J. Math. Anal. Appl. 365 (2010) 320–331

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The narrow fracture approximation by channeled flow ☆

Fernando Morales, R.E. Showalter *

Department of Mathematics, Oregon State University, Corvallis, OR 97331-4605, United States

ARTICLE INFO

Article history:
Received 14 April 2009
Available online 21 October 2009
Submitted by P. Sacks

Keywords:
Porous media
Heterogeneous
Narrow fracture
Singular Darcy system

ABSTRACT

The singular problem of non-stationary Darcy flow in a region containing a narrow channel of width $\mathcal{O}(\epsilon)$ and high permeability $O(\frac{1}{\epsilon})$ is shown to be well approximated by a problem with flow concentrated on a weighted Sobolev space over a lower-dimensional interface. The convergence of the solution as $\epsilon \to 0$ is proved for both the stationary case and the corresponding initial-boundary-value problem. The structure of the limiting problems is dependent on the rate of taper of the channel at its extremities.

Contents

1. Introduction

Fluid flow through a fully-saturated porous medium is altered in the vicinity of a rigid wall by the sharp rise in permeability due to the inefficiency of the packing of the particles in the vicinity of the wall. Consequently, in a narrow region close to the wall the velocity is substantially higher and the flow is predominantly parallel to it; this phenomenon is known as the *channeling effect* [9]. Related models were used previously to describe flow through a porous medium in the vicinity of a narrow fracture which is characterized similarly as a thin interior region of high permeability. Such problems arise

☆ This material is based in part upon work supported by grant 98089 from the Department of Energy, Office of Science.
* Corresponding author.
E-mail addresses: moralefe@math.oregonstate.edu (F. Morales), show@math.oregonstate.edu (R.E. Showalter).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.10.042

e.g. from hydraulic fracturing in which narrow channels of high permeability are created in the vicinity of a well to enhance the flow rate and consequently the production. The *narrow fracture approximation* leads to a model like the one above for thin channel flow, and by taking advantage of the symmetry about the center surface defining the fracture, one can reduce such a problem to one of the type considered here with the high-permeability region located on the boundary [2,6]. Analogous models of heat conduction arise from regions of high conductivity, and these may also include a *concentrated capacity*. We include these in the discussion for comparison.

For a final example, we mention saturated gravity-driven flow of subsurface water through a hillslope bounded below by sloping bedrock. A network of narrow channels of very high permeability develops in the vicinity of the impermeable bedrock, and it is observed that most of the fluid in the system flows through this region. Such systems with high flow rate over narrow regions greatly influence the transport and flow processes and are a topic of current study [16].

We shall describe such situations with Darcy flow for which the permeability is scaled to balance the channel width and model the higher flow rates in the channel. Due to the higher permeability, the fluid flows primarily into and then tangential to the channel. The resulting model captures the tangential boundary flow coupled to the interior flow by continuity of flux and pressure. It contains two sources of singularity: a geometric one from the thinness of the channel and a material one due to the higher permeability of the channel. With the appropriate scaling, these two singularities are balanced, and a fully-coupled model is obtained in the limit as an approximation. See [4,10] for asymptotic analysis and [6] for numerical analysis of these and related models.

An additional challenging issue is to account for the *shape* of the channel, especially for any *taper* near the edges or boundary of the channel. Such shapes are ubiquitous in applications, but they are not commonly included in the modeling process. They are important because the rate of the tapering at the edges determines the appropriate boundary conditions (or lack thereof) that describe the resulting model [8,12].

The geometry of the model is described first. Let Ω_1 be a bounded domain in $\mathbb{R}^n$ and denote by Γ a relatively-open connected portion of its boundary $\partial\Omega_1$ along the top of the domain. For simplicity of representation, we assume this portion of the boundary is *flat*, that is, $\Gamma \subset \mathbb{R}^{n-1} \times \{0\}$ and that $x_n < 0$ for each $x = (\tilde{x}, x_n) \in \Omega_1$, where $\tilde{x} \in \mathbb{R}^{n-1}$. The channel is realized as a region of the form $\Omega_2^\epsilon = \{(\tilde{x}, \omega(\tilde{x})x_n)\colon\ (\tilde{x}, x_n) \in \Gamma \times (0,\epsilon)\}$. The function $\omega(\cdot)$ shapes the width of the channel at each $\tilde{x} \in \Gamma$, and the parameter $\epsilon > 0$ denotes its scale. We assume that this width function satisfies $0 < a \leqslant \omega(\tilde{x}) \leqslant 1$ on each compact subset of Γ, where a depends on the set, but it may approach zero near $\partial\Gamma$ at a rate to be determined below. This assumption permits the channel to be tapered or to *pinch off* near its extremities.

For the single-phase flow of a slightly compressible fluid through $\Omega^\epsilon \equiv \Omega_1 \cup \Gamma \cup \Omega_2^\epsilon$, Darcy's law together with conservation of fluid mass lead to the interface problem

$$
\begin{aligned}
& m_1 \frac{\partial u_1^\epsilon}{\partial t} - \nabla \cdot k_1 \nabla u_1^\epsilon = m_1 f \quad \text{in } \Omega_1, \\
& u_1^\epsilon = 0 \quad \text{on } \partial\Omega_1 - \Gamma, \\
& u_1^\epsilon = u_2^\epsilon, \qquad k_1 \partial_z u_1^\epsilon - \frac{k_2}{\epsilon} \partial_z u_2^\epsilon = g \quad \text{on } \Gamma, \\
& m_2 \frac{\partial u_2^\epsilon}{\partial t} - \nabla \cdot \frac{k_2}{\epsilon} \nabla u_2^\epsilon = m_2 f \quad \text{in } \Omega_2^\epsilon, \\
& \frac{k_2}{\epsilon} \left(\nabla u_2^\epsilon\right) \cdot \hat{n} = 0 \quad \text{on } \partial\Omega_2^\epsilon - \Gamma,
\end{aligned}
\tag{1.1a}
$$

at each $t > 0$ for the fluid density $u_1^\epsilon(\cdot, t)$ in Ω_1 and $u_2^\epsilon(\cdot, t)$ in Ω_2^ϵ, and these satisfy the initial conditions

$$
u_1(\cdot, 0) = u_1^0(\cdot) \quad \text{on } \Omega_1, \qquad u_2(\cdot, 0) = u_2^0(\cdot) \quad \text{on } \Omega_2^\epsilon. \tag{1.1b}
$$

Thus, the region is drained along $\partial\Omega_1 - \Gamma$ and there is no flow across $\partial\Omega_2^\epsilon - \Gamma$, where the outward normal is indicated by $\hat{n}$. This latter condition would follow if the region were symmetric about $\Gamma \times \{\epsilon\}$. The given initial density distributions $u_j^0(\cdot)$ complete the initial-boundary-value problem. Corresponding non-homogeneous problems with known pressure on $\partial\Omega_1 - \Gamma$ and flow-rate along $\partial\Omega_2^\epsilon - \Gamma$ can be reduced to this case. The permeability in Ω_2^ϵ has been scaled by $\frac{1}{\epsilon}$ to indicate the high flow rate, and this will be shown to balance the width ϵ of the channel, so the flow in Ω_2^ϵ is closely approximated by surface flow along Γ. It will be seen below that k_2 is the *effective tangential permeability* and $\frac{k_2}{\epsilon^2}$ is the *effective normal permeability* for channel flow; see [6] for substantial discussion and further perspective. The coefficients m_1, m_2 are obtained from the *porosity* and from the *compressibility* of either the fluid or the medium. We include for comparison the concentrated capacity model in which also m_2 is scaled by $\frac{1}{\epsilon}$, but this has nothing to do with porous media.

