacing each such $X_n(\cdot)$ by the function obtained by dividing it by the corresponding *norm* $\|X_n(\cdot)\| = (X_n(\cdot), X_n(\cdot))^{\frac{1}{2}}$, we obtain an *orthonormal* set of functions in that space. That is, we have

$$(X_m(\cdot), X_n(\cdot)) = \delta_{mn} \text{ for } m,n \geq 1\,,$$

where we have scaled the eigenfunctions to get the *normalized eigenfunctions*

$$X_n(x) = \sqrt{\frac{2}{\ell}}\sin(\frac{n\pi}{\ell}x).$$

Now it is clear how to choose the coefficients A_n in (41): take the scalar product of that equation with $X_m(\cdot)$ to obtain

$$(u_0(\cdot), X_m(\cdot)) = A_m, \quad m \geq 1\,.$$

Thus we obtain

$$u(x,t) = \sum_{n=1}^{N} e^{-\lambda_n \alpha^2 t}(u_0(\cdot), X_n(\cdot))X_n(x), \tag{42}$$

when $u_0(\cdot)$ is appropriately restricted. We shall see below that we can go to the corresponding series with $N = +\infty$ as indicated with essentially no restriction on $u_0(\cdot)$.

The technique used in Example 1 is called the *method of separation of variables*. Since it depends on superposition, this technique is appropriate for solving linear initial-boundary-value problems with homogeneous boundary conditions. It can easily be modified to solve initial-boundary-value problems that contain *constant* non-homogeneous boundary conditions. In particular, for the problem

$$\begin{cases} u_t(x,t) = \alpha^2 u_{xx}(x,t), & 0 < x < \ell, t > 0, \\ u(0,t) = d_0,\ u(\ell,t) = d_\ell, & t > 0, \\ u(x,0) = u_0(x), & 0 < x < \ell, \end{cases} \tag{43}$$

define $w(x,t) = u(x,t) - \big(d_0 \frac{\ell - x}{\ell} + d_\ell \frac{x}{\ell}\big)$ and transform the above problem into an equivalent initial-boundary-value problem for $w(x,t)$. Note that $w(0,t) = w(\ell,t) = 0$, and so now we have a problem with homogeneous boundary conditions as in Example 1.

Exercise 5. *Compute the solution of* (43) *for the case of* $u_0(\cdot) = 0$, $d_0 = 0$ *and* $d_\ell = 1$.

However, if the boundary values are given by a pair of time dependent functions, $d_0(t)$, $d_\ell(t)$, then we are led to a non-homogeneous partial differential equation. More generally, we can start with a *non-homogeneous* initial-boundary-value problem of the form

$$\begin{aligned} u_t(x,t) = \alpha^2 u_{xx}(x,t) + f(x,t), \qquad & 0 < x < \ell, \quad t > 0, \\ u(0,t) = d_0(t),\ u(\ell,t) = d_\ell(t), \qquad & \qquad\qquad\quad t > 0, \\ u(x,0) = u_0(x), \qquad & 0 < x < \ell\,, \end{aligned}$$

and then define $w(x,t) = u(x,t) - \big(d_0(t)\frac{\ell - x}{\ell} + d_\ell(t)\frac{x}{\ell}\big)$ to transform the above problem into an equivalent initial-boundary-value problem of the form

$$\begin{aligned} w_t(x,t) = \alpha^2 w_{xx}(x,t) + \tilde{f}(x,t), \qquad & 0 < x < \ell, \quad t > 0, \\ w(0,t) = 0,\ w(\ell,t) = 0, \qquad & \qquad\qquad\quad t > 0, \\ w(x,0) = w_0(x), \qquad & 0 < x < \ell, \end{aligned}$$

where $w_0(x) = u_0(x) - \big(d_\ell(0)\frac{x}{\ell} + d_0(0)\frac{\ell - x}{\ell}\big)$ and $\tilde{f}(x,t) = f(x,t) - \big(d'_\ell(t)\frac{x}{\ell} + d'_0(t)\frac{\ell - x}{\ell}\big)$. Thus, by such a change of variable, we can always reduce the initial-boundary-value problem to the form with homogeneous boundary conditions.

