1. Homogenization

Let Ω be a smoothly bounded region and let Y be the unit cube in $\mathbb{R}^N$. Assume the coefficient function $a(\cdot)$ is defined on $\mathbb{R}^N$ and is Y-periodic. Let $\varepsilon > 0$. We want to approximate the solution to the *singular problem*

$$(1.1)\quad u^\varepsilon \in H_0^1(\Omega) : \quad \int_\Omega a(x/\varepsilon)\nabla u^\varepsilon(x)\cdot\nabla\varphi(x)\,dx = \int_\Omega F(x)\varphi(x)\,dx \text{ for all } \varphi \in H_0^1(\Omega).$$

We seek an approximation in the form

$$u^\varepsilon(x) = u(x, x/\varepsilon) + \varepsilon U(x, x/\varepsilon) + \mathcal{O}(\varepsilon^2)$$

in which each $u(x,\cdot)$ and $U(x,\cdot)$ is Y-periodic. We call x the *slow variable* and $y = x/\varepsilon$ the *fast variable*. Note that the gradient is given (formally) by

$$\nabla u^\varepsilon(x) = \nabla u(x, x/\varepsilon) + \frac{1}{\varepsilon}\nabla_y u(x, x/\varepsilon) + \nabla_y U(x, x/\varepsilon) + \mathcal{O}(\varepsilon)$$

by the chain rule $\nabla u(x,y) = \nabla_x u(x,y) + \nabla_y u(x,y)\frac{\partial y}{\partial x}$ with $y = x/\varepsilon$ and $\nabla = \nabla_x$. Since the solution u^ε of (1.1) has a bounded gradient, it follows that $\nabla_y u(x,y) = 0$, so we have

$$(1.2a)\quad u^\varepsilon(x) = u(x) + \varepsilon U(x, x/\varepsilon) + \mathcal{O}(\varepsilon^2)$$

$$(1.2b)\quad \nabla u^\varepsilon(x) = \nabla u(x) + \nabla_y U(x, x/\varepsilon) + \mathcal{O}(\varepsilon)$$

We will substute (1.2) into (1.1) with a test function of the same form, namely,

$$\varphi(x) + \varepsilon\Phi(x, x/\varepsilon)$$

where $\Phi(x,y)$ is Y-periodic for each $x \in \Omega$. Ignoring terms $\mathcal{O}(\varepsilon)$ but testing over these oscillations of order εY give the system

$$(1.3)\quad u \in H_0^1(\Omega),\quad U \in L^2(\Omega, H^1_\#(Y)) :$$
$$\int_\Omega\int_Y a(y)(\nabla u(x) + \nabla_y U(x,y))\cdot(\nabla\varphi(x) + \nabla_y\Phi(x,y))\,dy\,dx = \int_\Omega F(x)\varphi(x)\,dx$$
$$\text{for all } \varphi \in H_0^1(\Omega),\quad \Phi \in L^2(\Omega, H^1_\#(Y)).$$

Next we decouple the system (1.3). First set $\phi = 0$ to obtain the periodic boundary-value problem

$$(1.4a)\quad U \in L^2(\Omega, H^1_\#(Y)) :$$
$$\int_\Omega\int_Y a(y)(\nabla_y U(x,y) + \nabla u(x))\cdot\nabla_y\Phi(x,y)\,dy\,dx = 0$$
$$\text{for all } \Phi \in L^2(\Omega, H^1_\#(Y)).$$

Note that the input is $\nabla u(x)$, and this is independent of the fast variable, y. The solution $U(x,\cdot)$ is determined up to a constant (with respect to y), so the output $\nabla_y U(x,\cdot)$ is uniquely determined. Then by setting $\Phi = 0$ in (1.3) we obtain the variational boundary-value problem

(1.4b) $u \in H_0^1(\Omega)$:

$$\int_\Omega \int_Y a(y)(\nabla u(x) + \nabla_y U(x,y))\,dy \cdot \nabla\varphi(x)\,dx = \int_\Omega F(x)\varphi(x)\,dx$$

for all $\varphi \in H_0^1(\Omega)$.

The boundary-value problem (1.4a) is the *micro-problem* on $\Omega \times Y$, and (1.4b) is the *macro-problem* on Ω. This *homogenized system* (1.4) is equivalent to (1.3).

Finally we obtain a single equation to describe the solution u to the macro-problem by representing the solution U to the micro-problem in terms of u. The micro-problem (1.4a) is equivalent to requiring for almost every $x \in \Omega$ that

$$U(x,\cdot) \in H_\#^1(Y) : \int_Y a(y)(\nabla_y U(x,y) + \nabla u(x)) \cdot \nabla_y \Phi(y)\,dy = 0 \text{ for all } \Phi \in H_\#^1(Y).$$

In order to represent the solution, we define $W_i(y)$ for each $1 \leq i \leq N$ to be the solution of the *cell problem*

$$W_i \in H_\#^1(Y) : \int_Y a(y)(\nabla_y W_i(y) + \mathbf{e}_i) \cdot \nabla_y \Phi(y)\,dy = 0 \text{ for all } \Phi \in H_\#^1(Y), \tag{1.5}$$

where $\mathbf{e}_i$ is the indicated coordinate vector in $\mathbb{R}^N$. Then by linearity the solution of (1.4a) is given by

$$U(x,y) = \sum_{i=1}^{i=N} \partial_i u(x) W_i(y).$$

This is substituted into (1.4b) to obtain

$$u \in H_0^1(\Omega) : \quad \int_\Omega a_{ij}\partial_i u(x) \cdot \partial_j \varphi(x)\,dx = \int_\Omega F(x)\varphi(x)\,dx \quad \text{for all } \varphi \in H_0^1(\Omega), \tag{1.6}$$

where the *constant* coefficients are given by

$$a_{ij} = \int_Y a(y)\left(\delta_{ij} + \partial_j W_i(y)\right)\,dy\,. \tag{1.7}$$

The elliptic boundary-value problem (1.6) is the *homogenized equation* whose constant coefficients are given by (1.7). It's solution gives the first term in the expansion (1.2), and the second term is determined by the solution of (1.4a), so these together give the approximations

$$\begin{aligned} u^\varepsilon(x) &= u(x) + \varepsilon\,\nabla u(x) \cdot (W_1(x/\varepsilon), W_2(x/\varepsilon), \ldots, W_N(x/\varepsilon)) + \mathcal{O}(\varepsilon^2)\,, \\ \nabla u^\varepsilon(x) &= \nabla u(x) + \nabla u(x) \cdot (\nabla W_1(x/\varepsilon), \nabla W_2(x/\varepsilon), \ldots, \nabla W_N(x/\varepsilon)) + \mathcal{O}(\varepsilon) \end{aligned}$$

for the solution u^ε of the singular problem (1.1).

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