# INTERFACE APPROXIMATION OF DARCY FLOW IN A NARROW CHANNEL 

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#### Abstract

A mixed formulation is introduced for the singular problem of Darcy flow in a porous medium in a region containing a narrow fracture or channel with width $\mathcal{O}(\epsilon)$ and high permeability $\mathcal{O}\left(\frac{1}{\epsilon}\right)$. The solution converges as $\epsilon \rightarrow 0$ to that of Darcy flow coupled to tangential flow on the lower-dimensional interface or boundary.


## 1. Introduction

The flow of fluid through a fully-saturated porous medium is described by the constitutive law of Darcy,

$$
\begin{equation*}
a(x) \mathbf{u}(x, t)+\boldsymbol{\nabla} p(x, t)+\mathbf{g}(x)=\mathbf{0} \tag{1.1a}
\end{equation*}
$$

and the conservation law

$$
\begin{equation*}
c(x) \frac{\partial p(x, t)}{\partial t}+\boldsymbol{\nabla} \cdot \mathbf{u}(x, t)=F(x, t) . \tag{1.1b}
\end{equation*}
$$

Here $\mathbf{u}(x, t)$ is the fluid flux, $p(x, t)$ the pressure, and $\mathbf{g}(x)$ is the gravity force, the storeage rate term $c(x)$ is (slight) compressibility and porosity of the fluid and porous medium with sources $F(x, t)$. The density factor has been dropped from each term of (1.1b). The flow resistance $a(x)$ is fluid viscosity times the inverse of permeability of the porous medium. The system (1.1) is supplemented with appropriate boundary and initial conditions to make the initial-boundary-value problem [3]. The backward-difference approximation for $\frac{\partial p}{\partial t}$ leads to a corresponding

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boundary-value problem for the stationary system

$$
\begin{align*}
& a(x) \mathbf{u}(x)+\boldsymbol{\nabla} p(x)+\mathbf{g}(x)=\mathbf{0} \\
& c(x) \lambda p(x)+\boldsymbol{\nabla} \cdot \mathbf{u}(x)=F(x) \tag{1.2}
\end{align*}
$$

where $\lambda=h^{-1}$ is the reciprocal of the time increment $h>0$. Here we study such a problem for which the resistance coefficient $a(x)$ is of order $\epsilon>0$ on a thin region with width of order $\epsilon$ separated from the primary region by an interface, and we show that it is approximated for small $\epsilon$ by the problem on the primary region with tangential flow on the lower-dimensional interface. Such a situation provides a model for the relatively fast flow through an internal fracture in the porous medium or in a narrow channel of high permeability along a wall where the packing of particles is inefficient. See $[7,14,12]$ for further discussion and development of such models and [2] for a model of diffusion from an underground nuclear waste repository into surrounding geological layers.

The convergence of the singular narrow fracture problem to that of the interface problem has been studied before $[9,13]$ in the classical variational formulation on Sobolev spaces with linear transmission constraints posed on the interface. The usual constraints in this classical formulation are equality of pressures and the consequential matching of a linear combination of normal flux and pressure from each side for the complementary transmission condition.

We shall introduce a special mixed formulation $[6,5,8]$ which permits more general interface conditions of the form

$$
\begin{array}{r}
p^{1}-p^{2}=\alpha \mathbf{u}^{1} \cdot \mathbf{n} \\
-\mathbf{u}^{1} \cdot \mathbf{n}+\mathbf{u}^{2} \cdot \mathbf{n}=\beta \frac{\partial p^{2}}{\partial t}
\end{array}
$$

for the respective jumps in pressure and flux in (1.1). With the $L^{2}-H^{1}$ mixed formulation, which is equivalent to the classical formulation, we can obtain the case with $\alpha=0$. With the $H(\operatorname{div})-L^{2}$ mixed formulation we can obtain the case with $\beta=0$ which is needed here, but this introduces substantial difficulties in the convergence analysis. The formulation we introduce here is a combination of $H($ div $)-L^{2}$ in the primary region and $L^{2}-H^{1}$ in the narrow channel, that is, a mixed mixed formulation.

The mixed evolution problem $[4,10,17,16]$ is necessarily of degenerate type, since in the limit it becomes either elliptic $(\beta=0)$ or parabolic $(\beta>0)$ on the interface. We use the holomorphic semigroup representation of the solution to obtain rather general conditions on the data that yield existence of solutions and then use energy estimates of the $C^{0}$ semigroup representation to obtain the strong convergence of the solutions. This approach also permits inclusion of the parabolic-elliptic case of the system (1.1) with semidefinite $c(x) \geq 0$.

In the remainder of this section we describe the singular stationary problem with $\epsilon>0$ and show that the 'mixed mixed' formulation is well-posed. In Section 2 we rescale the domain to get the spaces independent of $\epsilon$ and corresponding estimates on the solutions. The limit interface problem, which is satisfied by the limit of the solutions of the singular problems as $\epsilon \rightarrow 0$, is displayed in Section 3. There we establish the convergence. In Section 4 similar results are obtained for the corresponding evolution problems.

The Singular Problem. Vectors are denoted by boldface letters, as are vector-valued functions and corresponding function spaces. We use $\tilde{\mathbf{x}}$ to denote a vector in $\mathbb{R}^{N-1}$. If $\mathbf{x} \in \mathbb{R}^{N}$, then the $\mathbb{R}^{N-1} \times\{0\}$ projection is identified with $\tilde{\mathbf{x}}=\left(x_{1}, x_{2}, \ldots, x_{N-1}\right)$ so that $\mathbf{x}=\left(\tilde{\mathbf{x}}, x_{N}\right)$. The $\mathbb{R}^{N-1}$ gradient $\widetilde{\boldsymbol{\nabla}}$ and divergence $\widetilde{\boldsymbol{\nabla}}$. are denoted similarly.

Consider a domain $\Omega^{\epsilon}=\Omega_{1} \cup \Gamma \cup \Omega_{2}^{\epsilon}$ in $\mathbb{R}^{N}$ representing a porous medium as the union of disjoint adjacent subdomains $\Omega_{1}, \Omega_{2}^{\epsilon}$ separated by a smooth domain $\Gamma=\partial \Omega_{1} \cap \partial \Omega_{2}^{\epsilon}$ in $\mathbb{R}^{N-1}$. Thus, we assume the interface $\Gamma$ is flat. Denote the thin fracture domain with width $\epsilon>0$ by

$$
\Omega_{2}^{\epsilon} \equiv\left\{\left(\tilde{\mathbf{x}}, x_{N}\right): 0<x_{N}<\epsilon, \tilde{x} \in \Gamma\right\} .
$$

It is bounded below by $\Gamma$ and above by its vertical $\epsilon$-translate, $\Gamma+\epsilon$. Let $\Omega_{1} \subset \mathbb{R}^{N}$ be a domain for which $\Omega_{1} \cap \Omega_{2}^{\epsilon}=\varnothing$ and $\partial \Omega_{1} \cap \partial \Omega_{2}^{\epsilon}=\Gamma$, and set $\Omega^{\epsilon}=\Omega_{1} \cup \Gamma \cup \Omega_{2}^{\epsilon}$. For any function on $\Omega^{\epsilon}$ we denote its restrictions to $\Omega_{1}$ and to $\Omega_{2}^{\epsilon}$ with superscripts 1 and 2 , respectively.

The singular stationary problem on $\Omega_{1} \cup \Gamma \cup \Omega_{2}^{\epsilon}$ is

$$
\begin{array}{r}
a_{1}(x) \mathbf{u}^{\epsilon, 1}+\boldsymbol{\nabla} p^{\epsilon, 1}+\mathbf{g}^{\epsilon}(x)=\mathbf{0} \text { and } \\
c_{1}(x) \lambda p^{\epsilon, 1}+\nabla \cdot \mathbf{u}^{\epsilon, 1}=F^{\epsilon} \text { in } \Omega_{1}, \\
p^{\epsilon, 1}=0 \quad \text { on } \partial \Omega_{1}-\Gamma, \\
p^{\epsilon, 1}-p^{\epsilon, 2}=\alpha \mathbf{u}^{\epsilon, 1} \cdot \mathbf{n} \text { and } \\
\lambda \beta p^{\epsilon, 2}-\mathbf{u}^{\epsilon, 1} \cdot \mathbf{n}+\mathbf{u}^{\epsilon, 2} \cdot \mathbf{n}=f_{\Gamma}^{\epsilon} \quad \text { on } \Gamma, \\
\epsilon a_{2}(x) \mathbf{u}^{\epsilon, 2}+\boldsymbol{\nabla} p^{\epsilon, 2}+\mathbf{g}^{\epsilon}(x)=\mathbf{0} \text { and } \\
c_{2}(x) \lambda p^{\epsilon, 2}+\boldsymbol{\nabla} \cdot \mathbf{u}^{\epsilon, 2}=F^{\epsilon} \text { in } \Omega_{2}^{\epsilon}, \\
\mathbf{u}^{\epsilon, 2} \cdot \mathbf{n}=0 \text { on } \partial \Omega_{2}^{\epsilon}-\Gamma, \tag{1.3g}
\end{array}
$$

for the fluid pressure $p^{\epsilon, 1}, p^{\epsilon, 2}$ and velocity $\mathbf{u}^{\epsilon, 1}, \mathbf{u}^{\epsilon, 2}$ on the respective domains $\Omega_{1}, \Omega_{2}^{\epsilon}$. The coefficients are $a_{1}, c_{1}$ on $\Omega_{1}$ and $\epsilon a_{2}, c_{2}$ on $\Omega_{2}^{\epsilon}$. The interface conditions on $\Gamma$ are that the normal fluid flux from $\Omega_{1}$ is driven by the pressure difference with resistance $\alpha \geq 0$ and that fluid is stored there at the rate $\beta \geq 0$.

With appropriate conditions on the data, we show that as $\epsilon \downarrow 0$, the solution of (1.3) converges to the solution of the interface problem on
the primary region, $\Omega_{1} \cup \Gamma$,

$$
\begin{gather*}
a_{1} \mathbf{u}^{1}+\boldsymbol{\nabla} p^{1}+\mathbf{g}=\mathbf{0} \text { and }  \tag{1.4a}\\
\lambda c_{1} p^{1}+\boldsymbol{\nabla} \cdot \mathbf{u}^{1}=F \text { in } \Omega_{1}, \tag{1.4b}
\end{gather*}
$$

$$
\begin{array}{r}
p^{1}=0 \text { on } \partial \Omega_{1}-\Gamma, \\
p^{1}-\alpha \mathbf{u}^{1} \cdot \mathbf{n}=p^{2} \text { on } \Gamma, \\
a_{2}(\tilde{\mathbf{x}}) \tilde{\mathbf{u}}^{2}+\widetilde{\boldsymbol{\nabla}} p^{2}+\tilde{\mathbf{g}}(\tilde{\mathbf{x}})=\tilde{\mathbf{0}} \text { and } \\
\lambda \beta p^{2}+\widetilde{\boldsymbol{\nabla}} \cdot \tilde{\mathbf{u}}^{2}=f_{\Gamma}+\mathbf{u}^{1} \cdot \mathbf{n} \text { on } \Gamma, \\
\tilde{\mathbf{u}}^{2} \cdot \tilde{\mathbf{n}}=0 \text { in } H^{-1 / 2}(\partial \Gamma) . \tag{1.4~g}
\end{array}
$$

The two-way coupling is attained by passing the pressure $p^{2}$ from $\Gamma$ to $\Omega_{1}$ with the Robin condition (1.4d) and the normal flux $\mathbf{u}^{1} \cdot \mathbf{n}$ from $\Omega_{1}$ to $\Gamma$ as a source in (1.4f).