Example 2. *Find the* eigenfunction expansion *of the solution of the initial-boundary-value problem*

$$u_t(x,t) = \alpha^2 u_{xx}(x,t) + f(x,t), \qquad 0 < x < \ell, \quad t > 0, \tag{46a}$$

$$u(0,t) = 0, \ u(\ell,t) = 0, \qquad t > 0, \tag{46b}$$

$$u(x,0) = u_0(x), \qquad 0 < x < \ell, \tag{46c}$$

with non-homogeneous partial differential equation and *homogeneous boundary conditions*. Recall that for $f(x,t) = 0$, the solution to this case was given by equation (40). For the problem with a non-homogeneous partial differential equation (46a), we look for the solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x). \tag{47}$$

If we assume that both $f(x,t)$ and $u_0(x)$ also have eigenfunction expansions given by

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) X_n(x) \quad \text{and} \quad u_0(x) = \sum_{n=1}^{\infty} u_0^n X_n(x),$$

respectively, where

$$f_n(t) \equiv \int_0^{\ell} f(\zeta,t) X_n(\zeta)\, d\zeta \quad \text{and} \quad u_0^n \equiv \int_0^{\ell} u_0(\zeta) X_n(\zeta)\, d\zeta,$$

then substituting each of these expansions into equation (46a) yields

$$\sum_{n=1}^{\infty} \left[\dot{u}_n(t) + \lambda_n \alpha^2 u_n(t)\right] X_n(x) = \sum_{n=1}^{\infty} f_n(t) X_n(x).$$

By equating the coefficients of the series given in this last equation, we are led to the initial-value problems

$$\dot{u}_n(t) + \lambda_n \alpha^2 u_n(t) = f_n(t), \qquad t > 0 \tag{48a}$$

$$u_n(0) = u_0^n. \tag{48b}$$

The solution to (48) is

$$u_n(t) = e^{-\lambda_n \alpha^2 t} u_0^n + \int_0^t e^{-\lambda_n \alpha^2 (t-\tau)} f_n(\tau)\, d\tau$$

Now, if we use this in (47), we find that the solution to our non-homogeneous initial-boundary-value problem (46) is

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\lambda_n \alpha^2 t} u_0^n X_n(x) + \sum_{n=1}^{\infty} \int_0^t e^{-\lambda_n \alpha^2 (t-\tau)} f_n(\tau) X_n(x)\, d\tau, \tag{49}$$

with the coefficients u_0^n and $f_n(\cdot)$ computed as above.

This formula has the tyical structure of the solution of an initial-value problem. That is, if we use the first term in this representation to define a family of operators $E(t),\ t \geq 0$, on the space of functions on the interval $[0, \ell]$ by

$$[E(t)u_0](x) = \sum_{n=1}^{\infty} e^{-\lambda_n \alpha^2 t} \big(u_0, X_n\big) X_n(x)\,, \tag{50}$$

then the solution (49) takes the form

$$u(\cdot, t) = E(t)u_0 + \int_0^t E(t-\tau) f(\tau)\, d\tau\,. \tag{51}$$

In particular, the operator $E(t)$ in this specific case is an *integral operator*

$$\begin{aligned}
[E(t)u_0](x) &= \sum_{n=1}^{\infty} e^{-\lambda_n \alpha^2 t} \int_0^\ell (u_0(s)\ \sin(\frac{n\pi}{\ell} s))\, ds \frac{2}{\ell} \sin(\frac{n\pi}{\ell} x) \\
&= \int_0^\ell \frac{2}{\ell} \big(\sum_{n=1}^{\infty} e^{-\lambda_n \alpha^2 t} \sin(\frac{n\pi}{\ell} s) \sin(\frac{n\pi}{\ell} x)\big) u_0(s)\, ds \\
&= \int_0^\ell G(x, s, t) u_0(s)\, ds
\end{aligned}$$

for which the kernel

$$G(x, s, t) = \frac{2}{\ell} \big(\sum_{n=1}^{\infty} e^{-\lambda_n \alpha^2 t} \sin(\frac{n\pi}{\ell} s) \sin(\frac{n\pi}{\ell} x)\big)$$

is the *Green's function* for the problem.

EXERCISE 6. *Compute the solution of* (37a) *with initial condition* $u(x, 0) = 0$ *and the boundary conditions* $u(0, t) = 0$ *and* $u(\ell, t) = t$.

