DUHAMEL

First Order Equation

The initial-value problem

$$\dot u(t) + Au(t) = 0, \qquad u(0) = \varphi, \qquad u|_\Gamma(t) = 0,$$

is well-posed. This defines the semigroup $\{E(t)\}$ by

$$u(t) \equiv E(t)\varphi$$

$$\dot u(t) + Au(t) = f(t), \qquad u(0) = \varphi, \qquad u|_\Gamma(t) = 0.$$

Note: $\frac{d}{d\tau}E(t-\tau)u(\tau) = E(t-\tau)\{\dot u(\tau) + Au(\tau)\} = E(t-\tau)f(\tau)$,
and an integration yields

$$u(t) = E(t)\varphi + \int_0^t E(t-\tau)f(\tau)\,d\tau.$$

$$\dot u(t) + Au(t) = 0, \qquad u(0) = 0, \qquad u|_\Gamma(t) = g(t).$$

Let $A(u_g(t)) = 0, \quad \gamma(u_g(t)) = g(t)$. Define $w(t) = u(t) - u_g(t)$ Then

$$\dot w(t) + Aw(t) = -\dot u_g(t), \qquad u(0) = -u_g(0), \qquad u|_\Gamma(t) = 0,$$

and we have from above

$$\begin{aligned} w(t) &= -E(t)u_g(0) - \int_0^t E(t-\tau)\dot u_g(\tau)\,d\tau \\ &= -u_g(t) - \int_0^t E'(t-\tau)u_g(\tau)\,d\tau\,. \end{aligned}$$

$$u(t) = -\int_0^t E'(t-\tau)u_g(\tau)\,d\tau = \int_0^t AE(t-\tau)u_g(\tau)\,d\tau$$

Second Order Equation

The initial-value problem

$$\ddot{w}(t) + Aw(t) = 0, \qquad w(0) = 0, \quad \dot{w}(0) = \psi, \qquad w|_\Gamma(t) = 0,$$

is well-posed. This defines the operators $\{S(t)\}$ by

$$w(t) \equiv S(t)\psi\,.$$

The derivative of these operators can be used to represent the solution of

$$\ddot{w}(t) + Aw(t) = 0, \qquad w(0) = \varphi, \quad \dot{w}(0) = 0, \qquad w|_\Gamma(t) = 0,$$

by the formula

$$w(t) = S'(t)\varphi\,.$$

$$\ddot{w}(t) + Aw(t) = f(t), \qquad w(0) = \varphi, \quad \dot{w}(0) = \psi, \qquad w|_\Gamma(t) = 0.$$

Note:

$$\frac{d}{d\tau}\{S(t-\tau)\dot{w}(\tau) + S'(t-\tau)w(\tau)\} = \\ S(t-\tau)\{\ddot{w}(\tau) + Aw(\tau)\} = S(t-\tau)f(\tau),$$

and an integration yields

$$w(t) = S'(t)\varphi + S(t)\psi + \int_0^t S(t-\tau)f(\tau)\,d\tau.$$

$$\ddot{w}(t) + Aw(t) = 0, \qquad w(0) = 0, \quad \dot{w}(0) = 0, \qquad w|_\Gamma(t) = g(t).$$

Let $A(w_g(t)) = 0, \quad \gamma(w_g(t)) = g(t)$. Define $u(t) = w(t) - w_g(t)$. Then

$$\ddot{u}(t) + Au(t) = -\ddot{w}_g(t), \qquad u(0) = -w_g(0), \quad \dot{u}(0) = -\dot{w}_g(0), \qquad u|_\Gamma(t) = 0,$$

so we have from above

$$\begin{aligned} u(t) &= -S'(t)w_g(0) - S(t)\dot{w}_g(0) - \int_0^t S(t-\tau)\ddot{w}_g(\tau)\,d\tau \\ &= -S'(t)w_g(0) - \int_0^t S'(t-\tau)\dot{w}_g(\tau)\,d\tau \\ &= -S'(0)w_g(t) + \int_0^t AS(t-\tau)w_g(\tau)\,d\tau \end{aligned}$$

$$w(t) = u(t) + w_g(t) = \int_0^t AS(t-\tau)w_g(\tau)\,d\tau$$