The Mixed Formulation. First we show that the stationary singular problem has a unique solution. For our weak formulation of the system (1.3) we use the spaces

$$
\begin{gathered}
\mathbf{V}^{\epsilon} \equiv\left\{\mathbf{v} \in \mathbf{L}^{2}\left(\Omega^{\epsilon}\right): \boldsymbol{\nabla} \cdot \mathbf{v}^{1} \in L^{2}\left(\Omega_{1}\right),\left.\mathbf{v}^{1} \cdot \mathbf{n}\right|_{\Gamma} \in L^{2}(\Gamma)\right\} \\
Q^{\epsilon} \equiv\left\{q \in L^{2}\left(\Omega^{\epsilon}\right): \nabla q^{2} \in \mathbf{L}^{2}\left(\Omega_{2}^{\epsilon}\right)\right\}
\end{gathered}
$$

with the norms

$$
\begin{gathered}
\|\mathbf{v}\|_{\mathbf{v}^{\epsilon}}=\left(\|\mathbf{v}\|_{\mathbf{L}^{2}\left(\Omega^{\epsilon}\right)}^{2}+\left\|\boldsymbol{\nabla} \cdot \mathbf{v}^{1}\right\|_{L^{2}\left(\Omega_{1}\right)}^{2}+\left\|\mathbf{v}^{1} \cdot \mathbf{n}\right\|_{L^{2}(\Gamma)}^{2}\right)^{1 / 2}, \\
\|q\|_{Q^{\epsilon}}=\left(\|q\|_{L^{2}\left(\Omega^{\epsilon}\right)}^{2}+\left\|\boldsymbol{\nabla} q^{2}\right\|_{\mathbf{L}^{2}\left(\Omega_{2}^{\epsilon}\right)}^{2}\right)^{1 / 2}
\end{gathered}
$$

The solution of the singular problem (1.3) satisfies
(1.5a) $\mathbf{u}^{\epsilon} \in \mathbf{V}^{\epsilon}, p^{\epsilon} \in Q^{\epsilon}: \int_{\Omega_{1}} a_{1} \mathbf{u}^{\epsilon} \cdot \mathbf{v} d x-\int_{\Omega_{1}} p^{\epsilon} \boldsymbol{\nabla} \cdot \mathbf{v} d x$

$$
\begin{array}{rl}
+\epsilon \int_{\Omega_{2}^{\epsilon}} a_{2} \mathbf{u}^{\epsilon} \cdot \mathbf{v} & d x+\int_{\Omega_{2}^{\epsilon}} \boldsymbol{\nabla} p^{\epsilon} \cdot \mathbf{v} d x+\int_{\Gamma} p^{\epsilon, 2} \mathbf{v}^{1} \cdot \mathbf{n} d S \\
& +\int_{\Gamma} \alpha\left(\mathbf{u}^{\epsilon, 1} \cdot \mathbf{n}\right)\left(\mathbf{v}^{1} \cdot \mathbf{n}\right) d S=-\int_{\Omega^{\epsilon}} \mathbf{g}^{\epsilon} \cdot \mathbf{v} d x
\end{array}
$$

$$
\begin{align*}
& \int_{\Omega_{1}} \lambda c_{1} p^{\epsilon} q d x+\int_{\Omega_{2}^{\epsilon}} \lambda c_{2} p^{\epsilon} q d x+\int_{\Gamma} \lambda \beta p^{\epsilon, 2} q^{2} d S  \tag{1.5b}\\
& +\int_{\Omega_{1}} \boldsymbol{\nabla} \cdot \mathbf{u}^{\epsilon} q d x-\int_{\Omega_{2}^{\epsilon}} \mathbf{u}^{\epsilon} \cdot \nabla q d x-\int_{\Gamma} \mathbf{u}^{\epsilon, 1} \cdot \mathbf{n} q^{2} d S \\
& \quad=\int_{\Omega^{\epsilon}} F^{\epsilon} q d x+\int_{\Gamma} f_{\Gamma}^{\epsilon} q^{2} d S \text { for all } \mathbf{v} \in \mathbf{V}^{\epsilon}, q \in Q^{\epsilon}
\end{align*}
$$

Remark 1.1. We have combined the $H($ div $)-L^{2}$ mixed formulation on $\Omega_{1}$ with the $L^{2}-H^{1}$ mixed formulation on $\Omega_{2}^{\epsilon}$. Each $q \in Q^{\epsilon}$ has a welldefined trace $\left.q^{2}\right|_{\Gamma} \in H^{1 / 2}\left(\partial \Omega_{2}\right)$ and similarly each $\mathbf{v} \in \mathbf{V}^{\epsilon}$ determines $a$ normal trace $\mathbf{v}^{1} \cdot \mathbf{n} \in H^{-1 / 2}\left(\partial \Omega_{1}\right)$. For each such $\mathbf{v}$ we additionally require that the restriction $\left.\mathbf{v}^{1} \cdot \mathbf{n}\right|_{\Gamma}$ to functions on $\Gamma$ belongs to $L^{2}(\Gamma)$. This permits the definition of the coupling terms on $\Gamma$ in (1.5). This non-standard choice of spaces permits the more general transmission conditions (1.3d). Moreover, there are no transmission constraints used to couple the spaces along $\Gamma$.

Define operators $\mathcal{A}^{\epsilon}: \mathbf{V}^{\epsilon} \rightarrow \mathbf{V}^{\epsilon^{\prime}}, \mathcal{B}^{\epsilon}: \mathbf{V}^{\epsilon} \rightarrow Q^{\epsilon^{\prime}}, C^{\epsilon}: Q^{\epsilon} \rightarrow Q^{\epsilon^{\prime}}$ by (1.6a)
$\mathcal{A}^{\epsilon} \mathbf{u}(\mathbf{v})=\int_{\Omega_{1}} a_{1} \mathbf{u} \cdot \mathbf{v} d x+\epsilon \int_{\Omega_{2}^{\epsilon}} a_{2} \mathbf{u} \cdot \mathbf{v} d x+\int_{\Gamma} \alpha\left(\mathbf{u}^{\epsilon, 1} \cdot \mathbf{n}\right)\left(\mathbf{v}^{1} \cdot \mathbf{n}\right) d S$,

$$
\begin{gather*}
\mathcal{B}^{\epsilon} \mathbf{u}(q)=-\int_{\Omega_{1}} \boldsymbol{\nabla} \cdot \mathbf{u} q d x+\int_{\Gamma} \mathbf{u}^{1} \cdot \mathbf{n} q^{2} d S+\int_{\Omega_{2}^{\epsilon}} \mathbf{u} \cdot \boldsymbol{\nabla} q d x  \tag{1.6b}\\
\mathcal{C}^{\epsilon} p(q)=\int_{\Omega_{1}} c_{1} p q d x+\int_{\Omega_{2}^{\epsilon}} c_{2} p q d x+\int_{\Gamma} \beta p^{2} q^{2} d S . \tag{1.6c}
\end{gather*}
$$

Then the system (1.5) is a mixed formulation for (1.3) of the form

$$
\begin{aligned}
\mathbf{u}^{\epsilon} \in \mathbf{V}^{\epsilon}, & p^{\epsilon} \in Q^{\epsilon}: \\
& \mathcal{A}^{\epsilon} \mathbf{u}^{\epsilon}(\mathbf{v})+\mathcal{B}^{\epsilon^{\prime}} p^{\epsilon}(\mathbf{v})=-\mathbf{g}^{\epsilon}(\mathbf{v}), \mathbf{v} \in \mathbf{V}^{\epsilon}, \\
& -\mathcal{B}^{\epsilon} \mathbf{u}^{\epsilon}(q)+\lambda \mathcal{C}^{\epsilon} p^{\epsilon}(q)=f^{\epsilon}(q), q \in Q^{\epsilon}
\end{aligned}
$$

Such problems are well-posed under rather general conditions. See $[8,6,16]$.

Theorem 1.1. Assume that $\mathbf{V}$ and $Q$ are Hilbert spaces and $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are continuous linear operators $\mathcal{A}: \mathbf{V} \rightarrow \mathbf{V}^{\prime}, \mathcal{B}: \mathbf{V} \rightarrow Q^{\prime}, C: Q \rightarrow Q^{\prime}$ such that

- $\mathcal{A}$ is non-negative and $\mathbf{V}$-coercive on $\operatorname{Ker} \mathcal{B}$,
- $\mathcal{C}$ is non-negative, symmetric, and
- $\mathcal{B}^{\prime}$ is bounding, i.e., it is 1-1 and

$$
\begin{equation*}
\inf _{q \in Q} \sup _{\mathbf{v} \in \mathbf{V}} \frac{|\mathcal{B} \mathbf{v}(q)|}{\|\mathbf{v}\|_{\mathbf{v}}\left\|_{q}\right\|_{Q}} \geq c_{0}>0 \tag{1.7}
\end{equation*}
$$

Then for every $f \in Q^{\prime}, \mathbf{g} \in \mathbf{V}^{\prime}$ and $\lambda \geq 0$ the system

$$
\mathbf{u} \in \mathbf{V}, p \in Q:
$$

$$
\begin{gather*}
\mathcal{A} \mathbf{u}+\mathcal{B}^{\prime} p=-\mathbf{g} \text { in } \mathbf{V}^{\prime}  \tag{1.8}\\
-\mathcal{B} \mathbf{u}+\lambda \mathcal{C} p=f \text { in } Q^{\prime}
\end{gather*}
$$

has a unique solution, and it satisfies the estimate

$$
\begin{equation*}
\|\mathbf{u}\|_{\mathbf{v}}+\|p\|_{Q} \leq K\left(\|\mathbf{g}\|_{\mathbf{v}^{\prime}}+\|f\|_{Q^{\prime}}\right) \tag{1.9}
\end{equation*}
$$

In order to apply Theorem 1.1, we use the following classical result.
Lemma 1.2. There is a $c_{\epsilon}>0$ for which

$$
\begin{equation*}
\|\nabla q\|_{L^{2}\left(\Omega_{2}^{\epsilon}\right)}^{2}+\|q\|_{L^{2}(\Gamma)}^{2} \geq c_{\epsilon}\|q\|_{L^{2}\left(\Omega_{2}^{\epsilon}\right)}^{2} \tag{1.10}
\end{equation*}
$$

for all $q \in H^{1}\left(\Omega_{2}^{\epsilon}\right)$.
(See Proposition 5.2 of [15].)
Lemma 1.3. For each $\epsilon>0$, the operator $\mathcal{B}^{\epsilon}$ satisfies the inf-sup condition (1.7) on $\mathbf{V}^{\epsilon}$ and $Q^{\epsilon}$.