EXAMPLE 3. *For the situation of Example 1, suppose the left end of the rod is insulated while the right end has a heat loss given by* $-u_x(\ell, t) = k_\ell\, u(\ell, t)$ *where* $k_\ell \geq 0$. *The initial-boundary-value problem for this situation is given by*

$$u_t(x, t) = \alpha^2 u_{xx}(x, t), \qquad 0 < x < \ell, \quad t > 0, \tag{52a}$$

$$u_x(0, t) = 0,\ \ k_\ell\, u(\ell, t) + u_x(\ell, t) = 0, \qquad t > 0, \tag{52b}$$

$$u(x, 0) = u_0(x), \qquad 0 < x < \ell. \tag{52c}$$

We seek a solution in the form

$$u(x, t) = X(x)T(t),$$

where the boundary conditions imply $X'(0) = 0$ and $k_\ell\, X(\ell) + X'(\ell) = 0$. The method of separation of variables leads us to the time-dependent problem

$$T'(t) + \lambda\alpha^2 T(t) = 0, \quad t > 0,$$

and the boundary-value problem

$$X''(x) + \lambda X(x) = 0, \qquad 0 < x < \ell, \tag{53a}$$

$$X'(0) = 0,\ \ k_\ell\, X(\ell) + X'(\ell) = 0. \tag{53b}$$

For values of $\lambda < 0$, we find that there are no non-null solutions. For the case of $\lambda = 0$, only if $k_\ell = 0$ do we get a non-zero solution, and this is $X_0(x) = (\frac{1}{\ell})^{1/2}$ with $\lambda_0 = 0$. But for $\lambda > 0$, the general solution to (53a) is

$$X(x) = C_1 \sin(\lambda^{1/2} x) + C_2 \cos(\lambda^{1/2} x),$$

and the boundary conditions (53b) imply that

$$C_1 = 0, \quad \text{and} \quad k_\ell C_2 \cos(\ell\lambda^{1/2}) - C_2 \lambda^{1/2} \sin(\ell\lambda^{1/2}) = 0.$$

Since we are only interested in non-null solutions, the latter equation is equivalent to solving $\lambda^{1/2} \tan(\ell\lambda^{1/2}) = k_\ell$. That is, we have $\tan(\ell\lambda^{1/2}) = \frac{k_\ell}{\lambda^{1/2}}$. The tangent function is π-periodic, so we obtain a sequence λ_n, $n \geq 0$, of solutions to this equation, and the corresponding eigenfunctions are given by

$$X_n(x) = \Big(\frac{2}{\ell}\Big)^{1/2} \cos(\lambda_n^{1/2} x), \quad n \geq 1.$$

Note that the eigenvalues belong to intervals determined by $\ell(\lambda_n)^{1/2} \in [n\pi,\, n\pi + \frac{1}{2}\pi]$, $n \geq 0$, and that for small $\frac{k_\ell}{(\lambda_n)^{1/2}}$ we have

$$\lambda_n \approx \Big(\frac{n\pi}{\ell}\Big)^2,$$

so the eigenvalues are asymptotically close to those of the preceding example. Combining these results with the time-dependent solutions $T_n(t) = e^{-\lambda_n \alpha^2 t}$ and using the orthogonality of the eigenfunctions, we find solutions of (52a) in the form

$$u(x,t) = \sum_{n=0}^{\infty} (u_0(\cdot), X_n(\cdot)) e^{-\lambda_n \alpha^2 t} X_n(x),$$

and it is understood that the sum starts at $n = 1$ if $k_\ell > 0$.

Exercise 7. *Compute the solution of* (37a) *with initial condition* $u(\cdot, 0) = u_0(\cdot)$ *and the boundary conditions* $u(0,t) = 0$ *and* $u_x(\ell, t) = 0$.

Exercise 8. *Consider the problem*

$$u''(x) + u'(x) = f(x)$$
$$u'(0) = u(0) = \frac{1}{2}[u'(\ell) + u(\ell)],$$

where $f(x)$ *is a given function.*

(a) *Is the solution unique?*
(b) *Does a solution necessarily exist, or is there a condition that* $f(x)$ *must satisfy for existence?*

Exercise 9. *Let the rod* G *be defined over the interval* $(0,1)$, *and suppose its lateral surface is perfectly insulated along its length. Furthermore, let's assume the material properties of the rod are constant, its thermal diffusivity is* α^2, *and there are no internal heat sources or sinks. Assuming both ends of the rod are insulated and the initial temperature distribution in the rod is given by* $u_0(x) = x$, *find the temperature distribution* $u(x,t)$ *within rod* G.