2. Preliminaries

We use standard notation and results on function spaces. $L^2(\Omega)$ is the Hilbert space of (equivalence classes of) Lebesgue square summable functions on Ω, and $H^m(\Omega)$, $m \geqslant 1$, with the norm $\|\cdot\|_{m,\Omega}$ is the Sobolev space of functions in $L^2(\Omega)$ for which each weak derivative up to order m belongs to $L^2(\Omega)$. The space $H_0^1(\Omega)$ is the closure in $H^1(\Omega)$ of those infinitely differentiable functions which have compact support in Ω. The *trace* $\gamma(v)$ of a $v \in H^1(\Omega)$ is its boundary value in $H^{1/2}(\partial\Omega)$. The spaces with fractional exponents are defined by interpolation. Corresponding spaces of vector-valued functions are denoted by bold-face symbols, $\mathbf{L}^2(\Omega)$, $\mathbf{H}^m(\Omega)$. The space of those functions of $\mathbf{L}^2(\Omega)$ whose divergence belongs to $L^2(\Omega)$ is denoted by $\mathbf{L}^2_{div}(\Omega)$. These have a *normal trace* on the boundary. See [1,13–15].

Assume the interface Γ is an open bounded connected subset of $\mathbb{R}^{n-1}$ and that it lies locally on one side of its boundary, $\partial\Gamma$, a C^1 manifold. Let $\delta(\tilde{x})$ be the distance from $\tilde{x} \in \Gamma$ to $\partial\Gamma$ and $0 \leqslant \alpha < 1$. Define $W(\alpha)$ to be the space obtained by completing $H^1(\Gamma)$ in the weaker norm

$$\|v\|_{W(\alpha)} = \left\{ \int_\Gamma \left(v(\tilde{x})^2 + \delta(\tilde{x})^\alpha \left\| \tilde{\nabla} v(\tilde{x}) \right\|^2 \right) d\tilde{x} \right\}^{1/2}.$$

Here and in the following, $\tilde{\nabla}$ denotes the $\mathbb{R}^{n-1}$-gradient in directions tangent to Γ. It is known that the embedding $W(\alpha) \to L^2(\Gamma)$ is compact and the trace operator $\gamma : W(\alpha) \to L^2(\partial\Gamma)$ is continuous [3,7]. Here we assume the width function satisfies

$$\omega(\tilde{x}) \geqslant c\delta^\alpha(\tilde{x}) \quad \text{a.e. } \tilde{x} \in \Gamma \tag{2.2}$$

for some $c > 0$, and we say Γ is *weakly tapered*. Then define $H^1_\omega(\Gamma)$ to be the completion of $H^1(\Gamma)$ with the norm

$$\|v\|_{H^1_\omega} = \left\{ \int_\Gamma \left(v(\tilde{x})^2 + \omega(\tilde{x}) \left\| \tilde{\nabla} v(\tilde{x}) \right\|^2 \right) d\tilde{x} \right\}^{1/2}.$$

As above, the embedding $H^1_\omega(\Gamma) \to L^2(\Gamma)$ is compact and the trace operator $\gamma : H^1_\omega(\Gamma) \to L^2(\partial\Gamma)$ is continuous. More generally, we have the following [12].

Theorem 2.1. *Let the bounded domain Γ be given as above and let $0 \leqslant \alpha < 1$. Suppose there is a function $\alpha(\cdot)$ on $\partial\Gamma$ for which $0 \leqslant \alpha(\tilde{x}) \leqslant \alpha$ for each $\tilde{x} \in \partial\Gamma$. Assume the function $\omega(\cdot)$ satisfies* (2.2) *and that at each point of $\partial\Gamma$ there is a neighborhood N in $\mathbb{R}^{n-1}$ and constants $0 < c(N) < C(N)$ such that*

(1) *for each $\tilde{x} \in N \cap \Gamma$ there is an $\tilde{x}_0 \in \partial\Gamma$ such that $\|\tilde{x}_0 - \tilde{x}\| = \delta(\tilde{x})$, and*
(2) *for each $\tilde{x} \in N \cap \Gamma$, $c(N) \leqslant \frac{\omega(\tilde{x})}{\delta(\tilde{x})^{\alpha(\tilde{x}_0)}} \leqslant C(N)$.*

Then the trace map is continuous from $H^1_\omega(\Gamma)$ into $L^2(\partial\Gamma)$, its kernel is the closure of $C_0^\infty(\Gamma)$ in $H^1_\omega(\Gamma)$, and the range is dense in $L^2(\partial\Gamma)$.

In the contrary case we call Γ *strongly tapered* if

$$\omega(\tilde{x}) \leqslant C\delta(\tilde{x}) \quad \text{a.e. } \tilde{x} \in \Gamma, \tag{2.3}$$

for some $C > 0$, and then $C_0^\infty(\Gamma)$ is dense in $H^1_\omega(\Gamma)$, so $H^1_\omega(\Gamma)'$ is a space of distributions on Γ and $L^2(\Gamma) \subset H^1_\omega(\Gamma)'$.

We recall some classical results for unbounded operators and the Cauchy problem; see [5,13] or the first chapter of [14] for details. Let V be a Hilbert space, and denote its dual space of continuous linear functionals by V'. A bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is *V-elliptic* if there is $c_0 > 0$ for which

$$a(u, u) \geqslant c_0 \|u\|_V^2, \quad u \in V.$$

The Lax–Milgram theorem shows this is a convenient sufficient condition for the associated problem to be well-posed.

Theorem 2.2. *If $a(\cdot,\cdot)$ is bilinear, continuous and V-elliptic, then for each $f \in V'$ there is a unique*

$$u \in V: \quad a(u, v) = f(v), \quad v \in V.$$

An unbounded linear operator $A : D \to H$ with domain D in the Hilbert space H is *accretive* if

$$(Ax, x)_H \geqslant 0, \quad x \in D,$$

and it is *m-accretive* if, in addition, $A + I$ maps D onto H. Sufficient conditions for the initial-value problem to be well-posed are provided by the Hille–Yoshida theorem.

Theorem 2.3. *Let the operator $A : D \to H$ be m-accretive on the Hilbert space H. Then for every $u^0 \in D(A)$ and $f \in C^1([0,\infty), H)$ there is a unique solution $u \in C^1([0,\infty), H)$ of the initial-value problem*

$$\frac{du}{dt}(t) + Au(t) = f(t), \quad t > 0, \quad u(0) = u^0. \tag{2.4}$$

If additionally A is self-adjoint, then for each $u^0 \in H$ and Hölder continuous $f \in C^\beta([0,\infty), H)$, $0 < \beta < 1$, there is a unique solution $u \in C([0,\infty), H) \cap C^1((0,\infty), H)$ of (2.4).

Finally, the standard finite-difference approximation of (2.4) leads to the *stationary problem* with $\lambda > 0$,

$$u \in D(A): \quad \lambda u + A(u) = \lambda F \quad \text{in } H,$$

for the resolvent of the operator A. It is precisely the m-accretive operators for which this problem is always solvable with $\|u\|_H \leqslant \|F\|_H$.

3. The stationary problem

With the family of domains $\Omega^\epsilon = \Omega_1 \cup \Gamma \cup \Omega_2^\epsilon$ given above for each value of the parameter with $0 < \epsilon \leqslant 1$, the stationary problem corresponding to the initial-value problem (1.1) takes the weak form

$$\begin{aligned} u^\epsilon \in V^\epsilon: \quad & \int_{\Omega_1} \lambda m_1 u^\epsilon v\,dx + \int_{\Omega_1} k_1 \nabla u^\epsilon \cdot \nabla v\,dx + \int_{\Omega_2^\epsilon} \lambda m_2 u^\epsilon v\,dx + \int_{\Omega_2^\epsilon} \frac{k_2}{\epsilon} \nabla u^\epsilon \cdot \nabla v\,d\tilde{x}\,dx_N \\ &= \int_{\Omega_1} \lambda m_1 F v\,dx + \int_{\Omega_2^\epsilon} \lambda m_2 F v\,dx + \int_{\Gamma} g\gamma(v)\,d\tilde{x}, \quad \forall v \in V^\epsilon, \end{aligned} \tag{3.5}$$

in the space $V^\epsilon \equiv \{v \in H^1(\Omega^\epsilon)\colon\ v = 0 \text{ on } \partial\Omega_1 - \Gamma\}$. This is the *exact* or ϵ-problem to be solved, and it depends on the thin domain Ω_2^ϵ and the high permeability $\frac{k_2}{\epsilon}$ through the scale parameter $\epsilon > 0$. We expect the last term on the left side to be approximated for small values of ϵ by averaging across the narrow channel,

$$\frac{1}{\epsilon} \int_{\Omega_2^\epsilon} k_2 \nabla u \cdot \nabla v\,dx_N\,d\tilde{x} \approx \int_{\Gamma} k_2 \tilde{\nabla} u \cdot \tilde{\nabla} v\, \omega(\tilde{x})\,d\tilde{x}, \tag{3.6}$$

where $\tilde{\nabla}$ denotes the gradient in the variable $\tilde{x}$ in Γ, and this will be established in our work below.