Proof. Let $q \in Q^{\epsilon}$ and denote by $\xi$ the unique solution of the mixed problem

$$
-\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \xi=q^{1} \text { in } \Omega_{1}, \boldsymbol{\nabla} \xi \cdot \mathbf{n}=q^{2} \text { on } \Gamma, \xi=0 \text { on } \partial \Omega_{1}-\Gamma .
$$

Set $\mathbf{v}^{1}=\boldsymbol{\nabla} \xi$. Then $-\boldsymbol{\nabla} \cdot \mathbf{v}^{1}=q^{1}$ and $\mathbf{v}^{1} \cdot \mathbf{n}=q^{2}$ on $\Gamma$ with $c_{1}\left\|\mathbf{v}^{1}\right\|_{\mathbf{L}_{\text {div }}^{2}\left(\Omega_{1}\right)} \leq\left\|q^{1}\right\|_{L^{2}\left(\Omega_{1}\right)}$ by the Poincaré inequality. Set $\mathbf{v}^{2}=\nabla q^{2}$. For $\mathbf{v}=\left[\mathbf{v}^{1}, \mathbf{v}^{2}\right]$ on $\Omega^{\epsilon}$ we have $\mathbf{v} \in \mathbf{V}^{\epsilon}$ and with (1.10) the estimate

$$
\begin{array}{r}
\mathcal{B}^{\epsilon} \mathbf{v}(q)=-\int_{\Omega_{1}} \boldsymbol{\nabla} \cdot \mathbf{v}^{1} q^{1} d x+\int_{\Gamma} \mathbf{v}^{1} \cdot \mathbf{n} q^{2} d S+\int_{\Omega_{2}^{\epsilon}} \mathbf{v}^{2} \cdot \boldsymbol{\nabla} q^{2} d x \\
=\int_{\Omega_{1}}\left|q^{1}\right|^{2} d x+\int_{\Gamma}\left|q^{2}\right|^{2} d x+\int_{\Omega_{2}^{\epsilon}}\left|\nabla q^{2}\right|^{2} d x \\
\geq \int_{\Omega_{1}}\left|q^{1}\right|^{2} d x+\frac{c_{\epsilon}}{2} \int_{\Omega_{2}^{\epsilon}}\left|q^{2}\right|^{2} d x+\frac{1}{2}\left(\int_{\Gamma}\left|q^{2}\right|^{2} d x+\int_{\Omega_{2}^{\epsilon}}\left|\nabla q^{2}\right|^{2} d x\right) \\
\geq c\|\mathbf{v}\|_{V^{\epsilon}}\|q\|_{Q^{\epsilon}}
\end{array}
$$

with $c_{0}=\min \left(c_{1}, \frac{1}{2}, \frac{c_{\epsilon}}{2}\right)$, and this yields the inf-sup condition (1.7).
Theorem 1.4. Assume that $0<\epsilon \leq 1,0 \leq \lambda, 0 \leq \alpha, 0 \leq \beta$, $a(\cdot), c(\cdot) \in L^{\infty}\left(\Omega^{\epsilon}\right), a(x) \geq a^{*}>0$ and $c(x) \geq 0$ on $\Omega^{\epsilon}, F^{\epsilon} \in L^{2}\left(\Omega^{\epsilon}\right)$, $\mathbf{g}^{\epsilon} \in \mathbf{L}^{2}\left(\Omega^{\epsilon}\right)$, and $f_{\Gamma}^{\epsilon} \in L^{2}(\Gamma)$. Then the system (1.5) has a unique solution.

We show below that the limit problem (1.4) is likewise well-posed in a mixed formulation (1.8).

## 2. The Scaled Problem

By scaling $\Omega_{2}^{\epsilon}$ in the vertical direction with $x_{N}=\epsilon z$, we reformulate the singular problem (1.5) on the domains

$$
\Omega_{2} \equiv\{(\tilde{\mathbf{x}}, z): 0<z<1, \tilde{\mathbf{x}} \in \Gamma\}, \quad \Omega \equiv \Omega_{1} \cup \Gamma \cup \Omega_{2} .
$$

These domains and the corresponding spaces

$$
\begin{gather*}
\mathbf{V} \equiv\left\{\mathbf{v} \in \mathbf{L}^{2}(\Omega): \boldsymbol{\nabla} \cdot \mathbf{v}^{1} \in L^{2}\left(\Omega_{1}\right),\left.\mathbf{v}^{1} \cdot \mathbf{n}\right|_{\Gamma} \in L^{2}(\Gamma)\right\} \\
Q \equiv\left\{q \in L^{2}(\Omega): \nabla \boldsymbol{\nabla} q^{2} \in \mathbf{L}^{2}\left(\Omega_{2}\right)\right\} \tag{2.1}
\end{gather*}
$$

are independent of $\epsilon$. The norms on the spaces $\mathbf{V}$ and $Q$ are given by

$$
\begin{aligned}
\|\mathbf{v}\|_{\mathbf{v}}= & \left(\|\mathbf{v}\|_{\mathbf{L}^{2}(\Omega)}^{2}+\left\|\boldsymbol{\nabla} \cdot \mathbf{v}^{1}\right\|_{L^{2}\left(\Omega_{1}\right)}^{2}+\left\|\mathbf{v}^{1} \cdot \mathbf{n}\right\|_{L^{2}(\Gamma)}^{2}\right)^{1 / 2} \\
& \|q\|_{Q}=\left(\|q\|_{L^{2}(\Omega)}^{2}+\left\|\nabla q^{2}\right\|_{\mathbf{L}^{2}\left(\Omega_{2}\right)}^{2}\right)^{1 / 2}
\end{aligned}
$$

The gradient is written as $\boldsymbol{\nabla}=\left(\widetilde{\boldsymbol{\nabla}}, \partial_{x_{N}}\right)$, and it becomes $\left(\widetilde{\boldsymbol{\nabla}}, \frac{1}{\epsilon} \partial_{z}\right)$ on $\Omega_{2}$ under the scaling above. The scaled singular problem is to find
(2.2a) $\mathbf{u}^{\epsilon} \in \mathbf{V}, p^{\epsilon} \in Q: \quad \int_{\Omega_{1}} a_{1} \mathbf{u}^{\epsilon} \cdot \mathbf{v} d x-\int_{\Omega_{1}} p^{\epsilon} \boldsymbol{\nabla} \cdot \mathbf{v} d x$ $+\epsilon^{2} \int_{\Omega_{2}} a_{2} \mathbf{u}^{\epsilon} \cdot \mathbf{v} d x+\epsilon \int_{\Omega_{2}} \widetilde{\nabla} p^{\epsilon} \cdot \tilde{\mathbf{v}} d x+\int_{\Omega_{2}} \partial_{z} p^{\epsilon} v_{N} d x$
$+\int_{\Gamma} p^{\epsilon, 2} \mathbf{v}^{1} \cdot \mathbf{n} d S+\int_{\Gamma} \alpha\left(\mathbf{u}^{\epsilon, 1} \cdot \mathbf{n}\right)\left(\mathbf{v}^{1} \cdot \mathbf{n}\right) d S$

$$
=-\int_{\Omega_{1}} \mathbf{g}^{\epsilon} \cdot \mathbf{v} d x-\epsilon \int_{\Omega_{2}} \mathbf{g}^{\epsilon} \cdot \mathbf{v} d x
$$

(2.2b) $\int_{\Omega_{1}} \lambda c_{1} p^{\epsilon} q d x+\epsilon \int_{\Omega_{2}} \lambda c_{2} p^{\epsilon} q d x+\int_{\Gamma} \lambda \beta p^{2} q^{2} d S$ $+\int_{\Omega_{1}} \boldsymbol{\nabla} \cdot \mathbf{u}^{\epsilon} q d x-\epsilon \int_{\Omega_{2}} \tilde{\mathbf{u}}^{\epsilon, 2} \cdot \widetilde{\nabla} q d x-\int_{\Omega_{2}} u_{N}^{\epsilon, 2} \partial_{z} q d x-\int_{\Gamma} \mathbf{u}^{\epsilon, 1} \cdot \mathbf{n} q^{2} d S$
$=\int_{\Omega_{1}} F^{\epsilon, 1} q d x+\epsilon \int_{\Omega_{2}} F^{\epsilon, 2} q d x+\int_{\Gamma} f_{\Gamma}^{\epsilon} q^{2} d S$ for all $\mathbf{v} \in \mathbf{V}, q \in Q$.
Theorem 1.4 shows that the system (2.2) has a unique solution for each $\epsilon, 0<\epsilon \leq 1$. This solution satisfies the equations

$$
\begin{gather*}
a_{1} \mathbf{u}^{\epsilon}+\boldsymbol{\nabla} p^{\epsilon}+\mathbf{g}^{\epsilon}=\mathbf{0} \text { and }  \tag{2.3a}\\
\lambda c_{1} p^{\epsilon}+\boldsymbol{\nabla} \cdot \mathbf{u}^{\epsilon}=F^{\epsilon} \text { in } \Omega_{1} \tag{2.3b}
\end{gather*}
$$

$$
\begin{gather*}
\epsilon a_{2} \tilde{\mathbf{u}}^{\epsilon, 2}+\widetilde{\boldsymbol{\nabla}} p^{\epsilon}+\tilde{\mathbf{g}}^{\epsilon}=\tilde{\mathbf{0}}, \epsilon^{2} a_{2} u_{N}^{\epsilon, 2}+\partial_{z} p^{\epsilon}+\epsilon g_{N}^{\epsilon}=0 \text { and }  \tag{2.3f}\\
\epsilon \lambda c_{2} p^{\epsilon}+\epsilon \widetilde{\boldsymbol{\nabla}} \cdot \tilde{\mathbf{u}}^{\epsilon, 2}+\partial_{z} u_{N}^{\epsilon, 2}=\epsilon F^{\epsilon} \text { in } \Omega_{2}
\end{gather*}
$$

$$
\begin{equation*}
\left(\epsilon \tilde{\mathbf{u}}^{\epsilon, 2}, u_{N}^{\epsilon, 2}\right) \cdot \mathbf{n}=0 \text { on } \partial \Omega_{2}-\Gamma \tag{2.3~g}
\end{equation*}
$$

The Estimates. We shall assume additionally that

$$
\begin{equation*}
\left\|F^{\epsilon}\right\|_{L^{2}\left(\Omega^{\epsilon}\right)} \text { is bounded and } F^{1, \epsilon} \xrightarrow{w} F^{1} \text { in } L^{2}\left(\Omega_{1}\right), \tag{2.4a}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{g}^{\epsilon} \stackrel{w}{\longrightarrow} \mathbf{g} \text { in } \mathbf{L}^{2}\left(\Omega_{1}\right), \mathbf{g}^{2, \epsilon}(\tilde{\mathbf{x}}, \epsilon z) \stackrel{w}{\rightharpoonup} \mathbf{g}(\tilde{\mathbf{x}}) \text { in } \mathbf{L}^{2}\left(\Omega_{2}\right), \tag{2.4b}
\end{equation*}
$$

$$
\begin{equation*}
\text { and } f_{\Gamma}^{\epsilon} \stackrel{w}{\rightharpoonup} f_{\Gamma} \text { in } L^{2}(\Gamma) \tag{2.4c}
\end{equation*}
$$