EXERCISE 10. *Given the same setup as in the previous example, find the temperature distribution* $u(x,t)$ *within rod* G*, under the following conditions:*

i. *The thermal diffusivity* α^2 *of the rod is some known constant.*
ii. *The left end of the rod is held at the fixed constant temperature* $u(0,t) = T_L$*, while the right end is held at the fixed constant temperature* $u(1,t) = T_R$*.*
iii. *The initial temperature distribution within the rod is given by* $u_0(x)$*.*

8. Transverse Vibrations

9. Longitudinal Vibrations

We recall the discussion of the longitudinal vibrations in a long narrow cylindrical rod of cross section area S. The rod is located along the $x-$axis, and we identify it with the interval $[a,b]$ in $\mathbb{R}$. The rod is assumed to stretch or contract in the horizontal direction, and we assume that the vertical plane cross-sections of the rod move only horizontally. Denote by $u(x,t)$ the *displacement* in the positive direction from the point $x \in [a,b]$ at the time $t > 0$. The corresponding *displacement rate* or *velocity* is denoted by $v(x,t) \equiv u_t(x,t)$.

Let $\sigma(x,t)$ denote the local *stress*, the force per unit area with which the part of the rod to the right of the point x acts on the part to the left of x. Since force is positive to the right, the stress is positive in conditions of *tension*. For a section of the rod, $x_1 < x < x_2$, the total (rightward) force acting on that section due to the remainder of the rod is given by

$$\big(\sigma(x_2,t) - \sigma(x_1,t)\big)S \, .$$

If the density of the rod at x is given by $\rho > 0$, the momentum of this section is just

$$\int_{x_1}^{x_2} \rho\, u_t(x,t)\, S\, dx \, .$$

If we let $F(x,t)$ denote any external applied force per unit of volume in the positive $x-$direction, then we obtain from Newton's second law that

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho\, u_t(x,t)\, S\, dx = \big(\sigma(x_2,t) - \sigma(x_1,t)\big)S + \int_{x_1}^{x_2} F(x,t)\, S\, dx$$

for any such $x_1 < x_2$. For a sufficiently smooth displacement $u(x,t)$, we obtain the *conservation of momentum* equation

$$\rho\, u_{tt}(x,t) - \sigma_x(x,t) = F(x,t) \, , \qquad a < x < b, \ t > 0 \, . \tag{54}$$

The *stress* $\sigma(x,t)$ is determined by the type of material of which the rod is composed and the amount by which the neighboring region is stretched or compressed, i.e., on the *elongation* or *strain*, $\varepsilon(x,t)$. In order to define this, first note that a section $[x, x+h]$ of the rod is deformed by the displacement to the new position $[x+u(x), (x+h)+u(x+h)]$. The *elongation* is the limiting increment of the change in the length due to the deformation as given by

$$\lim_{h\to 0} \frac{[u(x+h)+(x+h)] - [u(x)+x] - h}{h} = \frac{d\,u(x)}{dx} \, ,$$

so the strain is given by $\varepsilon(x,t) \equiv u_x(x,t)$.

The relation between the stress and strain is a *constitutive law*, usually determined by experiment, and it depends on the type of material. In the simplest case, with small displacements, we find by experiment that $\sigma(x,t)$ is proportional to $\varepsilon(x,t)$, i.e., that there is a constant k called *Young's modulus* for which

$$\sigma(x,t) = k\,\varepsilon(x,t)\ .$$

The constant k is a property of the material, and in this case we say the material is purely *elastic*.

A rate-dependent component of the stress-strain relationship arises when the force generated by the elongation depends not only on the magnitude of the strain but also on the speed at which it is changed, i.e., on the *strain rate* $\varepsilon_t(x,t) = v_x(x,t)$. The simplest such case is that of a *visco-elastic* material defined by the linear constitutive equation

$$\sigma(x,t) = k\,\varepsilon(x,t) + \mu\,\varepsilon_t(x,t)\ ,$$

in which the material constant μ is the *viscosity* or internal friction of the material. Finally, if we include the effect of the transverse motions of the rod that result from the elongations under conditions of constant volume or mass, we will get an additional term to represent the transverse inertia. If the constant P denotes *Poisson's ratio*, and r is the average radius of that cross section, then the corresponding *stress-strain* relationship is given as before by

$$\sigma(x,t) = k\,\varepsilon(x,t) + \mu\,\varepsilon_t(x,t) + \rho\,r^2\,P\,\varepsilon_{tt}(x,t)\,.$$

In terms of displacement, the total stress is

$$\sigma(x,t) = ku_x(x,t) + \mu u_{xt}(x,t) + \rho\,r^2\,Pu_{xtt}(x,t)\,. \tag{55}$$

The partial differential equation for the longitudinal vibrations of the rod is obtained by substituting (55) into (54).