3.1. The scaled problem

Since our primary interest is the dependence of the solution on ϵ, we shall reformulate the problem in a space that is independent of this parameter. In order to eliminate this dependence on the domain, we scale Ω_2^ϵ in the direction normal to Γ by $x_N = \epsilon z$ to get an equivalent problem on the domain $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$ with $\Omega_2 \equiv \Omega_2^1 = \{(\tilde{x}, \omega(\tilde{x})z) \in \mathbb{R}^n\colon\ (\tilde{x}, z) \in \Gamma \times (0,1)\}$. The corresponding bilinear form is

$$a^\epsilon(u, v) \equiv \int_{\Omega_1} k_1 \nabla u \cdot \nabla v\,dx + \int_{\Omega_2} k_2 \tilde{\nabla} u \cdot \tilde{\nabla} v\,d\tilde{x}\,dz + \int_{\Omega_2} \frac{k_2}{\epsilon^2} \partial_z u\, \partial_z v\,d\tilde{x}\,dz. \tag{3.7}$$

This form is continuous on $V \equiv \{v \in H^1(\Omega)\colon\ v = 0 \text{ on } \partial\Omega_1 - \Gamma\}$, and the *scaled problem* is

$$\begin{aligned} u^\epsilon \in V: \quad & \int_{\Omega_1} \lambda m_1 u^\epsilon v\,dx + \epsilon \int_{\Omega_2} \lambda m_2 u^\epsilon v\,d\tilde{x}\,dz + a^\epsilon\big(u^\epsilon, v\big) \\ &= \int_{\Omega_1} \lambda m_1 F v\,dx + \epsilon \int_{\Omega_2} \lambda m_2 F v\,dx + \int_{\Gamma} g\gamma(v)\,d\tilde{x}, \quad \forall v \in V. \end{aligned} \tag{3.8}$$

For each $\epsilon > 0$ the bilinear form (3.7) is clearly V-elliptic, so the problem (3.8) is well-posed. Moreover, the solution u^ϵ satisfies

$$\begin{aligned}
&\lambda m_1 u_1^\epsilon - \nabla\cdot k_1 \nabla u_1^\epsilon = \lambda m_1 F \quad \text{in } \Omega_1,\\
&u_1^\epsilon = 0 \quad \text{on } \partial\Omega_1 - \Gamma,\\
&u_1^\epsilon = u_2^\epsilon, \qquad k_1 \partial_z u_1^\epsilon - \frac{k_2}{\epsilon^2}\partial_z u_2^\epsilon = g \quad \text{on } \Gamma,\\
&\epsilon\lambda m_2 u_2^\epsilon - \tilde\nabla\cdot k_2 \tilde\nabla u_2^\epsilon - \frac{k_2}{\epsilon^2}\partial_z\partial_z u_2^\epsilon = \epsilon\lambda m_2 F \quad \text{in } \Omega_2,\\
&\left(\tilde\nabla u_2^\epsilon, \frac{1}{\epsilon^2}\partial_z u_2^\epsilon\right)\cdot \hat n = 0 \quad \text{on } \partial\Omega_2 - \Gamma.
\end{aligned} \tag{3.9}$$

This is the stationary form of the interface problem (1.1) after the rescaling. Here we see the role of the effective tangential permeability k_2 and the effective normal permeability $\frac{k_2}{\epsilon^2}$.

The estimates. Denote by χ_j the *characteristic function* of Ω_j, $j = 1, 2$, and set $u^\epsilon \equiv u_1^\epsilon \chi_1 + u_2^\epsilon \chi_2$. Due to the boundary conditions of the space V, the gradient controls the entire $H^1(\Omega)$ norm on V. Testing (3.8) with $v = u^\epsilon$, we obtain

$$\begin{aligned}
C_1\big(\|u_1^\epsilon\|_{0,\Omega_1}^2 + \|u_2^\epsilon\|_{0,\Omega_2}^2\big) &\leqslant C_2\left(\|u_1^\epsilon\|_{0,\Omega_1}^2 + \|\nabla u_1^\epsilon\|_{0,\Omega_1}^2 + \epsilon\|u_2^\epsilon\|_{0,\Omega_2}^2 + \|\tilde\nabla u_2^\epsilon\|_{0,\Omega_2}^2 + \left\|\frac{1}{\epsilon}\partial_z u_2^\epsilon\right\|_{0,\Omega_2}^2\right)\\
&\leqslant \|F\|_{0,\Omega_1}\|u_1^\epsilon\|_{0,\Omega_1} + \|g\|_{0,\Gamma}\|u_1^\epsilon\|_{1,\Omega_1} \leqslant \tilde C\|u_1^\epsilon\|_{1,\Omega_1}
\end{aligned}$$

where $C_1, C_2, \tilde C$ are positive constants. It follows that

$$\|u_1^\epsilon\|_{0,\Omega_1}^2 + \|\nabla u_1^\epsilon\|_{0,\Omega_1}^2 + \epsilon\|u_2^\epsilon\|_{0,\Omega_2}^2 + \|\tilde\nabla u_2^\epsilon\|_{0,\Omega_2}^2 + \left\|\frac{1}{\epsilon}\partial_z u_2^\epsilon\right\|_{0,\Omega_2}^2 \leqslant C \tag{3.10}$$

for some generic positive constant C.

The limit. The estimate (3.10) implies that there is a subsequence, which we denote again by $\{u^\epsilon\}$, and a $u^* = u_1\chi_1 + u_2\chi_2 \in V$ such that $u^\epsilon \overset{w}{\rightharpoonup} u^*$ in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. For any $v \in V$, as $\epsilon \to 0$ we have

$$\begin{aligned}
&\int_{\Omega_1} k_1 \nabla u_1^\epsilon \cdot \nabla v\, dx \to \int_{\Omega_1} k_1 \nabla u_1 \cdot \nabla v\, dx, \quad \text{and}\\
&\int_{\Omega_2} k_2 \tilde\nabla u_2^\epsilon \cdot \tilde\nabla v\, d\tilde x\, dz \to \int_{\Omega_2} k_2 \tilde\nabla u_2 \cdot \tilde\nabla v\, d\tilde x\, dz.
\end{aligned}$$

Since the right side of (3.8) is bounded for $v \in V$ fixed, we conclude the existence of the limit

$$\ell(v) \equiv \lim_{\epsilon\downarrow 0} \int_{\Omega_2} \frac{k_2}{\epsilon^2}\partial_z u_2^\epsilon \partial_z v\, d\tilde x\, dz,$$

and due to the a priori estimates we conclude $\ell \in V'$. In addition, there must exist $\zeta \in L^2(\Omega_2)$ such that $\epsilon^{-1}\partial_z u_2^\epsilon \overset{w}{\rightharpoonup} \zeta$ in $L^2(\Omega_2)$. Also $\|\partial_z u_2^\epsilon\|_{0,\Omega_2} \leqslant \epsilon C$, so $\|\partial_z u_2^\epsilon\|_{0,\Omega_2} \to 0$, and we know $\partial_z u_2^\epsilon \overset{w}{\rightharpoonup} \partial_z u_2$ in $L^2(\Omega_2)$, so $\partial_z u_2 \equiv 0$ and u_2 is independent of z in Ω_2. Taking the limit in (3.8), we find that $u^* = u_1\chi_1 + u_2\chi_2$ satisfies

$$\begin{aligned}
u^* \in V\colon\quad &\partial_z u_2 = 0 \quad \text{in } \Omega_2, \quad \text{and}\\
&\int_{\Omega_1} \lambda m_1 u_1 v\, dx + \int_{\Omega_1} k_1 \nabla u_1 \cdot \nabla v\, dx + \int_{\Omega_2} k_2 \tilde\nabla u_2 \cdot \tilde\nabla v\, dx + \ell(v) = \int_{\Omega_1} \lambda m_1 F v\, dx + \int_\Gamma g\gamma(v)\, d\tilde x, \quad \forall v \in V.
\end{aligned} \tag{3.11}$$