Note that $\epsilon^{1 / 2} F^{2, \epsilon}$ is bounded in $L^{2}\left(\Omega_{2}\right)$, so $\epsilon F^{2, \epsilon} \rightarrow 0$.
Set $\mathbf{v}=\mathbf{u}^{\epsilon}, q=p^{\epsilon}$ in (2.2) and add to obtain

$$
\begin{align*}
& \text { (2.5) } a^{*}\left(\left\|\mathbf{u}^{\epsilon, 1}\right\|_{0, \Omega_{1}}^{2}+\left\|\epsilon \mathbf{u}^{\epsilon, 2}\right\|_{0, \Omega_{2}}^{2}\right)+\alpha\left\|\mathbf{u}^{\epsilon, 1} \cdot \mathbf{n}\right\|_{L^{2}(\Gamma)}^{2}+\lambda \beta\left\|p^{\epsilon, 2}\right\|_{L^{2}(\Gamma)}^{2}  \tag{2.5}\\
& +\lambda\left\|c_{1}^{1 / 2} p^{\epsilon}\right\|_{0, \Omega_{1}}^{2}+\lambda\left\|c_{2}^{1 / 2} \epsilon^{1 / 2} p^{\epsilon}\right\|_{0, \Omega_{2}}^{2}=\int_{\Omega_{1}} F^{\epsilon} p^{\epsilon} d x \\
& +\int_{\Omega_{2}} \epsilon F^{\epsilon} p^{\epsilon} d x+\int_{\Gamma} f_{\Gamma}^{\epsilon} p^{\epsilon, 2} d S-\int_{\Omega_{1}} \mathbf{g}^{\epsilon} \cdot \mathbf{u}^{\epsilon} d x-\int_{\Omega_{2}} \mathbf{g}^{\epsilon} \cdot \epsilon \mathbf{u}^{\epsilon} d x \\
& \leq C\left(\left\|F^{\epsilon}\right\|_{0, \Omega}+\left\|f_{\Gamma}^{\epsilon}\right\|_{0, \Gamma}\right)\left\|p^{\epsilon}\right\|_{Q}+\left\|\mathbf{g}^{\epsilon}\right\|_{0, \Omega}\left(\left\|\mathbf{u}^{\epsilon, 1}\right\|_{0, \Omega_{1}}+\left\|\epsilon \mathbf{u}^{\epsilon, 2}\right\|_{0, \Omega_{2}}\right)
\end{align*}
$$

The constant $C$ is independent of $\epsilon \leq 1$. From (2.3f) we have

$$
\begin{gather*}
\left\|\widetilde{\nabla} p^{\epsilon, 2}\right\|_{0, \Omega_{2}} \leq \epsilon\left\|a_{2}\right\|_{L^{\infty}\left(\Omega_{2}\right)}\left\|\tilde{\mathbf{u}}^{\epsilon, 2}\right\|_{0, \Omega_{2}}+\left\|\tilde{\mathbf{g}}^{\epsilon}\right\|_{0, \Omega_{2}}  \tag{2.6a}\\
\left\|\partial_{z} p^{\epsilon, 2}\right\|_{0, \Omega_{2}} \leq \epsilon^{2}\left\|a_{2}\right\|_{L^{\infty}\left(\Omega_{2}\right)}\left\|u_{N}^{\epsilon, 2}\right\|_{0, \Omega_{2}}+\epsilon\left\|g_{N}^{\epsilon}\right\|_{0, \Omega_{2}} \tag{2.6b}
\end{gather*}
$$

so we obtain for $0<\epsilon \leq 1$

$$
\begin{equation*}
\left\|\nabla p^{\epsilon, 2}\right\|_{0, \Omega_{2}} \leq\left\|a_{2}\right\|_{L^{\infty}\left(\Omega_{2}\right)}\left\|\epsilon \mathbf{u}^{\epsilon, 2}\right\|_{0, \Omega_{2}}+\left\|\mathbf{g}^{\epsilon}\right\|_{0, \Omega_{2}} \tag{2.7}
\end{equation*}
$$

From (2.3a) we obtain

$$
\left\|\nabla p^{\epsilon, 1}\right\|_{0, \Omega_{1}} \leq\left\|a_{1}\right\|_{L^{\infty}\left(\Omega_{1}\right)}\left\|\mathbf{u}^{\epsilon, 1}\right\|_{0, \Omega_{1}}+\left\|\mathbf{g}^{\epsilon}\right\|_{0, \Omega_{1}}
$$

With the boundary condition (2.3c) and the Poincaré inequality, this shows the left side of (2.5) bounds $\left\|p^{\epsilon, 1}\right\|_{H^{1}\left(\Omega_{1}\right)}^{2}$. The transmission condition (2.3d) and (2.7) in (1.10) show that the left side of (2.5) bounds $\left\|p^{\epsilon, 2}\right\|_{H^{1}\left(\Omega_{2}\right)}^{2}$. We conclude from these together with (2.6b) and (2.3b) that each of the sequences

$$
\begin{gather*}
\left\|\mathbf{u}^{\epsilon, 1}\right\|_{0, \Omega_{1}},\left\|\epsilon \mathbf{u}^{\epsilon, 2}\right\|_{0, \Omega_{2}}, \alpha^{1 / 2}\left\|\mathbf{u}^{\epsilon, 1} \cdot \mathbf{n}\right\|_{L^{2}(\Gamma)},  \tag{2.8}\\
\left\|p^{\epsilon, 1}\right\|_{H^{1}\left(\Omega_{1}\right)},\left\|p^{\epsilon, 2}\right\|_{H^{1}\left(\Omega_{2}\right)},\left\|\frac{1}{\epsilon} \partial_{z} p^{\epsilon}\right\|_{0, \Omega_{2}},\left\|\boldsymbol{\nabla} \cdot \mathbf{u}^{\epsilon, 1}\right\|_{L^{2}\left(\Omega_{1}\right)} \tag{2.9}
\end{gather*}
$$

is bounded. In $L^{2}\left(\Omega_{2}\right)$ we know only that the combination $\widetilde{\boldsymbol{\nabla}} \cdot \tilde{\mathbf{u}}^{\epsilon, 2}+$ $\frac{1}{\epsilon} \partial_{z} u_{N}^{\epsilon, 2}$ is bounded due to (2.3g).
Remark 2.1. The preceding can be done with the boundary condition (2.3c) replaced by a Neumann condition if the coefficient $c_{1}(\cdot)$ is not identically zero and $\lambda>0$. This would use the following result.

Lemma 2.1. Assume the nonnegative function $c_{1}(\cdot)$ is non-zero in $L^{\infty}\left(\Omega_{1}\right)$. There is a $c>0$ for which

$$
\|\nabla q\|_{\mathbf{L}^{2}\left(\Omega_{1}\right)}^{2}+\left\|c_{1}^{1 / 2} q\right\|_{L^{2}\left(\Omega_{1}\right)}^{2} \geq c\|q\|_{H^{1}\left(\Omega_{1}\right)}^{2}
$$

for $q \in H^{1}\left(\Omega_{1}\right)$.
The Weak Limits. We have bounds on $\mathbf{u}^{\epsilon}=\left[\mathbf{u}^{\epsilon, 1}, \epsilon \mathbf{u}^{\epsilon, 2}\right]$ in $\mathbf{V}$ and on $p^{\epsilon}=\left[p^{\epsilon, 1}, p^{\epsilon, 2}\right]$ in $H^{1}\left(\Omega_{1}\right) \times H^{1}\left(\Omega_{2}\right)$, hence, in $Q$. Therefore, there must exist $p \in Q, \mathbf{u}=\left[\mathbf{u}^{1}, \mathbf{u}^{2}\right] \in \mathbf{V}, \eta \in L^{2}\left(\Omega_{2}\right)$ such that for some subsequence, hereafter denoted the same, we have weak convergence

$$
\begin{equation*}
p^{\epsilon} \stackrel{w}{\rightharpoonup} p \text { in } Q, \text { strongly in } L^{2}(\Omega), \tag{2.10a}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{u}^{\epsilon, 1} \xrightarrow{w} \mathbf{u}^{1} \text { in } \mathbf{L}^{2}\left(\Omega_{1}\right) \text { and } \boldsymbol{\nabla} \cdot \mathbf{u}^{\epsilon, 1} \stackrel{w}{\longrightarrow} \boldsymbol{\nabla} \cdot \mathbf{u}^{1} \text { in } L^{2}\left(\Omega_{1}\right), \tag{2.10b}
\end{equation*}
$$

$$
\begin{gather*}
\alpha^{1 / 2} \mathbf{u}^{\epsilon, 1} \cdot \mathbf{n} \stackrel{w}{\rightharpoonup} \alpha^{1 / 2} \mathbf{u}^{1} \cdot \mathbf{n} \text { in } L^{2}(\Gamma),  \tag{2.10c}\\
\epsilon \mathbf{u}^{\epsilon, 2} \stackrel{w}{\rightharpoonup} \mathbf{u}^{2} \text { in } \mathbf{L}^{2}\left(\Omega_{2}\right),  \tag{2.10d}\\
\frac{1}{\epsilon} \partial_{z} p^{\epsilon} \stackrel{w}{\sim} \eta, \quad \partial_{z} p^{\epsilon} \rightarrow 0 \text { strongly in } L^{2}\left(\Omega_{2}\right) . \tag{2.10e}
\end{gather*}
$$

In the equation (2.2b), take limits with $q=\epsilon \phi \in C_{0}^{\infty}\left(\Omega_{2}\right)$; then from (2.10d) we conclude $\left\langle\epsilon \partial_{z} u_{N}^{\epsilon, 2}, \phi\right\rangle_{D^{\prime}\left(\Omega_{2}\right), D\left(\Omega_{2}\right)} \rightarrow\left\langle\partial_{z} u_{N}^{2}, \phi\right\rangle_{D^{\prime}\left(\Omega_{2}\right), D\left(\Omega_{2}\right)}=$ 0 , so the component $u_{N}^{2}=u_{N}^{2}(\tilde{\mathbf{x}})$ is independent of $z$ in $\Omega_{2}$. Again with $\epsilon q$ in (2.2b) with a general $q \in Q$, take limits and use (2.10d) to conclude

$$
\begin{aligned}
& 0=\lim _{\epsilon \downarrow 0} \int_{\Omega_{2}} \epsilon u_{N}^{\epsilon, 2} \partial_{z} q d x=\int_{\Omega_{2}} u_{N}^{2}(\tilde{\mathbf{x}}) \partial_{z} q(\tilde{\mathbf{x}}, z) d x \\
&=\int_{\Gamma} u_{N}^{2}(\tilde{\mathbf{x}})\left(\int_{0}^{1} \partial_{z} q(\widetilde{x}, z) d z\right) d \tilde{\mathbf{x}} \\
&=\int_{\Gamma} u_{N}^{2}(\widetilde{x})(q(\widetilde{x}, 1)-q(\widetilde{x}, 0)) d \widetilde{x} .
\end{aligned}
$$

Since this holds for all $q \in Q$, in particular with $q(\widetilde{x}, 0)=\left.q\right|_{\Gamma}=0$ and $q(\widetilde{x}, 1)=\left.q\right|_{\Gamma+1}=\phi(\widetilde{x})$ for $\phi \in C_{0}^{\infty}(\Gamma)$ arbitrary, we obtain $u_{N}^{2}=0$.