Initial and Boundary conditions. Since the momentum equation is second–order in time, one may expect that in order to have a well–posed problem, two initial conditions should be specified. Thus, we shall specify the *initial conditions*

$$u(x,0) = u_0(x), \quad u_t(x,t) = v_0(x), \qquad a < x < b\,,$$

where $u_0(\cdot)$ and $v_0(\cdot)$ are the initial displacement and the initial velocity, respectively.

We list a number of typical possibilities for determining the two boundary conditions. Each of these is illustrated as before with a condition at the right end, $x = b$, and we note that another such condition will also be prescribed at the left end, $x = a$.

1. The displacement could be specified at the end point:

$$u(b,t) = d_b(t), \quad t > 0\,.$$

This is the *Dirichlet* boundary condition, or boundary condition of *first type*. It could be obtained from observation of the endpoint position, or it could be imposed directly on the endpoint. The homogeneous case $d_b(t) = 0$ corresponds to a *clamped end*.

2. The horizontal force on the rod could be specified at the end point:

$$\sigma(b,t) = f_b(t), \quad t > 0\,.$$

For the purely elastic case, $\sigma = ku_x$, this is the *Neumann* boundary condition, or boundary condition of *second type*. The homogeneous condition with $f_b(t) = 0$ corresponds to a *free end*.

3. The force on the end is determined by an elastic constraint, a restoring force proportional to the displacement:

$$\sigma(b,t) + k_0(u(b,t) - d_b(t)) = f_b(t), \quad t > 0\,.$$

For the purely elastic case this is the *Robin* boundary condition, or boundary condition of *third type*. Here both $d_b(\cdot)$ and $f_b(\cdot)$ are prescribed. The first is a prescribed displacement of the spring reference, and the second is a horizontal force concentrated on the right end point. For $k_0 \to \infty$, we obtain formally the Dirichlet boundary condition, while for $k_0 \to 0$ we get the Neumann condition. Thus the *effective tension* k_0 interpolates between the first two types.

4. Another type of boundary condition arises if there is a *concentrated mass* at the end point. Then $u(b,t)$ is the displacement of this mass, and we have the *dynamic* boundary condition

$$\rho_0 u_{tt}(b,t) + \sigma(b,t) = f_b(t), \quad t > 0\,,$$

which is the boundary condition of *fourth type* for the elastic case.

10. The Eigenfunction Expansion, II

We shall apply the method of *separation of variables* to the initial-boundary-value problem for longitudinal vibrations with Dirichlet boundary conditions.

EXAMPLE 4. *Suppose the rod $(0,\ell)$ is perfectly elastic and set $\alpha^2 = \frac{k}{\rho}$. Assume there are no internal forces,* i.e., *$f(x,t) = 0$, and that both ends of the rod are fixed. The initial displacement and velocity are given by $u_0(x)$ and $v_0(x)$, respectively. The initial-boundary-value problem for this scenario is*

$$u_{tt}(x,t) = \alpha^2 u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0, \tag{56a}$$

$$u(0,t) = 0,\ u(\ell,t) = 0, \qquad t > 0, \tag{56b}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = v_0(x), \quad 0 < x < \ell. \tag{56c}$$

We look for non-null solutions of the form $u(x,t) = X(x)T(t)$ and find as before that

$$\frac{T''(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}, \qquad 0 < x < \ell,\ t > 0\,.$$

Each side must be equal a common constant, denoted by $-\lambda$, and this leads to the pair of ordinary differential equations

$$\begin{aligned} T''(t) \quad &+ \quad \lambda\,\alpha^2\,T(t) = 0, \quad t > 0, \\ X''(x) \quad &+ \quad \lambda\,X(x) = 0, \quad 0 < x < \ell. \end{aligned}$$

The boundary conditions given in (56b) imply that $X(0) = X(\ell) = 0$.

Note that if $X(\cdot)$ and $T(\cdot)$ are solutions of these respective equations, then it follows directly that their product is a solution (56a). We have already found the non-null solutions of the *boundary-value problem*

$$\left\{\begin{array}{l} X''(x)+\lambda X(x)=0, \quad 0<x<\ell, \\ X(0)=0,\ X(\ell)=0. \end{array}\right. \tag{57}$$