Define now the subspace $W \equiv \{v \in V\colon\ \partial_z v = 0 \text{ on } \Omega_2\}$. We have shown that for some subsequence we obtain a weak limit, $u^\epsilon \overset{w}{\rightharpoonup} u^*$ in V with $u^* \in W$, and since the linear functional $\ell(\cdot)$ vanishes on W, this limit satisfies

$$u^* \in W\colon\quad \int_{\Omega_1} \lambda m_1 u^* v\, dx + a^0\big(u^*, v\big) = \int_{\Omega_1} \lambda m_1 F v\, dx + \int_\Gamma g\gamma(v)\, d\tilde x \quad \text{for all } v \in W, \tag{3.12}$$

where the limit bilinear form on W is defined by

$$a^0(u, v) \equiv \int_{\Omega_1} k_1 \nabla u \cdot \nabla v \, dx + \int_{\Omega_2} k_2 \tilde{\nabla} u \cdot \tilde{\nabla} v \, d\tilde{x} \, dz. \tag{3.13}$$

This continuous bilinear form is W-elliptic, so we see that u^* is the only solution and the original sequence $\{u^\epsilon\}$ converges weakly to u^*. In summary, the problem (3.12) characterizes the limit u^* of the stationary problems (3.8).

3.2. Strong convergence

On the space V we take the scalar product

$$\langle v, w \rangle \equiv \int_{\Omega_1} k_1 \nabla v \cdot \nabla w \, dx + \int_{\Omega_2} k_2 \nabla v \cdot \nabla w \, dx. \tag{3.14}$$

This scalar product $\langle \cdot,\cdot \rangle$ is equivalent to the usual $H^1(\Omega)$ scalar product, that is, the V-norm $\|v\|_V \equiv \langle v, v\rangle^{1/2}$ is equivalent to the $H^1(\Omega)$ norm, so from the weak convergence $u^\epsilon \overset{w}{\rightharpoonup} u^*$ in $H^1(\Omega)$ we know

$$\|u^*\|_V \leqslant \liminf_{\epsilon \downarrow 0} \|u^\epsilon\|_V.$$

Now, for $0 < \epsilon \leqslant 1$, the solution u^ϵ of (3.8) satisfies

$$\|u^\epsilon\|_V^2 \leqslant \epsilon \int_{\Omega_2} \lambda m_2 (u^\epsilon)^2 \, d\tilde{x} \, dz + a^\epsilon(u^\epsilon, u^\epsilon) = -\int_{\Omega_1} \lambda m_1 (u^\epsilon)^2 \, dx + \int_{\Omega_1} \lambda m_1 F u^\epsilon \, dx + \int_{\Omega_2} \epsilon \lambda m_2 F u^\epsilon \, dx + \int_{\Gamma} g \gamma u^\epsilon \, d\tilde{x},$$

so from weak lower-semicontinuity of the first term we obtain

$$\limsup_{\epsilon \downarrow 0} \|u^\epsilon\|_V^2 \leqslant -\int_{\Omega_1} \lambda m_1 (u^*)^2 \, dx + \int_{\Omega_1} \lambda m_1 F u^* \, dx + \int_{\Gamma} g \gamma (u^*) \, d\tilde{x}.$$

But with (3.12) this gives

$$\limsup_{\epsilon \downarrow 0} \|u^\epsilon\|_V^2 \leqslant a^0(u^*, u^*) = \|u^*\|_V^2,$$

so $\lim_{\epsilon \downarrow 0} \|u^\epsilon\|_V = \|u^*\|_V$. Together with the weak convergence of the sequence, this implies $\|u^\epsilon - u^*\|_V \to 0$, and so we have strong convergence $u^\epsilon \to u^*$ in $H^1(\Omega)$.

An alternative system. The solution of the limiting problem can be characterized by a boundary-value problem on Ω_1 and Γ. First we rewrite (3.11). Since $C_0^\infty(\Omega_1) \subseteq V$, for any $\varphi \in C_0^\infty(\Omega_1)$ we obtain

$$\int_{\Omega_1} \lambda m_1 u_1 \varphi \, dx + \int_{\Omega_1} k_1 \nabla u_1 \cdot \nabla \varphi \, dx = \int_{\Omega_1} \lambda m_1 F \varphi \, dx,$$

i.e., $\lambda m_1 u_1 - \nabla \cdot k_1 \nabla u_1 = \lambda m_1 F$ in $L^2(\Omega_1)$, so $k_1 \nabla u_1 \in \mathbf{L}^2_{div}(\Omega_1)$ and the normal trace $k_1 \nabla u_1 \cdot \hat{n} \in H^{-1/2}(\partial \Omega_1)$ is well defined. Moreover, we know that for any $v \in V$ the Stokes' formula [15]

$$\langle k_1 \nabla u_1 \cdot \hat{n}, \gamma v \rangle_{H^{-1/2}(\partial\Omega_1), H^{1/2}(\partial\Omega_1)} = \int_{\Omega_1} k_1 \nabla u_1 \cdot \nabla v \, dx + \int_{\Omega_1} \nabla \cdot (k_1 \nabla u_1) v \, dx$$

must hold. Substituting these into (3.11), we conclude

$$\langle k_1 \nabla u_1 \cdot \hat{n}, \gamma v \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} + \int_{\Omega_2} k_2 \tilde{\nabla} u_2 \cdot \tilde{\nabla} v \, dx + \ell(v) = \int_{\Gamma} g \gamma v \, d\tilde{x} \quad \text{for all } v \in V. \tag{3.15}$$

Since the functions in W are independent of z for $(\tilde{x}, z) \in \Omega_2$, we have for each pair $u, v \in W$

$$\begin{aligned}(u, v)_{H^1(\Omega_2)} &= \int_{\Omega_2} \big(u(\tilde{x}) v(\tilde{x}) + \nabla u(\tilde{x}) \cdot \nabla v(\tilde{x})\big) \, d\tilde{x} \, dz \\ &= \int_{\Gamma} \big(u(\tilde{x}) v(\tilde{x}) + \tilde{\nabla} u(\tilde{x}) \cdot \tilde{\nabla} v(\tilde{x})\big) \omega(\tilde{x}) \, d\tilde{x}.\end{aligned}$$

This is equivalent to the scalar product

$$(u,v)_{H^1_\omega(\Gamma)} \equiv \int\limits_\Gamma \big(u(\tilde{x})v(\tilde{x}) + \omega(\tilde{x})\tilde{\nabla}u(\tilde{x})\cdot\tilde{\nabla}v(\tilde{x})\big)\,d\tilde{x}$$

of the weighted Sobolev space

$$H^1_\omega(\Gamma) \equiv \big\{u \in L^2(\Gamma):\ \omega^{1/2}\tilde{\nabla}u \in \mathbf{L}^2(\Gamma)\big\}.$$