Now consider a function $\tilde{\mathbf{v}} \in\left(C_{0}^{\infty}\left(\Omega_{2}\right)\right)^{N-1}$, set $\mathbf{v}=\left(\frac{1}{\epsilon} \tilde{\mathbf{v}}, 0\right)$ in (2.2a) and let $\epsilon \downarrow 0$ to obtain

$$
\begin{aligned}
\epsilon \int_{\Omega_{2}} a_{2}(x) \tilde{\mathbf{u}}^{\epsilon, 2} \cdot \tilde{\mathbf{v}} d x+ & \int_{\Omega_{2}}\left(\tilde{\nabla} p^{\epsilon}+\tilde{\mathbf{g}}^{\epsilon}\right) \cdot \tilde{\mathbf{v}} d x \rightarrow \\
& \int_{\Omega_{2}} a_{2} \tilde{\mathbf{u}}^{2} \cdot \tilde{\mathbf{v}} d x+\int_{\Omega_{2}}(\widetilde{\boldsymbol{\nabla}} p+\tilde{\mathbf{g}}) \cdot \tilde{\mathbf{v}} d x=0 .
\end{aligned}
$$

This holds for all $\tilde{\mathbf{v}} \in\left(C_{0}^{\infty}\left(\Omega_{2}\right)\right)^{N-1}$, so we conclude the lower-dimensional Darcy-type constitutive law

$$
a_{2}(x) \tilde{\mathbf{u}}^{2}+\widetilde{\boldsymbol{\nabla}} p^{2}+\tilde{\mathbf{g}}=\tilde{\mathbf{0}} \text { in } \Omega_{2}
$$

From (2.10e) it is clear that $p^{2}$ does not depend on the variable $z$, i.e. $p^{2}=p^{2}(\tilde{\mathbf{x}})$. Therefore if we assume

$$
\begin{equation*}
a_{2}=a_{2}(\tilde{\mathbf{x}}), \tilde{\mathbf{g}}=\tilde{\mathbf{g}}(\tilde{\mathbf{x}}) \text { in } \Omega_{2}, \tag{2.11}
\end{equation*}
$$

we conclude $\tilde{\mathbf{u}}^{2}=\tilde{\mathbf{u}}^{2}(\tilde{\mathbf{x}})$ is independent of $z$ in $\Omega_{2}$.

## 3. The Limit Problem

Define the subspaces
(3.1a) $\mathbf{V}_{0} \equiv\left\{\mathbf{v} \in \mathbf{V}: \partial_{z} \mathbf{v}^{2}=\mathbf{0}\right.$ and $v_{N}^{2}=0$ in $\left.\Omega_{2}\right\}$

$$
=\left\{\left[\mathbf{v}^{1}, \tilde{\mathbf{v}}^{2}\right] \in \mathbf{L}^{2}\left(\Omega_{1}\right) \times \mathbf{L}^{2}(\Gamma): \boldsymbol{\nabla} \cdot \mathbf{v}^{1} \in L^{2}\left(\Omega_{1}\right), \mathbf{v}^{1} \cdot \mathbf{n} \in L^{2}(\Gamma)\right\},
$$

$$
\begin{equation*}
Q_{0} \equiv\left\{q \in Q: \partial_{z} q=0 \text { in } \Omega_{2}\right\}=\left\{\left[q^{1}, q^{2}\right] \in L^{2}\left(\Omega_{1}\right) \times H^{1}(\Gamma)\right\} . \tag{3.1b}
\end{equation*}
$$

That is, $\mathbf{v}^{2}=\left[\tilde{\mathbf{v}}^{2}(\tilde{\mathbf{x}}), 0\right]$ when $\mathbf{v} \in \mathbf{V}_{0}$ and $q^{2}=q^{2}(\tilde{\mathbf{x}})$ when $q \in Q_{0}$.
If $\mathbf{v} \in \mathbf{V}_{0}$ then we have $\left[\mathbf{v}^{1}, \frac{1}{\epsilon} \mathbf{v}^{2}\right] \in \mathbf{V}_{0}$. Using the latter and a $q \in Q_{0}$ as test functions in (2.2), we obtain

$$
\begin{aligned}
& \int_{\Omega_{1}} a_{1} \mathbf{u}^{\epsilon} \cdot \mathbf{v} d x-\int_{\Omega_{1}} p^{\epsilon} \boldsymbol{\nabla} \cdot \mathbf{v} d x+\epsilon \int_{\Omega_{2}} a_{2} \tilde{\mathbf{u}}^{\epsilon, 2} \cdot \tilde{\mathbf{v}} d x \\
& +\int_{\Omega_{2}} \tilde{\nabla} p^{\epsilon} \cdot \tilde{\mathbf{v}} d x+\int_{\Gamma} p^{\epsilon, 2} \mathbf{v}^{1} \cdot \mathbf{n} d S+\int_{\Gamma} \alpha\left(\mathbf{u}^{\epsilon, 1} \cdot \mathbf{n}\right)\left(\mathbf{v}^{1} \cdot \mathbf{n}\right) d S \\
& \\
& =-\int_{\Omega_{1}} \mathbf{g} \cdot \mathbf{v} d x-\int_{\Omega_{2}} \tilde{\mathbf{g}} \cdot \tilde{\mathbf{v}}^{2} d x, \\
& \begin{array}{r}
\int_{\Omega_{1}} \lambda c_{1} p^{\epsilon} q d x+\epsilon \int_{\Omega_{2}} \lambda c_{2} p^{\epsilon} q d x+\int_{\Gamma} \lambda \beta p^{\epsilon, 2} q^{2} d S \\
+\int_{\Omega_{1}} \boldsymbol{\nabla} \cdot \mathbf{u}^{\epsilon} q d x-\epsilon \int_{\Omega_{2}} \tilde{\mathbf{u}}^{\epsilon, 2} \cdot \widetilde{\boldsymbol{\nabla}} q d x-\int_{\Gamma} \mathbf{u}^{\epsilon, 1} \cdot \mathbf{n} q^{2} d S \\
\\
=\int_{\Omega_{1}} F^{\epsilon} q d x+\int_{\Omega_{2}} \epsilon F^{\epsilon} q d x+\int_{\Gamma} f_{\Gamma}^{\epsilon} q^{2} d S .
\end{array}
\end{aligned}
$$

Letting $\epsilon \downarrow 0$ we find that the limits $\left[\mathbf{u}^{\epsilon, 1}, \epsilon \mathbf{u}^{\epsilon, 2}\right] \rightarrow \mathbf{u}$ and $p^{\epsilon} \rightarrow p$ of the indicated subsequences are a solution of the limit problem

$$
\begin{align*}
\mathbf{u} \in & \mathbf{V}_{0}, p \in Q_{0}: \quad \int_{\Omega_{1}} a_{1} \mathbf{u} \cdot \mathbf{v} d x-\int_{\Omega_{1}} p \boldsymbol{\nabla} \cdot \mathbf{v} d x  \tag{3.2a}\\
& +\int_{\Omega_{2}} a_{2} \tilde{\mathbf{u}}^{2} \cdot \tilde{\mathbf{v}} d x+\int_{\Omega_{2}} \widetilde{\boldsymbol{\nabla}} p \cdot \tilde{\mathbf{v}} d x+\int_{\Gamma} p^{2} \mathbf{v}^{1} \cdot \mathbf{n} d S \\
& +\int_{\Gamma} \alpha\left(\mathbf{u}^{1} \cdot \mathbf{n}\right)\left(\mathbf{v}^{1} \cdot \mathbf{n}\right) d S=-\int_{\Omega_{1}} \mathbf{g} \cdot \mathbf{v} d x-\int_{\Omega_{2}} \tilde{\mathbf{g}} \cdot \tilde{\mathbf{v}} d x,
\end{align*}
$$

$$
\begin{align*}
& \int_{\Omega_{1}} \lambda c_{1} p q d x+\int_{\Omega_{1}} \boldsymbol{\nabla} \cdot \mathbf{u} q d x+\int_{\Gamma} \lambda \beta p^{2} q^{2} d S  \tag{3.2b}\\
&- \int_{\Omega_{2}} \tilde{\mathbf{u}} \cdot \widetilde{\boldsymbol{\nabla}} q d x-\int_{\Gamma} \mathbf{u}^{1} \cdot \mathbf{n} q^{2} d S=\int_{\Omega_{1}} F q d x+\int_{\Gamma} f_{\Gamma} q^{2} d S \\
& \quad \text { for all } \mathbf{v} \in \mathbf{V}_{0}, q \in Q_{0} .
\end{align*}
$$

This problem is a mixed formulation (1.8) with the operators (3.3a)
$\mathcal{A}^{0} \mathbf{u}(\mathbf{v})=\int_{\Omega_{1}} a_{1}(x) \mathbf{u} \cdot \mathbf{v} d x+\int_{\Omega_{2}} a_{2}(\tilde{\mathbf{x}}) \tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}} d x+\int_{\Gamma} \alpha\left(\mathbf{u}^{1} \cdot \mathbf{n}\right)\left(\mathbf{v}^{1} \cdot \mathbf{n}\right) d S$,

$$
\begin{gather*}
\mathcal{B}^{0} \mathbf{u}(q)=-\int_{\Omega_{1}} \boldsymbol{\nabla} \cdot \mathbf{u} q d x+\int_{\Omega_{2}} \tilde{\mathbf{u}} \cdot \widetilde{\boldsymbol{\nabla}} q d x+\int_{\Gamma} \mathbf{u}^{1} \cdot \mathbf{n} q^{2} d S  \tag{3.3b}\\
\mathcal{C}^{0} p(q)=\int_{\Omega_{1}} c_{1} p q d x+\int_{\Gamma} \beta p^{2} q^{2} d S \tag{3.3c}
\end{gather*}
$$

Surface area on $\Gamma$ is $d S=d \tilde{\mathbf{x}}$, and functions of $\tilde{\mathbf{x}}$ can be regarded as functions on $\Omega_{2}$ or $\Gamma$. Thus the second terms in $\mathcal{A}^{0}$ and $\mathcal{B}^{0}$ can be written as integrals over $\Gamma$, namely, $\int_{\Gamma} a_{2} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}} d S$ and $\int_{\Gamma} \tilde{\mathbf{u}} \cdot \widetilde{\boldsymbol{\nabla}} q d S$. Note the degeneracy in $\mathcal{C}^{0}$ : the $c_{2}$-terms on $\Omega_{2}$ have vanished in the limit. Theorem 1.1 applies to these operators on $\mathbf{V}_{0}$ and $Q_{0}$. The infsup condition follows from the proof of Lemma 1.3. As a consequence of the uniqueness of the solution of the limit problem (3.2), not only a subsequence but the original sequences $\left[\mathbf{u}^{\epsilon, 1}, \epsilon \mathbf{u}^{\epsilon, 2}\right], p^{\epsilon}$ converge as indicated to $\left[\mathbf{u}^{1}, \mathbf{u}^{2}\right], p$.