The solutions are the *normalized eigenfunctions*

$$X_n(x)=\sqrt{\frac{2}{\ell}}\sin(\frac{n\pi}{\ell}x)$$

corresponding to the eigenvalues $\lambda_n=(n\pi/\ell)^2$. If we combine these with the corresponding time-dependent solutions $\cos(\alpha\sqrt{\lambda_n}t)$ and $\sin(\alpha\sqrt{\lambda_n}t)$ of the first differential equation and take linear combinations, we obtain a large class of solutions of the partial differential equation (56a) and boundary conditions (56b) in the form of a series

$$u(x,t)=\sum_{n=1}^{\infty}\big(A_n\cos(\alpha\sqrt{\lambda_n}t)+B_n\sin(\alpha\sqrt{\lambda_n}t)\big)X_n(x),$$

where the sequences $\{A_n\}$ and $\{B_n\}$ are to be determined. From the initial conditions (56c), it follows that these coefficients must satisfy

$$\sum_{n=1}^{\infty}A_nX_n(x)=u_0(x),\ \sum_{n=1}^{\infty}B_n\alpha\sqrt{\lambda_n}X_n(x)=v_0(x), \quad 0<x<\ell,$$

so we obtain

$$A_m=(u_0(\cdot),X_m(\cdot)),\ B_m=\frac{(v_0(\cdot),X_m(\cdot))}{\alpha\sqrt{\lambda_m}}, \quad m\geq 1\,.$$

In summary, the solution of the initial-boundary-value problem (56) is given by the series

$$u(x,t)=\sum_{n=1}^{\infty}\Big(\cos(\alpha\sqrt{\lambda_n}t)(u_0(\cdot),X_n(\cdot))+\sin(\alpha\sqrt{\lambda_n}t)\frac{(v_0(\cdot),X_n(\cdot))}{\alpha\sqrt{\lambda_n}}\Big)X_n(x).$$

Denote the second term in the preceding formula by

$$[S(t)v_0](x)=\sum_{n=1}^{\infty}\Big(\sin(\alpha\sqrt{\lambda_n}t)\frac{(v_0(\cdot),X_n(\cdot))}{\alpha\sqrt{\lambda_n}}\Big)X_n(x).$$

This defines the operator $S(t)$ on the space of functions on $[0,\ell]$. We can use this operator to represent the solution by

$$u(\cdot,t)=S'(t)u_0+S(t)v_0\,. \tag{58}$$

EXAMPLE 5. *Suppose the rod $(0,\ell)$ is perfectly elastic and set $\alpha^2=\frac{k}{\rho}$. Assume that both ends of the rod are fixed, the initial displacement and velocity are both null, and that* *there are distributed forces $f(x,t)$ along its length. The initial-boundary-value problem for this case is*

$$u_{tt}(x,t) = \alpha^2 u_{xx}(x,t) + f(x,t), \quad 0 < x < \ell, \quad t > 0, \tag{59a}$$

$$u(0,t) = 0, \; u(\ell,t) = 0, \qquad t > 0, \tag{59b}$$

$$u(x,0) = 0, \; u_t(x,0) = 0, \quad 0 < x < \ell. \tag{59c}$$

For this problem with a non-homogeneous partial differential equation (59a), we look for the solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x). \tag{60}$$

If we assume that $f(x,t)$ has the eigenfunction expansion

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) X_n(x)\,,$$

then the coefficients are given by

$$f_n(t) \equiv \int_0^{\ell} f(s,t) X_n(s)\, ds\,,$$

and substituting this expansion into equation (59a) yields

$$\sum_{n=1}^{\infty} \left[\ddot{u}_n(t) + \lambda_n \alpha^2 u_n(t) \right] X_n(x) = \sum_{n=1}^{\infty} f_n(t) X_n(x)\,.$$

By equating the coefficients of the series given in this last equation, we are led to the sequence of initial-value problems

$$\ddot{u}_n(t) + \lambda_n \alpha^2 u_n(t) = f_n(t), \qquad t > 0 \tag{61a}$$

$$u_n(0) = 0, \; \dot{u}_n(0) = 0\,. \tag{61b}$$

The solution to (61) is

$$u_n(t) = \int_0^t \frac{\ell}{n\pi\alpha} \sin\left(\frac{n\pi\alpha}{\ell}(t-\tau)\right) f_n(\tau)\, d\tau$$

Now, if we use this in (60), we find that the solution to our non-homogeneous initial-boundary-value problem (46) is

$$u(x,t) = \int_0^t \sum_{n=1}^{\infty} \frac{\ell}{n\pi\alpha} \sin\left(\frac{n\pi\alpha}{\ell}(t-\tau)\right) f_n(\tau) X_n(x)\, d\tau. \tag{62}$$