Furthermore, we see W is equivalent to the space

$$V_\Gamma \equiv \big\{v \in H^1(\Omega_1):\ v|_\Gamma \in H^1_\omega(\Gamma),\ v|_{\partial\Omega_1-\Gamma} = 0\big\}$$

in the sense of boundary trace. Thus, the solution of problem (3.12) is characterized by

$$u^* \in V_\Gamma:\quad \int\limits_{\Omega_1} \lambda m_1 u^* v\,dx + \int\limits_{\Omega_1} k_1\nabla u^*\cdot\nabla v\,dx + \int\limits_\Gamma k_2\omega\tilde{\nabla}u^*\cdot\tilde{\nabla}v\,d\tilde{x}$$
$$= \int\limits_{\Omega_1} \lambda m_1 F v\,dx + \int\limits_\Gamma g\gamma(v)\,d\tilde{x}\quad \text{for all } v \in V_\Gamma, \tag{3.16}$$

and this means it determines a pair $u_1 = \chi_1 u^* \in H^1(\Omega_1)$, $u_2 = \gamma(u^*) \in H^1_\omega(\Gamma)$ which satisfies the system

$$\lambda m_1 u_1 - \nabla\cdot k_1\nabla u_1 = \lambda m_1 F \quad \text{in } \Omega_1, \tag{3.17a}$$
$$u_1 = 0 \quad \text{on } \partial\Omega_1 - \Gamma, \tag{3.17b}$$
$$u_1 = u_2 \quad \text{on } \Gamma, \quad \text{and} \tag{3.17c}$$
$$\int\limits_\Gamma k_2\omega\tilde{\nabla}u_2\cdot\tilde{\nabla}\gamma v\,d\tilde{x} + \langle k_1\nabla u_1\cdot\hat{n}, \gamma v\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = \int\limits_\Gamma g\gamma v\,d\tilde{x}\quad \text{for all } v \in V. \tag{3.17d}$$

In the situation of Theorem 2.1, the variational identity (3.17d) is equivalent to

$$-\tilde{\nabla}\cdot k_2\omega\tilde{\nabla}u_2 + k_1\partial_z u_1 = g \quad \text{in } \Gamma, \tag{3.17e}$$
$$u_2 = 0 \quad \text{on } \partial\Gamma. \tag{3.17f}$$

However, in the strongly tapered case of (2.3), the last condition (3.17f) is deleted, since the trace is meaningless and the variational equation is equivalent to Eq. (3.17e) in $H^1_\omega(\Gamma)'$. See [12] for such examples. Thus, the limiting form of the singular problem (3.8) is the elliptic boundary-value problem on Ω_1 with the (non-local and possibly degenerate) elliptic boundary constraint.

We summarize the above as follows.

Theorem 3.1. *Let the regions Ω^ϵ and the rescaled Ω, the constants $k_1, k_2, m_1, m_2 > 0$, $\lambda \geqslant 0$, and functions $F \in L^2(\Omega)$, $g \in L^2(\Gamma)$ be given. Define the bilinear form* (3.7) *for each $0 < \epsilon \leqslant 1$ on the space V. Then each scaled problem* (3.8) *has a unique solution, u^ϵ, these satisfy the estimates* (3.10) *and converge strongly $u^\epsilon \to u^*$ in V, where u^* satisfies* (3.11)*. Finally, the limit u^* is characterized as the solution of the well-posed limit problem* (3.12) *or its equivalent form* (3.16)*.*

3.2.1. Remarks on minimization and penalty

Set $f^\epsilon(v) = \int_{\Omega_1} \lambda m_1 F v\,dx + \epsilon\int_{\Omega_2} \lambda m_2 F v\,dx + \int_\Gamma g\gamma(v)\,d\tilde{x}$. Eq. (3.8) shows that u^ϵ is characterized by the minimization of

$$\varphi^\epsilon(v) \equiv \frac{1}{2}\bigg(\int\limits_{\Omega_1} \lambda m_1 v^2\,dx + \int\limits_{\Omega_2} \epsilon\lambda m_2 v^2\,dx + a^\epsilon(v,v)\bigg) - f^\epsilon(v), \quad v \in V.$$

According to (3.11), the limit u^* satisfies

$$u^* \in W:\quad \int\limits_{\Omega_1} \lambda m_1 u^* v\,dx + \langle u^*, v\rangle_V + \ell(v) = f^0(v)\quad \text{for all } v \in V$$

and is characterized by (3.12), that is,

$$u^* \in W:\quad \int\limits_{\Omega_1} \lambda m_1 u^* v\,dx + \langle u^*, v\rangle_V = f^0(v)\quad \text{for all } v \in W.$$

This shows that u^* is obtained by the minimization of

$$\varphi(v) \equiv \frac{1}{2}\left(\int_{\Omega_1} \lambda m_1 v^2\,dx + \langle v, v\rangle_V\right) - f^0(v), \quad v \in V, \tag{3.18}$$

over the subspace W. This is the same as minimizing $\varphi(v) + I_W(v)$ over all of V, where

$$I_W(v) \equiv \begin{cases} 0 & \text{if } v \in W, \\ +\infty & \text{if } v \notin W, \end{cases}$$

is the *indicator function* of W.

Furthermore, if $\partial I_W(\cdot)$ denotes the subgradient of the convex $I_W(\cdot)$, then $\ell \in \partial I_W(u^*)$ is the Lagrange multiplier that realizes the constraint $u^* \in W$. The last term in (3.7) is the *penalty* term and (3.8) is a *penalty method* to approximate (3.12).

3.3. The concentrated capacity model

Suppose that in the interface problem (1.1), we assume that not only the permeability k_2 but also m_2 is scaled by $\frac{1}{\epsilon}$ in Ω_2. Such an assumption is meaningless for porous media, since the porosity is bounded by 1, but it is appropriate in analogous heat conduction problems with a concentrated capacity along the highly-conducting interface or boundary. However, the problem (3.8) with the factor ϵ deleted from the two terms can be used as a fracture model with highly anisotropic permeability. We include this case to show what assumptions are required to arrive at the narrow fracture model described in [2].

Theorem 3.2. *Let the region Ω, the constants $k_1, k_2, \lambda m_1 > 0$, and functions $F \in L^2(\Omega)$, $g \in L^2(\Gamma)$ be given. For each $0 < \epsilon \leqslant 1$, consider the problem*

$$u^\epsilon \in V\colon \quad \int_{\Omega_1} \lambda m_1 u^\epsilon v\,dx + \int_{\Omega_2} \lambda m_2 u^\epsilon v\,dx + a^\epsilon\big(u^\epsilon, v\big)$$
$$= \int_{\Omega_1} \lambda m_1 F v\,dx + \int_{\Omega_2} \lambda m_2 F v\,dx + \int_{\Gamma} g\gamma(v)\,d\tilde{x}, \quad \forall v \in V. \tag{3.19}$$

This problem has a unique solution, u^ϵ, these satisfy the estimates (3.10) *and converge strongly $u^\epsilon \to u^*$ in V, where the limit u^* satisfies*

$$u^* \in W\colon \quad \int_{\Omega_1} \lambda m_1 u^* v\,dx + \int_{\Gamma} \lambda m_2 \omega u^* v\,d\tilde{x} + \int_{\Omega_1} k_1 \nabla u^* \cdot \nabla v\,dx + \int_{\Gamma} k_2 \omega \tilde{\nabla} u^* \cdot \tilde{\nabla} v\,d\tilde{x}$$
$$= \int_{\Omega_1} \lambda m_1 F v\,dx + \int_{\Gamma} \lambda m_2 \omega \tilde{F} v\,d\tilde{x} + \int_{\Gamma} g\gamma(v)\,d\tilde{x} \quad \textit{for all } v \in W, \tag{3.20}$$

and the channel average *of F in Ω_2 is given by*

$$\tilde{F}(\tilde{x}) = \frac{1}{\omega(\tilde{x})} \int_0^{\omega(\tilde{x})} F(\tilde{x}, z)\,dz, \quad \tilde{x} \in \Gamma.$$