We summarize the above as follows.
Theorem 3.1. Assume the conditions of Theorem 1.4 and (2.4) and (2.11). Then the sequence $\left[\mathbf{u}^{\epsilon, 1}, \epsilon \mathbf{u}^{\epsilon, 2}\right]$, $p^{\epsilon}$ of solutions of the corresponding scaled problems (2.2) converges weakly in $\mathbf{V} \times Q$ to the solution $\left[\mathbf{u}^{1}, \mathbf{u}^{2}\right] \in \mathbf{V}_{0}, p \in Q_{0}$ of the limit problem (3.2), and

$$
p^{\epsilon} \stackrel{w}{\rightharpoonup} p \text { weakly in } H^{1}\left(\Omega_{1}\right) \times H^{1}\left(\Omega_{2}\right), \text { strongly in } L^{2}(\Omega) .
$$

The Strong Form. From (3.2a) we obtain (1.4a), (1.4e), and

$$
-\left\langle p^{1}, \mathbf{v}^{1} \cdot \mathbf{n}\right\rangle_{\partial \Omega_{1}}+\int_{\Gamma}\left\{p^{2} \mathbf{v}^{1} \cdot \mathbf{n}+\alpha\left(\mathbf{u}^{1} \cdot \mathbf{n}\right)\left(\mathbf{v}^{1} \cdot \mathbf{n}\right)\right\} d S=0
$$

for all $\mathbf{v} \in \mathbf{V}$. Note that (1.4a) shows $p^{1} \in H^{1}\left(\Omega_{1}\right)$, so its trace is in $H^{1 / 2}\left(\partial \Omega_{1}\right)$, and so we have (1.4d) and (1.4c). Choosing $q \in$ $C_{0}^{\infty}\left(\Omega_{1}\right) \times C_{0}^{\infty}(G)$ in (3.2b), we first obtain (1.4b) and (1.4f). Since $\widetilde{\boldsymbol{\nabla}} \cdot \tilde{\mathbf{u}}^{2} \in L^{2}(G)$, the third term in (3.2b) can be rewritten
$-\int_{\Omega_{2}} \tilde{\mathbf{u}} \cdot \tilde{\nabla} q d x=-\int_{\Gamma} \tilde{\mathbf{u}} \cdot \widetilde{\nabla} q d \tilde{\mathbf{x}}=\int_{\Gamma} \widetilde{\boldsymbol{\nabla}} \cdot \tilde{\mathbf{u}} q d \tilde{\mathbf{x}}-<\tilde{\mathbf{u}} \cdot \tilde{\mathbf{n}}, q>_{\partial \Gamma}$ for $q \in Q_{0}$,
so we obtain also (1.4g). Thus, the system (1.4) is the strong form of the limit problem (3.2).

Strong Convergence. Assume additionally the strong convergence

$$
\begin{equation*}
\mathbf{g}^{\epsilon} \rightarrow \mathbf{g} \text { in } \mathbf{L}^{2}(\Omega) \text { and } f_{\Gamma}^{\epsilon} \rightarrow f_{\Gamma} \text { in } L^{2}(\Gamma) \tag{3.4}
\end{equation*}
$$

Set $\mathbf{v}=\mathbf{u}^{\epsilon}, q=p^{\epsilon}$ in (2.2) and add to obtain the identity

$$
\begin{aligned}
& \left\|a_{1}^{1 / 2} \mathbf{u}^{\epsilon}\right\|_{0, \Omega_{1}}^{2}+\epsilon^{2}\left\|a_{2}^{1 / 2} \mathbf{u}^{\epsilon}\right\|_{0, \Omega_{2}}^{2}+\alpha\left\|\mathbf{u}^{\epsilon, 1} \cdot \mathbf{n}\right\|_{L^{2}(\Gamma)}^{2}+\lambda \beta\left\|p^{\epsilon, 2}\right\|_{L^{2}(\Gamma)}^{2} \\
& \quad+\lambda\left\|c_{1}^{1 / 2} p^{\epsilon}\right\|_{0, \Omega_{1}}^{2}+\epsilon \lambda\left\|c_{2}^{1 / 2} p^{\epsilon}\right\|_{0, \Omega_{2}}^{2}=\int_{\Omega_{1}} F^{\epsilon} p^{\epsilon} d x \\
& \quad+\epsilon \int_{\Omega_{2}} F^{\epsilon} p^{\epsilon} d x+\int_{\Gamma} f_{\Gamma}^{\epsilon} p^{\epsilon, 2} d S-\int_{\Omega_{1}} \mathbf{g}^{\epsilon} \cdot \mathbf{u}^{\epsilon} d x-\epsilon \int_{\Omega_{2}} \mathbf{g}^{\epsilon} \cdot \mathbf{u}^{\epsilon} d x
\end{aligned}
$$

From the strong convergence of the source terms (3.4) and the strong convergence of the sequence $\left\{p^{\epsilon}: \epsilon>0\right\}$ in $L^{2}(\Omega)$, we can estimate

$$
\begin{align*}
& \limsup _{\epsilon \rightarrow 0}\left\{\left\|a_{1}^{1 / 2} \mathbf{u}^{\epsilon}\right\|_{0, \Omega_{1}}^{2}+\left\|a_{2}^{1 / 2} \epsilon \mathbf{u}^{\epsilon}\right\|_{0, \Omega_{2}}^{2}+\alpha\left\|\mathbf{u}^{\epsilon, 1} \cdot \mathbf{n}\right\|_{L^{2}(\Gamma)}^{2}+\lambda \beta\left\|p^{\epsilon, 2}\right\|_{L^{2}(\Gamma)}^{2}\right\}  \tag{3.5}\\
\leq & -\lambda\left\|c_{1}^{1 / 2} p\right\|_{0, \Omega_{1}}^{2}+\int_{\Omega_{1}} F^{1} p d x+\int_{\Gamma} f_{\Gamma} p^{2} d S-\int_{\Omega_{1}} \mathbf{g} \cdot \mathbf{u} d x-\int_{\Omega_{2}} \tilde{\mathbf{g}} \cdot \tilde{\mathbf{u}}^{2} d x
\end{align*}
$$

Set $\mathbf{v}=\mathbf{u}, q=p$ in the limit problem (3.2) and add. Using the resulting identity to evaluate the right side of (3.5), and then using the weak lower semicontinuity of the norms, we obtain

$$
\begin{gathered}
\limsup _{\epsilon \rightarrow 0}\left\{\left\|a_{1}^{1 / 2} \mathbf{u}^{\epsilon}\right\|_{0, \Omega_{1}}^{2}+\left\|a_{2}^{1 / 2} \epsilon \mathbf{u}^{\epsilon}\right\|_{0, \Omega_{2}}^{2}+\alpha\left\|\mathbf{u}^{\epsilon, 1} \cdot \mathbf{n}\right\|_{L^{2}(\Gamma)}^{2}+\lambda \beta\left\|p^{\epsilon, 2}\right\|_{L^{2}(\Gamma)}^{2}\right\} \\
\leq\left\|a_{1}^{1 / 2} \mathbf{u}^{1}\right\|_{0, \Omega_{1}}^{2}+\left\|a_{2}^{1 / 2} \tilde{\mathbf{u}}^{2}\right\|_{0, \Omega_{2}}^{2}+\alpha\left\|\mathbf{u}^{1} \cdot \mathbf{n}\right\|_{L^{2}(\Gamma)}^{2}+\lambda \beta p^{2} \|_{L^{2}(\Gamma)}^{2} \\
\leq \liminf _{\epsilon \rightarrow 0}\left\{\left\|a_{1}^{1 / 2} \mathbf{u}^{\epsilon}\right\|_{0, \Omega_{1}}^{2}+\left\|a_{2}^{1 / 2} \epsilon \mathbf{u}^{\epsilon}\right\|_{0, \Omega_{2}}^{2}+\alpha\left\|\mathbf{u}^{\epsilon, 1} \cdot \mathbf{n}\right\|_{L^{2}(\Gamma)}^{2}+\lambda \beta\left\|p^{,, 2}\right\|_{L^{2}(\Gamma)}^{2}\right\} .
\end{gathered}
$$

But since these norms converge to their value at the weak limit, it follows that the convergence is strong in the indicated norm.
Theorem 3.2. Under the assumptions of Theorem 3.1 and (3.4), we have strong convergence

$$
\begin{align*}
\mathbf{u}^{\epsilon, 1} & \rightarrow \mathbf{u}^{1} \text { in } \mathbf{L}^{2}\left(\Omega_{1}\right), \epsilon \mathbf{u}^{\epsilon, 2} \rightarrow \mathbf{u}^{2} \text { in } \mathbf{L}^{2}\left(\Omega_{2}\right),  \tag{3.6a}\\
p^{\epsilon, 1} & \rightarrow p^{1} \text { in } H^{1}\left(\Omega_{1}\right), \text { and } p^{\epsilon, 2} \rightarrow p^{2} \text { in } H^{1}\left(\Omega_{2}\right) . \tag{3.6b}
\end{align*}
$$

An important remark is that $\left\|\epsilon u_{N}^{\epsilon, 2}-u_{N}^{2}\right\|_{0, \Omega_{2}}=\left\|\epsilon u_{N}^{\epsilon, 2}\right\|_{0, \Omega_{2}} \rightarrow 0$ implies that we could have the normal component blowing up with the rate $\left\|u_{N}^{\epsilon, 2}\right\|_{0, \Omega_{2}} \sim \epsilon^{-p}, 0<p<1$ without any contradiction. Still we can conclude some information about the order of magnitude of the normal component. Suppose first that $\mathbf{u}_{T}^{2} \neq 0$ and consider the quotients

$$
\frac{\left\|\tilde{\mathbf{u}}^{\epsilon, 2}\right\|_{0, \Omega_{2}}}{\left\|u_{N}^{\epsilon, 2}\right\|_{0, \Omega_{2}}}=\frac{\left\|\epsilon \tilde{\mathbf{u}}^{\epsilon, 2}\right\|_{0, \Omega_{2}}}{\left\|\epsilon u_{N}^{\epsilon, 2}\right\|_{0, \Omega_{2}}}>\frac{\left\|\mathbf{u}_{T}^{2}\right\|_{0, \Omega_{2}}-\delta}{\left\|\epsilon u_{N}^{\epsilon, 2}\right\|_{0, \Omega_{2}}}>0
$$

The lower bound holds true for $\epsilon>0$ small enough and adequate $\delta>0$ then we conclude the quotient of tangent component over normal component $\mathbf{L}^{2}$-norms blows-up, i.e. the tangential velocity is much faster than the normal one in the thin channel.