Note that we can use the operator $S(t)$ to represent this formula as

$$u(\cdot,t) = \int_0^t S(t-\tau) f(\tau)\, d\tau. \tag{63}$$

As before, each $S(t)$ is an integral operator of the form

$$\begin{aligned}
[S(t)v_0](x) &= \sum_{n=1}^{\infty} \frac{\ell}{n\pi\alpha} \sin(\frac{n\pi\alpha}{\ell}t) \int_0^\ell (v_0(s)\ \sin(\frac{n\pi}{\ell}s))\, ds \frac{2}{\ell} \sin(\frac{n\pi}{\ell}x) \\
&= \int_0^\ell \frac{2}{\ell}\Big(\sum_{n=1}^{\infty} \frac{\ell}{n\pi\alpha} \sin(\frac{n\pi\alpha}{\ell}t) \sin(\frac{n\pi}{\ell}s) \sin(\frac{n\pi}{\ell}x)\Big) v_0(s)\, ds \\
&= \int_0^\ell H(x,s,t) v_0(s)\, ds
\end{aligned}$$

for which the kernel

$$H(x,s,t) = \frac{2}{\ell}\Big(\sum_{n=1}^{\infty} \frac{\ell}{n\pi\alpha} \sin(\frac{n\pi\alpha}{\ell}t) \sin(\frac{n\pi}{\ell}s) \sin(\frac{n\pi}{\ell}x)\Big)$$

is the *Green's function* for the problem.

EXAMPLE 6. *Suppose the left end of the elastic rod is free while the right end has an elastic constraint given by* $u(\ell,t) + u_x(\ell,t) = 0$. *The initial-boundary-value problem for this situation is*

(64a) $$u_{tt}(x,t) = \alpha^2 u_{xx}(x,t), \qquad 0 < x < \ell, \quad t > 0,$$

(64b) $$u_x(0,t) = 0, \quad u(\ell,t) + u_x(\ell,t) = 0, \qquad t > 0,$$

(64c) $$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), \qquad 0 < x < \ell.$$

We seek a solution in the form

$$u(x,t) = X(x)T(t),$$

where the boundary conditions imply that $X_x(0) = 0$ and $X(\ell)+X_x(\ell) = 0$. The method of separation of variables leads us to the boundary-value problem

$$\begin{aligned}
X_{xx}(x) + \lambda X(x) &= 0, \qquad 0 < x < \ell, \\
X'(0) = 0,\ X(\ell) + X'(\ell) &= 0.
\end{aligned}$$

We obtain a sequence of eigenvalues λ_n and corresponding eigenfunctions given by

(65) $$X_n(x) = \Big(\frac{2}{\ell}\Big)^{1/2} \cos(\lambda_n^{1/2} x).$$

Note that for large λ_n we have

$$\lambda_n \approx (\frac{n\pi}{\ell})^2,$$

so the eigenvalues are asymptotically close to those of the preceding example. Combining these results with the time-dependent solutions and using the orthogonality of the eigenfunctions, we find solutions of the initial-boundary-value problem (64) in the form

$$u(x,t) = \sum_{n=1}^{\infty} \Big(\cos(\alpha\sqrt{\lambda_n}t)(u_0(\cdot), X_n(\cdot)) + \sin(\alpha\sqrt{\lambda_n}t)\frac{(v_0(\cdot), X_n(\cdot))}{\alpha\sqrt{\lambda_n}}\Big) X_n(x),$$

with the eigenfunctions given by (65). Once again, this can be represented in the form (58) for an appropriate family of operators $\{S(t) : t \geq 0\}$.

EXERCISE 11. *Suppose the rod* $(0, \ell)$ *is elastic and that we account for the inertia of lateral extension. Set* $\alpha^2 = \frac{k}{\rho}$ *and* $\beta^2 = r^2 P$*, where* P *is Poisson's ratio and* r *is the average radius of a cross section as above. Assume there are no internal forces,* i.e., $f(x,t) = 0$*, and that both ends of the rod are fixed. The initial displacement and velocity are given by* $u_0(x)$ *and* $v_0(x)$*, respectively. The initial-boundary-value problem is*

$$u_{tt}(x,t) = \alpha^2 u_{xx}(x,t) + \beta^2 u_{xxtt}(x,t), \quad 0 < x < \ell, \quad t > 0, \tag{66a}$$

$$u(0,t) = 0,\ u(\ell,t) = 0, \qquad t > 0, \tag{66b}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = v_0(x), \quad 0 < x < \ell. \tag{66c}$$

Find the solution by separation of variables. Find the family of operators $\{S(t) : t \geq 0\}$ *for which this can be represented in the form* (58).