Note as before that the limit $u^* \in V_\Gamma$ determines a pair $u_1 \in H^1(\Omega_1)$, $u_2 \in H^1_\omega(\Gamma)$ which satisfies

$$\lambda m_1 u_1 - \nabla \cdot k_1 \nabla u_1 = \lambda m_1 F \quad \text{in } \Omega_1, \tag{3.21a}$$

$$u_1 = 0 \quad \text{on } \partial\Omega_1 - \Gamma, \tag{3.21b}$$

$$u_1 = u_2 \quad \text{on } \Gamma, \quad \text{and}$$
$$\int_{\Gamma} \lambda m_2 \omega u_2 v\,d\tilde{x} + \int_{\Gamma} k_2 \omega \tilde{\nabla} u_2 \cdot \tilde{\nabla} v\,d\tilde{x} + \langle k_1 \nabla u_1 \cdot \hat{n}, \gamma v\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}$$
$$= \int_{\Gamma} \lambda m_2 \omega \tilde{F} v\,d\tilde{x} + \int_{\Gamma} g\gamma v\,d\tilde{x} \quad \text{for all } v \in V_\Gamma. \tag{3.21c}$$

In the weakly tapered situation of Theorem 2.1, the variational identity is equivalent to

$$\lambda m_2 \omega u_2 - \tilde{\nabla} \cdot k_2 \omega \tilde{\nabla} u_2 + k_1 \partial_z u_1 = \lambda m_2 \omega \tilde{F} + g \quad \text{in } \Gamma, \tag{3.21d}$$

$$u_2 = 0 \quad \text{on } \partial\Gamma, \tag{3.21e}$$

and in the strongly tapered case of (2.3), the last condition (3.21e) is deleted.

4. The evolution problems

We apply Theorem 3.1 to show the dynamics of the initial-boundary-value problem (1.1) is governed by an analytic semigroup on the Hilbert space $H = L^2(\Omega)$, and the limiting form corresponds similarly to an analytic semigroup on the Hilbert space $H_0 = L^2(\Omega_1)$. Then we establish the convergence as $\epsilon \to 0$ of solutions of the corresponding evolution problems.

4.1. Well-posed problems

Let H_ϵ denote H with the norm $\|u\|_{H_\epsilon} = \|m_1^{1/2}\chi_1 u + (\epsilon m_2)^{1/2}\chi_2 u\|_{L^2(\Omega)}$, so its Riesz map is the multiplication function $m_\epsilon = m_1\chi_1 + \epsilon m_2 \chi_2$ from H_ϵ to H_ϵ'. Similarly, $m_0 = m_1$ is the Riesz map from H_0 to H_0', where $\|u\|_{H_0} = \|m_1^{1/2} u\|_{L^2(\Omega_1)}$. Note that $V \subset H_\epsilon$ and $W \subset H_0$ are dense and continuous inclusions.

Define the operators $A^\epsilon : D^\epsilon \to H_\epsilon'$ with domains $D^\epsilon \subset V$ by $u^\epsilon \in D^\epsilon$ and $A^\epsilon(u^\epsilon) = F \in H_\epsilon'$ if $u^\epsilon \in V$: $a^\epsilon(u^\epsilon, v) = F(v)$ for all $v \in V$. Similarly the operator $A^0 : D^0 \to H_0'$ with domain $D^0 \subset W$ is determined by $u^0 \in D^0$ and $A^0(u^0) = F \in H_0'$ if $u^0 \in W$: $a^0(u^0, w) = F(w)$ for all $w \in W$. If we set $g = 0$, then the scaled problem (3.8) is equivalent to $A^\epsilon(u^\epsilon) = \lambda m_\epsilon (F - u^\epsilon)$ for $F \in H$, and the limit problem (3.12) is equivalent to $A^0(u^*) = \lambda m_0 (F - u^*)$ when $F \in H_0$.

Each of the operators $m_\epsilon^{-1} A^\epsilon$ is m-accretive on H_ϵ, that is, $\|(I + \alpha m_\epsilon^{-1} A^\epsilon)^{-1} F\|_{H_\epsilon} \leqslant \|F\|_{H_\epsilon}$ for each $\alpha > 0$ and $F \in H_\epsilon$. Likewise $(I + \alpha m_0^{-1} A^0)^{-1}$ is a contraction on H_0 for each $\alpha > 0$. These operators are also self-adjoint, since the corresponding bilinear forms are symmetric, so $m_\epsilon^{-1} A^\epsilon$ and $m_0^{-1} A^0$ generate analytic semigroups on H_ϵ and H_0, respectively.

The Hille–Yoshida Theorem 2.3 shows that the corresponding initial-value problems are well-posed. Applying it to the operator $m_\epsilon^{-1} A^\epsilon$ in H_ϵ, we obtain the scaled problem.

Theorem 4.1. *For every $u_0 \in L^2(\Omega)$ and $F \in C^\beta([0,\infty), L^2(\Omega))$, there is a unique $u^\epsilon \in C([0,\infty), L^2(\Omega)) \cap C^1((0,\infty), L^2(\Omega))$ with $u^\epsilon(t) \in D^\epsilon$ for each $t > 0$ such that $u^\epsilon(t) = \chi_1 u_1^\epsilon(t) + \chi_2 u_2^\epsilon(t)$ satisfies the scaled problem*

$$\begin{aligned}
&m_1 \frac{\partial u_1^\epsilon}{\partial t} - \nabla \cdot k_1 \nabla u_1^\epsilon = m_1 F \quad \text{in } \Omega_1,\\
&u_1^\epsilon = 0 \quad \text{on } \partial\Omega_1 - \Gamma,\\
&u_1^\epsilon = u_2^\epsilon, \qquad k_1 \partial_z u_1^\epsilon - \frac{k_2}{\epsilon^2} \partial_z u_2^\epsilon = 0 \quad \text{on } \Gamma,\\
&\epsilon m_2 \frac{\partial u_2^\epsilon}{\partial t} - \tilde{\nabla} \cdot k_2 \tilde{\nabla} u_2^\epsilon - \frac{k_2}{\epsilon^2} \partial_z \partial_z u_2^\epsilon = \epsilon m_2 F \quad \text{in } \Omega_2,\\
&\left(k_2 \tilde{\nabla} u_2^\epsilon, \frac{k_2}{\epsilon^2} \partial_z u_2^\epsilon\right) \cdot \hat{n} = 0 \quad \text{on } \partial\Omega_2 - \Gamma,
\end{aligned} \tag{4.22a}$$

at each $t > 0$, and these satisfy the initial conditions

$$u_1^\epsilon(\cdot, 0) = u_0(\cdot) \quad \text{on } \Omega_1, \qquad u_2^\epsilon(\cdot, 0) = u_0(\cdot) \quad \text{on } \Omega_2. \tag{4.22b}$$

Note that this is a rather strong solution, since $\nabla \cdot k_j \nabla u_1^\epsilon(t) \in L^2(\Omega_j)$ for each $t > 0$, $j = 1, 2$.

Similarly from the operator $m_0^{-1} A^0$ in H_0 we obtain the limiting problem. When the fracture is weakly tapered, this takes the following form.

Theorem 4.2. *For every $u_0 \in L^2(\Omega_1)$ and $F \in C^\beta([0,\infty), L^2(\Omega_1))$, there is a unique $u^* \in C([0,\infty), L^2(\Omega_1)) \cap C^1((0,\infty), L^2(\Omega_1))$ with $u^*(t) \in D^0$ for each $t > 0$, such that the functions $u_1(t) = u^*(t)|_{\Omega_1} \in H^1(\Omega_1)$, $u_2(t) = \gamma(u^*(t)) \in H^1_\omega(\Gamma)$ satisfy*

$$m_1 \frac{\partial u_1}{\partial t} - \nabla \cdot k_1 \nabla u_1 = m_1 F \quad \text{in } \Omega_1, \tag{4.23a}$$

$$u_1 = 0 \quad \text{on } \partial\Omega_1 - \Gamma, \tag{4.23b}$$

$$u_1 = u_2 \quad \text{on } \Gamma, \quad \text{and} \tag{4.23c}$$

$$-\tilde{\nabla} \cdot k_2 \omega \tilde{\nabla} u_2 + k_1 \partial_z u_1 = 0 \quad \text{in } \Gamma, \tag{4.23d}$$

$$u_2 = 0 \quad \text{on } \partial\Gamma, \tag{4.23e}$$

at each $t > 0$ *and the initial condition*

$$u_1(\cdot, 0) = u_0(\cdot) \quad \text{on } \Omega_1. \tag{4.23f}$$

In particular, each term of Eq. (4.23a) belongs to $L^2(\Omega_1)$, so the solution is rather strong. As before, in the strongly tapered case, the last condition (4.23e) is deleted.