If $\mathbf{u}_{T}^{2}=0$ we can not conclude the same reasoning, so a further analysis has to be made. Suppose then that the solution $\mathbf{u}, p$ of (3.2) is such that $\mathbf{u}^{2}=0$; then (3.2b) takes the form

$$
\int_{\Omega_{1}} \lambda c p q d x-\int_{\Omega_{1}} \mathbf{u} \cdot \nabla q d x=\int_{\Omega_{1}} F q d x+\int_{\Omega_{2}} F q d x+\int_{\Gamma} f_{\Gamma} \gamma(q) d S
$$

$$
\text { for all } q \in M
$$

and we conclude

$$
\begin{align*}
& -\langle\mathbf{u} \cdot \mathbf{n}, \gamma(q)\rangle_{H^{-1 / 2}(\Gamma), H^{1 / 2}(\Gamma)}  \tag{3.7}\\
& \quad=\int_{\Gamma}\left(\int_{0}^{1} F(\widetilde{x}, z) d z\right) \gamma(q) d \widetilde{x}+\int_{\Gamma} f_{\Gamma} \gamma(q) d \widetilde{x}
\end{align*}
$$

i.e. $\mathbf{u} \cdot \mathbf{n}=\left(\int_{0}^{1} F(\widetilde{x}, z) d z\right)+f_{\Gamma}$. But on the other hand, if $\mathbf{u}^{2}=0$ then it is clearly in $\mathbf{L}_{d i v}^{2}\left(\Omega_{2}\right)$ and so $\mathbf{u} \in \mathbf{L}_{d i v}^{2}(\Omega)$. Then, in particular it must hold $\mathbf{u} \cdot \mathbf{n}=\mathbf{u}^{2} \cdot \mathbf{n}=0$ on $\Gamma$, then, if we impose the condition $\left(\int_{0}^{1} F(\widetilde{x}, z) d z\right)+f_{\Gamma} \neq 0(3.7)$ is impossible and $\left\|\tilde{\mathbf{u}}^{\epsilon, 2}\right\|_{0, \Omega_{2}} \gg$ $\left\|u_{N}^{\epsilon, 2}\right\|_{0, \Omega_{2}}$ for $\epsilon>0$ small enough as discussed above.

## 4. The evolution problems

We shall resolve the initial-boundary-value problem for the equations (1.1) with the coefficients, interface and boundary conditions as given above, that is, for the singular evolution system

$$
\begin{array}{r}
a_{1}(x) \mathbf{u}^{\epsilon, 1}+\boldsymbol{\nabla} p^{\epsilon, 1}+\mathbf{g}^{\epsilon, 1}(x)=\mathbf{0} \text { and } \\
c_{1}(x) \frac{\partial p^{\epsilon, 1}}{\partial t}+\boldsymbol{\nabla} \cdot \mathbf{u}^{\epsilon, 1}=F^{\epsilon} \text { in } \Omega_{1}, \\
p^{\epsilon, 1}=0 \quad \text { on } \partial \Omega_{1}-\Gamma, \\
p^{\epsilon, 1}-p^{\epsilon, 2}=\alpha \mathbf{u}^{\epsilon, 1} \cdot \mathbf{n} \text { and } \\
\beta \frac{\partial p^{\epsilon, 2}}{\partial t}-\mathbf{u}^{\epsilon, 1} \cdot \mathbf{n}+\mathbf{u}^{\epsilon, 2} \cdot \mathbf{n}=f_{\Gamma}^{\epsilon} \quad \text { on } \Gamma, \\
\epsilon a_{2}(x) \mathbf{u}^{\epsilon, 2}+\boldsymbol{\nabla} p^{\epsilon, 2}+\mathbf{g}^{\epsilon, 2}(x)=\mathbf{0} \text { and } \\
c_{2}(x) \frac{\partial p^{\epsilon, 2}}{\partial t}+\boldsymbol{\nabla} \cdot \mathbf{u}^{\epsilon, 2}=F^{\epsilon} \text { in } \Omega_{2}^{\epsilon}, \\
\mathbf{u}^{\epsilon, 2} \cdot \mathbf{n}=0 \text { on } \partial \Omega_{2}^{\epsilon}-\Gamma, \tag{4.1g}
\end{array}
$$

with initial conditions

$$
\begin{align*}
c_{1}(x) p^{\epsilon, 1}(\cdot, 0) & =c_{1}(x) p_{0}^{1} \text { in } \Omega_{1}  \tag{4.1h}\\
c_{2}(x) p^{\epsilon, 2}(\cdot, 0) & =c_{2}(x) p_{0}^{2} \text { in } \Omega_{2}^{\epsilon}  \tag{4.1i}\\
\beta p^{\epsilon, 2}(\cdot, 0) & =\beta p_{0}^{3} \text { on } \Gamma . \tag{4.1j}
\end{align*}
$$

Since we shall describe the limit as $\epsilon \rightarrow 0$, we first rescale as in Section 2 to get an evolution system in mixed form

$$
\begin{align*}
& \mathbf{u}^{\epsilon}(t) \in \mathbf{V}, p^{\epsilon}(t) \in Q: \\
& \mathcal{A}^{\epsilon} \mathbf{u}^{\epsilon}(t)+\mathcal{B}^{\epsilon^{\prime}} p^{\epsilon}(t)=-\mathbf{g}^{\epsilon} \text { in } \mathbf{V}^{\prime}  \tag{4.2a}\\
&-\mathcal{B}^{\epsilon} \mathbf{u}^{\epsilon}(t)+\frac{d}{d t} \mathcal{C}^{\epsilon} p^{\epsilon}(t)=f^{\epsilon}(t) \text { in } Q^{\prime}  \tag{4.2b}\\
& \mathcal{C}^{\epsilon} p^{\epsilon}(0)=\mathcal{C}^{\epsilon} p_{0}, \tag{4.2c}
\end{align*}
$$

in which the spaces (2.1) are independent of $\epsilon$ and the operators are chosen as for the scaled singular problem (2.2), namely,

$$
\begin{align*}
\mathcal{A}^{\epsilon} \mathbf{u}(\mathbf{v})= & \int_{\Omega_{1}} a_{1}(x) \mathbf{u} \cdot \mathbf{v} d x  \tag{4.3a}\\
& +\epsilon^{2} \int_{\Omega_{2}} a_{2}(x) \mathbf{u} \cdot \mathbf{v} d x+\int_{\Gamma} \alpha\left(\mathbf{u}^{1} \cdot \mathbf{n}\right)\left(\mathbf{v}^{1} \cdot \mathbf{n}\right) d S
\end{align*}
$$

$$
\begin{equation*}
\mathcal{C}^{\epsilon} p(q)=\int_{\Omega_{1}} c_{1}(x) p q d x+\epsilon \int_{\Omega_{2}} c_{2}(x) p q d x .+\int_{\Gamma} \beta p^{2} q^{2} d S \tag{4.3c}
\end{equation*}
$$

The functionals are given by

$$
\begin{aligned}
& \mathbf{g}^{\epsilon}(\mathbf{v})=\int_{\Omega_{1}} \mathbf{g}^{\epsilon, 1}(x) \cdot \mathbf{v}^{1}(x) d x+\epsilon \int_{\Omega_{2}} \mathbf{g}^{\epsilon, 2}(x) \cdot \mathbf{v}^{2}(x) d x, \quad \mathbf{v} \in \mathbf{V} \\
& f^{\epsilon}(t)(q)=\int_{\Omega_{1}} F^{\epsilon, 1}(x, t) q^{1}(x) d x+\epsilon \int_{\Omega_{2}} F^{\epsilon, 2}(x, t) q^{2}(x) d x \\
&+\int_{\Gamma} f_{\Gamma}^{\epsilon}(s) q^{2}(s) d S, \quad q \in Q
\end{aligned}
$$

where each $\mathbf{g}^{\epsilon, j} \in \mathbf{L}^{2}\left(\Omega_{j}\right)$, and $f_{\Gamma}^{\epsilon} \in L^{2}(\Gamma) ; F^{\epsilon}$ will be determined below.
We shall show that the elliptic-parabolic system (4.2) is governed by an analytic semigroup on a Hilbert space determined by $\mathcal{C}^{\epsilon}$ and that the limiting form corresponds similarly to an analytic semigroup which realizes an elliptic-parabolic equation in $\Omega_{1}$ constrained by an elliptic equation on the part $\Gamma$ of its boundary. Then we establish the convergence as $\epsilon \rightarrow 0$ of these solutions of the evolution problems.
4.1. The Scaled Problem. Assume the conditions of Theorem 1.4 on all data. Additional assumptions on $F^{\epsilon}(t)$ will be prescribed below.

First we simplify the problem (4.2) by a translation. Let $\mathbf{u}_{*}^{\epsilon}, p_{*}^{\epsilon}$ be the solution of the stationary problem (2.2) with $\mathbf{g}^{\epsilon}$ and $f_{\Gamma}^{\epsilon}$ as given (independent of $t$ ), $F^{\epsilon, 1}=F^{\epsilon, 2}=0$, and $\lambda=0$. Subtract $\mathbf{u}_{*}^{\epsilon}$ from $\mathbf{u}^{\epsilon}(t)$ and $p_{*}^{\epsilon}$ from $p^{\epsilon}(t)$ to get the problem (4.2) for the differences, but with $\mathbf{g}^{\epsilon}=\mathbf{0}, f_{\Gamma}^{\epsilon}=0$, and initial value $p_{0}-p_{*}^{\epsilon}$.

Consider the case of $\beta=0$. The parabolic region for the system (4.1) is $\Omega_{0}=\operatorname{int}\{x \in \Omega: c(x)>0\}$. Then $\int_{\Omega_{0}} c(x) p(x) q(x) d x$ is a continuous scalar-product on the restrictions $\left\{\left.q\right|_{\Omega_{0}}: q \in Q\right\}$, and we denote by $Q_{c}$ the completion of these restrictions in that scalar-product. This is the state space for (4.1), the weighted space $L^{2}\left(\Omega_{0}, c d x\right)$ with the measure $d y=c(x) d x$. We let $Q_{c}^{\epsilon}$ be the space $Q_{c}$ with the (equivalent) scalar product $\mathcal{C}^{\epsilon} p(q)$ for $0<\epsilon \leq 1$. Then we have the uniformly bounded and dense restrictions $Q \rightarrow Q_{c}^{\epsilon}$ and uniformly bounded inclusions $Q_{c}^{\epsilon \prime} \hookrightarrow Q^{\prime}$.

Define the (unbounded) operator $L^{\epsilon}$ on $Q_{c}^{\epsilon}$ by $L^{\epsilon} p=f \in Q_{c}^{\epsilon}$ if $p \in Q$ and there exists a $\mathbf{u} \in \mathbf{V}$ such that $\mathcal{A}^{\epsilon} \mathbf{u}+\mathcal{B}^{\epsilon} p=\mathbf{0},-\mathcal{B}^{\epsilon} \mathbf{u}=\mathcal{C}^{\epsilon} f$. Then $\mathbf{u}$ and $p$ are unique, and we have

$$
\left(L^{\epsilon} p, p\right)_{Q_{\epsilon}^{\epsilon}}=\mathcal{C}^{\epsilon} f(p)=-\mathcal{B}^{\epsilon} \mathbf{u}(p)=-\mathcal{B}^{\epsilon^{\prime}} p(\mathbf{u})=\mathcal{A}^{\epsilon} \mathbf{u}(\mathbf{u}) \geq 0
$$

so $L^{\epsilon}$ is accretive on $Q_{c}^{\epsilon}$. A similar calculation shows that $L^{\epsilon}$ is symmetric, because $\mathcal{A}^{\epsilon}$ is symmetric. Theorem 1.4 shows that $\left(\begin{array}{cc}\mathcal{A}^{\epsilon} & \mathcal{B}^{\epsilon^{\prime}} \\ -\mathcal{B}^{\epsilon} & \mathcal{C}^{\epsilon}\end{array}\right)$ is onto $\{0\} \times Q_{c}^{\epsilon \prime}$, so $I+L^{\epsilon}$ is onto $Q_{c}^{\epsilon}$ and $L^{\epsilon}$ is m-accretive on $Q_{c}^{\epsilon}$. The evolution system (4.2) is equivalent to the initial-value problem

$$
\frac{d p^{\epsilon}(t)}{d t}+L^{\epsilon} p^{\epsilon}(t)=\left(\mathcal{C}^{\epsilon}\right)^{-1} f^{\epsilon}(t), 0<t \leq T, p^{\epsilon}(0)=p_{0}
$$

in the Hilbert space $Q_{c}^{\epsilon}$, so we get existence and uniqueness of the solution from the Hille-Yosida-Phillips theorem. (See Theorem IX-1.19 and Theorem IX-1.27 of [11] or Theorem I.5.2 and Theorem IV.4.1 of [15].) The case of $\beta>0$ is similar; it follows as above but with $Q_{c}$ replaced by $Q_{c, \beta}=Q_{c} \bigoplus L^{2}(\Gamma)$.