11. Duhamel Formulae: Variation of Parameters

We would like to investigate further the structure of the solutions of both the first and second order equations that we found above. The point is that the forms we found above are indeed quite general, and we find in each case that the formula for a solution of the general non-homogeneous problem can be written down immediately, once we know the formula for a *basic* problem.

11.1. First Order Equations. Suppose that we know the *basic initial-value problem* for the first-order evolution equation

$$\dot{u}(t) + Au(t) = 0, \qquad u(0) = \varphi,$$

is well-posed, that is, that there exists exactly one solution of this problem for each choice of the initial function φ. This defines the *semigroup* of operators $\{E_A(t)\}$ by

$$u(t) \equiv E_A(t)\varphi, \quad t \geq 0\,.$$

In particular cases, this permits us to write a formula for $E_A(\cdot)$ as an integral operator with an explicit kernel. Now suppose that we have a solution $u(\cdot)$ of the more general problem with a nonhomogeneous equation,

$$\dot{u}(t) + Au(t) = f(t), \qquad u(0) = \varphi.$$

From the formal computation

$$\frac{d}{d\tau} E_A(t-\tau)u(\tau) = E_A(t-\tau)\{\dot{u}(\tau) + Au(\tau)\} = E_A(t-\tau)f(\tau)\,,$$

and an integration in time, we obtain

$$u(t) = E_A(t)\varphi + \int_0^t E_A(t-\tau)f(\tau)\,d\tau\,.$$

This is just the form (51). In particular, we can use our formula for the integral operators $E_A(\cdot)$ to obtain an explicit representation and verify directly that this formula gives the solution of the non-homogeneous problem.

11.2. Second Order Equation. Now for the second-order evolution equation, we suppose that the *basic initial-value problem*

$$\ddot{w}(t) + Aw(t) = 0, \qquad w(0) = 0, \quad \dot{w}(0) = \psi,$$

is well-posed. This defines the operators $\{S_A(t)\}$ by

$$w(t) \equiv S_A(t)\psi\,.$$

Note first that the derivative of these operators can be used to represent the solution of the corresponding problem

$$\ddot{w}(t) + Aw(t) = 0, \qquad w(0) = \psi, \quad \dot{w}(0) = 0,$$

by the formula

$$w(t) = S'_A(t)\psi\,.$$

Now consider the general initial-value problem with non-homogeneous data in the form

$$\ddot{w}(t) + Aw(t) = f(t), \qquad w(0) = \varphi, \quad \dot{w}(0) = \psi.$$

We make the formal computation

$$\frac{d}{d\tau}\{S_A(t-\tau)\dot{w}(\tau) + S'_A(t-\tau)w(\tau)\} = \\ S_A(t-\tau)\{\ddot{w}(\tau) + Aw(\tau)\} = S_A(t-\tau)f(\tau),$$

and then an integration in time yields the representation

$$w(t) = S'_A(t)\varphi + S_A(t)\psi + \int_0^t S_A(t-\tau)f(\tau)\,d\tau.$$

This is just the combination of the two formulae (58) and (63). Again, we find that if we find the formula for the integral operators $S_A(\cdot)$ which gives the explicit representation of the solution to the *basic* initial-value problem, then the corresponding formula for the solution of the general non-homogeneous problem can be written immediately.

EXERCISE 12. *Suppose the rod $(0,\ell)$ is perfectly elastic and set $\alpha^2 = \frac{k}{\rho}$. Assume that the left end of the rod is fixed, the right end is free, and the initial displacement is null. The initial-boundary-value problem for this situation is*

$$\begin{aligned} u_{tt}(x,t) &= \alpha^2 u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0, \\ u(0,t) &= 0,\ u_x(\ell,t) = 0, \qquad\qquad\quad t > 0, \\ u(x,0) &= 0,\ u_t(x,0) = v_0(x), \quad 0 < x < \ell. \end{aligned}$$

Find the kernel $H(x,s,t)$ for which the solution is given by the integral operator

$$u(x,t) = \int_0^\ell H(x,s,t)v_0(s)\,ds\,.$$

Find a corresponding formula for the solution of the non-homogeneous problem with given initial displacement, velocity, and distributed forces $f(x,t)$ along its length:

$$\begin{aligned} u_{tt}(x,t) &= \alpha^2 u_{xx}(x,t) + f(x,t), \quad 0 < x < \ell, \quad t > 0, \\ u(0,t) &= 0,\ u_x(\ell,t) = 0, \qquad\qquad\qquad t > 0, \\ u(x,0) &= u_0(x),\ u_t(x,0) = v_0(x), \quad 0 < x < \ell. \end{aligned}$$