4.2. Strong convergence

For the stationary problems, we have shown that $(m_\epsilon + A^\epsilon)^{-1} m_\epsilon F \to (m_0 + A^0)^{-1} m_0 F$ in the V-norm, hence, in $H^1(\Omega)$ so also in H. However, for the corresponding dynamic problems, with $\epsilon > 0$ we have an evolution in $H_\epsilon = L^2(\Omega)$ whereas the limit is an evolution in $H_0 = L^2(\Omega_1)$, and these are not immediately comparable, so we shall work directly in the corresponding evolution spaces, $\mathcal{V} \equiv L^2(0, T; V)$ and $\mathcal{W} \equiv L^2(0, T; W)$. The Cauchy problem leads to the Hilbert space

$$W^{1.2}(0, T) \equiv \left\{ u \in \mathcal{V}\colon\ \frac{du}{dt} \in \mathcal{V}' \right\}$$

with the norm $\|u\|_{W^{1.2}(0,T)} = (\|u\|_{\mathcal{V}}^2 + \|\frac{du}{dt}\|_{\mathcal{V}'}^2)^{1/2}$, and this space is contained in $C([0, T], H)$ with continuous imbedding, that is,

$$\|u\|_{C([0,T],H)} \leqslant C \|u\|_{W^{1.2}(0,T)}, \quad u \in W^{1.2}(0, T).$$

See any one of [1,14,15].

The solution of (4.22) satisfies

$$u^\epsilon \in \mathcal{V}\colon\quad \forall v \in \mathcal{V} \cap W^{1.2}(0, T; H) \text{ with } v(T) = 0,$$
$$-\int_0^T \left(m_\epsilon u^\epsilon(t), \frac{dv}{dt}(t) \right)_{L^2(\Omega)} dt + \int_0^T a^\epsilon\big(u^\epsilon(t), v(t)\big) = \int_0^T \big(m_\epsilon F(t), v(t)\big)_{L^2(\Omega)} dt + \big(m_\epsilon u_0, v(0)\big)_{L^2(\Omega)}.$$

This is the weak formulation of the Cauchy problem

$$u^\epsilon \in \mathcal{V}\colon\quad m_\epsilon \frac{du^\epsilon}{dt}(\cdot) + A^\epsilon\big(u^\epsilon(\cdot)\big) = m_\epsilon F(\cdot) \quad \text{in } \mathcal{V}', \quad u^\epsilon(0) = u_0,$$

and the solution u^ϵ satisfies the identity

$$\frac{1}{2}\big(m_\epsilon u^\epsilon(T), u^\epsilon(T)\big)_{L^2(\Omega)} + \int_0^T a^\epsilon\big(u^\epsilon(t), u^\epsilon(t)\big)\, dt = \int_0^T \big(m_\epsilon F(t), u^\epsilon(t)\big)_{L^2(\Omega)} dt + \frac{1}{2}(m_\epsilon u_0, u_0)_{L^2(\Omega)}. \tag{4.24}$$

This implies that $\|u^\epsilon\|_{\mathcal{V}}$, $\|\frac{1}{\epsilon}\partial_z u^\epsilon\|_{L^2(0,T;H_0)}$ are bounded, so there is a weakly convergent subsequence, $u^\epsilon \overset{w}{\rightharpoonup} u^*$ in $\mathcal{V}$ with limit $u^* \in \mathcal{W}$. Then the evolution equation shows that $\frac{du^\epsilon}{dt} \overset{w}{\rightharpoonup} \frac{du^*}{dt}$ in $\mathcal{W}'$, so we obtain

$$u^* \in \mathcal{W}\colon\quad \forall v \in \mathcal{W} \cap W^{1.2}(0, T; H_0) \text{ with } v(T) = 0,$$
$$-\int_0^T \left(m_0 u^*(t), \frac{dv}{dt}(t) \right)_{L^2(\Omega_1)} dt + \int_0^T a^0\big(u^*(t), v(t)\big) = \int_0^T \big(m_0 F(t), v(t)\big)_{L^2(\Omega_1)} dt + \big(m_0 u_0, v(0)\big)_{L^2(\Omega_1)}.$$

As before, this characterizes the solution of

$$u^* \in \mathcal{W}\colon\quad m_0 \frac{du^*}{dt}(\cdot) + A^0\big(u^*(\cdot)\big) = m_0 F(\cdot) \quad \text{in } \mathcal{W}', \quad u^*(0) = \chi_1 u_0,$$

which has *only* one solution [11], so the original sequence converges weakly to u^* and this is also the solution of (4.23). Moreover, we have

$$\frac{1}{2}\big(m_0 u^*(T), u^*(T)\big)_{L^2(\Omega_1)} + \int_0^T a^0\big(u^*(t), u^*(t)\big)\, dt = \int_0^T \big(m_0 F(t), u^*(t)\big)_{L^2(\Omega_1)} dt + \frac{1}{2}(m_0 u_0, u_0)_{L^2(\Omega_1)}, \tag{4.25}$$

and this will be used to show strong convergence $u^\epsilon \to u^*$ in $\mathcal{V}$. From the weak convergence, we have

$$\int_0^T \langle u^*(t), u^*(t)\rangle\,dt \leqslant \liminf_{\epsilon\downarrow 0}\int_0^T \langle u^\epsilon, u^\epsilon\rangle\,dt.$$

This follows since the V-norm from the scalar product (3.14) is equivalent to the $H^1(\Omega)$-norm. Also from (4.24) we have

$$\int_0^T \langle u^\epsilon, u^\epsilon\rangle\,dt \leqslant \int_0^T a^\epsilon\left(u^\epsilon, u^\epsilon\right)dt = -\frac{1}{2}\left(m_\epsilon u^\epsilon(T), u^\epsilon(T)\right)_{L^2(\Omega)} + \int_0^T \left(m_\epsilon F(t), u^\epsilon(t)\right)_{L^2(\Omega)}dt + \frac{1}{2}(m_\epsilon u_0, u_0)_{L^2(\Omega)}.$$

Then using the (weak) continuity of the linear map $u \to u(T)$ from $\{u \in \mathcal{W}\colon m_0^{1/2}\frac{du}{dt} \in \mathcal{W}'\}$ to H_0, we take the lim sup above to get

$$\limsup_{\epsilon\downarrow 0}\int_0^T \langle u^\epsilon, u^\epsilon\rangle\,dt \leqslant -\frac{1}{2}\left(m_0 u^*(T), u^*(T)\right)_{L^2(\Omega_1)} + \int_0^T \left(m_0 F(t), u^*(t)\right)_{L^2(\Omega_1)}dt + \frac{1}{2}(m_0 u_0, u_0)_{L^2(\Omega_1)}.$$

Together with the limiting identity (4.25) this shows

$$\limsup_{\epsilon\downarrow 0}\int_0^T \langle u^\epsilon, u^\epsilon\rangle\,dt \leqslant \int_0^T a^0\left(u^*(t), u^*(t)\right)dt = \int_0^T \langle u^*(t), u^*(t)\rangle\,dt,$$

so we have established $\lim_{\epsilon\downarrow 0}\int_0^T \langle u^\epsilon, u^\epsilon\rangle\,dt = \int_0^T \langle u^*(t), u^*(t)\rangle\,dt$ and, hence, strong convergence in $\mathcal{V}$. Recalling that from the evolution equation we have the strong convergence $m_\epsilon \frac{du^\epsilon}{dt} \to m_0\frac{du^*}{dt}$ in $\mathcal{W}'$, we have

Theorem 4.3. *In the situation of Theorems* 4.1 *and* 4.2, *the sequence converges strongly* $u^\epsilon \to u^*$ *in* $\mathcal{V} = L^2(0, T; V)$ *and in* $C([0, T], H_0)$.

4.3. The concentrated capacity model

We obtain the analogous results for the evolution problem corresponding to Theorem 3.2. The approximation evolves in $H = L^2(\Omega)$ with the norm $\|u\|_H = \|(m_1^{1/2}\chi_1 + m_2^{1/2}\chi_2)u\|_{L^2(\Omega)}$; its Riesz map is the multiplication function $m_1\chi_1 + m_2\chi_2$ from H to H'. Similarly, H_0 is defined to be the closure of W in H, and as above we find it is equivalent to the weighted L^2 space with the scalar product

$$(u, v)_{L^2_\omega(\Omega)} = \int_{\Omega_1} m_1 u(x)v(x)\,dx + \int_\Gamma m_2 u(\tilde{x})v(\tilde{x})\omega(\tilde{x})\,d\tilde{x}.$$

Note that $V \subset H$ and $W \subset H_0$ are dense and continuous inclusions.

By the same arguments given previously, we obtain the following.

Theorem 4.4. *For every* $u_0 \in L^2(\Omega)$ *and* $F \in C^\beta([0, \infty), L^2(\Omega))$, *there is a unique* $u^\epsilon \in C([0, \infty), L^2(\Omega)) \cap C^1((0, \infty), L^2(\Omega))$ *with* $u^\epsilon(t) \in D^\epsilon$ *for each* $t > 0$ *such that* $u^\epsilon(t) = \chi_1 u_1^\epsilon(t) + \chi_2 u_2^\epsilon(t)$ *satisfies the scaled problem*

$$\begin{aligned}
&m_1\frac{\partial u_1^\epsilon}{\partial t} - \nabla\cdot k_1\nabla u_1^\epsilon = m_1 F \quad \text{in } \Omega_1,\\
&u_1^\epsilon = 0 \quad \text{on } \partial\Omega_1 - \Gamma,\\
&u_1^\epsilon = u_2^\epsilon, \qquad k_1\partial_z u_1^\epsilon - \frac{k_2}{\epsilon^2}\partial_z u_2^\epsilon = 0 \quad \text{on } \Gamma,\\
&m_2\frac{\partial u_2^\epsilon}{\partial t} - \tilde{\nabla}\cdot k_2\tilde{\nabla}u_2^\epsilon - \frac{k_2}{\epsilon^2}\partial_z\partial_z u_2^\epsilon = m_2 F \quad \text{in } \Omega_2,\\
&\left(k_2\tilde{\nabla}u_2^\epsilon, \frac{k_2}{\epsilon^2}\partial_z u_2^\epsilon\right)\cdot\hat{n} = 0 \quad \text{on } \partial\Omega_2 - \Gamma,
\end{aligned}\tag{4.26a}$$

at each $t > 0$, *and these satisfy the initial conditions*

$$u_1^\epsilon(\cdot, 0) = u_0(\cdot) \quad \text{on } \Omega_1, \qquad u_2^\epsilon(\cdot, 0) = u_0(\cdot) \quad \text{on } \Omega_2. \tag{4.26b}$$

Also, there is a unique $u^* \in C([0,\infty), L^2(\Omega)) \cap C^1((0,\infty), L^2(\Omega))$ with $u^*(t) \in D^0$ for each $t > 0$, such that the functions $u_1(t) = u^*(t)|_{\Omega_1} \in H^1(\Omega_1)$, $u_2(t) = \gamma(u^*(t)) \in H^1_\omega(\Gamma)$ satisfy

$$m_1 \frac{\partial u_1}{\partial t} - \nabla \cdot k_1 \nabla u_1 = m_1 F \quad \textit{in } \Omega_1, \tag{4.27a}$$

$$u_1 = 0 \quad \textit{on } \partial\Omega_1 - \Gamma, \tag{4.27b}$$

$$u_1 = u_2 \quad \textit{on } \Gamma, \quad \textit{and} \tag{4.27c}$$

$$m_2\omega \frac{\partial u_2}{\partial t} - \tilde{\nabla} \cdot k_2 \omega \tilde{\nabla} u_2 + k_1 \partial_z u_1 = m_2 \omega \tilde{F} \quad \textit{in } \Gamma, \tag{4.27d}$$

$$u_2 = 0 \quad \textit{on } \partial\Gamma, \tag{4.27e}$$

at each $t > 0$ and the initial condition

$$u_1(\cdot, 0) = u_0(\cdot) \quad \textit{on } \Omega_1, \qquad u_2(\cdot, 0) = \tilde{u}_0(\cdot) \quad \textit{on } \Gamma. \tag{4.27f}$$

Finally, we have strong convergence $u^\epsilon \to u^*$ in $\mathcal{V} = L^2(0, T; V)$ and in $C([0, T], H_0)$.

References

[1] Robert A. Adams, Sobolev Spaces, Pure Appl. Math., vol. 65, Academic Press (a subsidiary of Harcourt Brace Jovanovich Publishers), New York, London, 1975.
[2] John R. Cannon, G.H. Meyer, Diffusion in a fractured medium, SIAM J. Appl. Math. 20 (1971) 434–448.
[3] P. Grisvard, Espaces interméiaires entre espaces de Sobolev avec poids, Ann. Sc. Norm. Super. Pisa (3) 17 (1963) 255–296.
[4] Pham Huy Hung, Enrique Sánchez-Palencia, Phénomènes de transmission à travers des couches minces de conductivité élevée, J. Math. Anal. Appl. 47 (1974) 284–309.
[5] Tosio Kato, Perturbation Theory for Linear Operators, Classics Math., Springer-Verlag, Berlin, 1995, reprint of the 1980 edition.
[6] Vincent Martin, Jérôme Jaffré, Jean E. Roberts, Modeling fractures and barriers as interfaces for flow in porous media, SIAM J. Sci. Comput. 26 (5) (2005) 1667–1691 (electronic).
[7] Richard D. Meyer, Some embedding theorems for generalized Sobolev spaces and applications to degenerate elliptic differential operators, J. Math. Mech. 16 (1967) 739–760.
[8] Richard D. Meyer, Degenerate elliptic differential systems, J. Math. Anal. Appl. 29 (1970) 436–442.
[9] Donald A. Nield, Adrian Bejan, Convection in Porous Media, second ed., Springer-Verlag, New York, 1999.
[10] Enrique Sánchez-Palencia, Nonhomogeneous Media and Vibration Theory, Lecture Notes in Phys., vol. 127, Springer-Verlag, Berlin, 1980.
[11] R.E. Showalter, Degenerate evolution equations and applications, Indiana Univ. Math. J. 23 (1973/74) 655–677.
[12] R.E. Showalter, Degenerate parabolic initial-boundary value problems, J. Differential Equations 31 (3) (1979) 296–312.
[13] R.E. Showalter, Hilbert Space Methods for Partial Differential Equations, Monographs and Studies in Mathematics, vol. 1, Pitman, London–San Farncisco, CA–Melbourne, 1977, Reprinted in Electronic Monographs in Differential Equations, Electronic Journal of Differential Equations, San Marcos, TX, 1994 (http://ejde.math.txstate.edu/).
[14] R.E. Showalter, Monotone Operators in Banach Space and Nonlinear Partial Differential Equations, Math. Surveys Monogr., vol. 49, American Mathematical Society, Providence, RI, 1997.
[15] Roger Temam, Navier–Stokes Equations, Stud. Math. Appl., vol. 2, North-Holland Publishing Co., Amsterdam, 1979.
[16] Markus Weiler, J.J. McDonnell, Conceptualizing lateral preferential flow and flow networks and simulating the effects on gauged and ungauged hillslopes, Water Resour. Res. 43 (W03403) (2007), doi:10.1029/2006WR004867.