Theorem 4.1. Assume the hypotheses of Theorem 1.4. Then for every $p_{0} \in Q_{c}, T>0$, and Hölder continuous $F^{\epsilon} \in C^{r}\left([0, T], Q_{c}^{\prime}\right)$ for some $0<r<1$, there is a unique solution $\mathbf{u}^{\epsilon}:(0, T] \rightarrow \mathbf{V}$ and $p^{\epsilon}:[0, T] \rightarrow Q$ with $p^{\epsilon} \in C\left([0, T], Q_{c}\right) \cap C^{1}\left((0, T], Q_{c}\right)$ of the scaled evolution problem (4.2). This solution satisfies

$$
\begin{array}{r}
a_{1} \mathbf{u}^{\epsilon, 1}+\boldsymbol{\nabla} p^{\epsilon, 1}+\mathbf{g}^{\epsilon, 1}(x)=\mathbf{0} \text { and } \\
c_{1} \frac{\partial p^{\epsilon, 1}}{\partial t}+\boldsymbol{\nabla} \cdot \mathbf{u}^{\epsilon, 1}=F^{\epsilon, 1}(x, t) \text { in } \Omega_{1}, \\
p^{\epsilon, 1}=0 \text { on } \partial \Omega_{1}-\Gamma, \tag{4.4c}
\end{array}
$$

$$
\begin{equation*}
p^{\epsilon, 1}-p^{\epsilon, 2}=\alpha \mathbf{u}^{\epsilon, 1} \cdot \mathbf{n} \text { and } \tag{4.4~d}
\end{equation*}
$$

$$
\begin{gather*}
\beta \frac{\partial p^{\epsilon, 2}}{\partial t}-\mathbf{u}^{\epsilon, 1} \cdot \mathbf{n}+\left(\epsilon \tilde{\mathbf{u}}^{\epsilon, 2}, u_{N}^{\epsilon, 2}\right) \cdot \mathbf{n}=f_{\Gamma}^{\epsilon}(s) \text { on } \Gamma,  \tag{4.4e}\\
\epsilon a_{2} \tilde{\mathbf{u}}^{\epsilon, 2}+\widetilde{\nabla} p^{\epsilon, 2}+\tilde{\mathbf{g}}^{\epsilon, 2}(x)=\tilde{\mathbf{0}}, \\
\epsilon^{2} a_{2} u_{N}^{\epsilon, 2}+\partial_{z} p^{\epsilon, 2}+\epsilon g_{N}^{\epsilon, 2}(x)=\mathbf{0},  \tag{4.4f}\\
\epsilon c_{2} \frac{\partial p^{\epsilon, 2}}{\partial t}+\epsilon \widetilde{\boldsymbol{\nabla}} \cdot \tilde{\mathbf{u}}^{\epsilon, 2}+\partial_{z} u_{N}^{\epsilon, 2}=\epsilon F^{\epsilon, 2}(x, t) \text { in } \Omega_{2}  \tag{4.4~g}\\
\text { and } \quad\left(\epsilon \tilde{\mathbf{u}}^{\epsilon, 2}, u_{N}^{\epsilon, 2}\right) \cdot \mathbf{n}=0 \text { on } \partial \Omega_{2}-\Gamma \tag{4.4h}
\end{gather*}
$$

at each time $t>0$ and the initial conditions

$$
\begin{align*}
c_{1}(\cdot) p^{\epsilon, 1}(\cdot, 0) & =c_{1}(\cdot) p_{0}^{1}(\cdot) \text { in } \Omega_{1},  \tag{4.4i}\\
c_{2}(\cdot) p^{\epsilon, 2}(\cdot, 0) & =c_{2}(\cdot) p_{0}^{2}(\cdot) \text { in } \Omega_{2},  \tag{4.4j}\\
\beta p^{\epsilon, 2}(\cdot, 0) & =\beta p_{0}^{3} \text { on } \Gamma . \tag{4.4k}
\end{align*}
$$

If $p_{0} \in Q$ and $F^{\epsilon}:[0, T] \rightarrow Q_{c}^{\prime}$ is absolutely continuous, then $p^{\epsilon}:$ $[0, T] \rightarrow Q_{c}$ is Lipschitz continuous and $\boldsymbol{\nabla} \cdot \mathbf{u}^{\epsilon, 1} \in L^{\infty}\left(0, T ; L^{2}\left(\Omega_{1}\right)\right)$.

The system (4.4) is the strong form of (4.2). Note that since $F^{\epsilon}(\cdot, t) \in$ $Q_{c}^{\prime}$, necessarily $F^{\epsilon}(x, t)=0$ wherever $c(x)=0$. The last statement of the theorem follows from the inclusion $Q_{c}^{\prime} \hookrightarrow Q^{\prime}$ with unit norm and (4.4b).

A similar construction applies to the spaces (3.1) and operators (3.3), so we obtain the corresponding result for the limit problem

$$
\begin{align*}
\mathbf{u}(t) \in \mathbf{V}_{0}, p(t) & \in Q_{0}: \\
& \mathcal{A}^{0} \mathbf{u}(t)+\mathcal{B}^{0^{\prime}} p(t)=-\mathbf{g} \text { in } \mathbf{V}_{0}^{\prime}, \\
- & \mathcal{B}^{0} \mathbf{u}(t)+\frac{d}{d t} \mathcal{C}^{0} p(t)=f(t) \text { in } Q_{0}^{\prime},  \tag{4.5}\\
\mathcal{C}^{0} p(0)= & \mathcal{C}^{0} p_{0} .
\end{align*}
$$

Theorem 4.2. Assume the hypotheses of Theorem 1.4 and (2.11). Set $\Omega_{0}^{1}=\Omega_{0} \cap \Omega_{1}$ and define $Q_{c}^{1}$ to be the completion of the restrictions $\left\{\left.q\right|_{\Omega_{0}^{1}}: q \in Q_{0}\right\}$ in the scalar product $\int_{\Omega_{0}^{1}} c_{1}(x) p(x) q(x) d x$. Then for every $p_{0}^{1} \in Q_{c}^{1}, T>0$, and $F^{1} \in C^{r}\left([0, T], Q_{c}^{1^{\prime}}\right)$ with $0<r<1$, there is a unique solution $\mathbf{u}:(0, T] \rightarrow \mathbf{V}_{0}$ and $p:[0, T] \rightarrow Q_{0}$ with $p \in C\left([0, T], Q_{c}^{1}\right) \cap C^{1}\left((0, T], Q_{c}^{1}\right)$ of the limit evolution problem (4.5). This solution satisfies

$$
\begin{array}{r}
a_{1}(x) \mathbf{u}^{1}+\boldsymbol{\nabla} p^{1}+\mathbf{g}^{1}(x)=\mathbf{0} \text { and } \\
c_{1}(x) \frac{\partial p^{1}}{\partial t}+\boldsymbol{\nabla} \cdot \mathbf{u}^{1}=F^{1} \quad \text { in } \Omega_{1}, \\
p^{1}=0 \text { on } \partial \Omega_{1}-\Gamma, \\
p^{1}-\alpha \mathbf{u}^{1} \cdot \mathbf{n}=p^{2} \text { on } \Gamma, \tag{4.6d}
\end{array}
$$

$$
\begin{array}{r}
a_{2}(\tilde{\mathbf{x}}) \tilde{\mathbf{u}}^{2}+\widetilde{\boldsymbol{\nabla}} p^{2}+\tilde{\mathbf{g}}^{2}(\tilde{\mathbf{x}})=\mathbf{0} \text { and } \\
\beta \frac{\partial p^{2}}{\partial t}+\widetilde{\boldsymbol{\nabla}} \cdot \tilde{\mathbf{u}}^{2}=f_{\Gamma}+\left.\mathbf{u}^{1} \cdot \mathbf{n}\right|_{\Gamma} \text { on } \Gamma, \\
\mathbf{u}^{2} \cdot \mathbf{n}=0 \text { on } \partial \Omega_{2}-\Gamma \tag{4.6~g}
\end{array}
$$

at each time $t>0$ and the initial condition

$$
\begin{align*}
c^{1}(\cdot) p^{1}(\cdot, 0) & =c^{1}(\cdot) p_{1}^{0}(\cdot) \text { on } \Omega_{1},  \tag{4.6h}\\
\beta p^{2}(\cdot, 0) & =\beta p_{0}^{3} \text { on } \Gamma . \tag{4.6i}
\end{align*}
$$

4.2. Convergence. Assume the situation of Theorem 4.1. By the translation above, we reduce to the case of $\mathbf{g}^{\epsilon}=\mathbf{0}$ and $f_{\Gamma}^{\epsilon}=0$, since these functions are independent of time. Also we assume that $p_{0} \in Q$, each $F^{\epsilon}:[0, T] \rightarrow Q_{c}^{\prime}$ is absolutely continuous, and

$$
\begin{align*}
& \left\|F^{\epsilon}\right\|_{L^{2}\left(0, T ; Q_{c}^{\prime}\right.},\left\|\frac{d}{d t} F^{\epsilon}\right\|_{L^{1}\left(0, T ; Q_{c}^{\prime}\right)} \text { are bounded, }  \tag{4.7a}\\
& F^{1, \epsilon} \stackrel{w}{\rightharpoonup} F^{1} \text { in } L^{2}\left(0, T ; \Omega_{1}\right),  \tag{4.7b}\\
& \mathbf{g}^{\epsilon} \stackrel{w}{\rightharpoonup} \mathbf{g} \text { in } \mathbf{L}^{2}(\Omega), \text { and }  \tag{4.7c}\\
& f_{\Gamma}^{\epsilon} \stackrel{w}{\rightharpoonup} f_{\Gamma} \text { in } L^{2}(\Gamma) . \tag{4.7d}
\end{align*}
$$

Theorem 4.3. Assume the conditions of Theorem 1.4, (2.4), (2.11) and (4.7). Then the sequence $\mathbf{u}^{\epsilon}=\left[\mathbf{u}^{\epsilon, 1}, \epsilon \mathbf{u}^{\epsilon, 2}\right]$, $p^{\epsilon}$ of solutions of the corresponding scaled problems (4.4) converges weakly in $\mathbb{V} \times \mathbb{Q}$ to the solution $\left[\mathbf{u}^{1}, \mathbf{u}^{2}\right] \in \mathbb{V}_{0}, p \in \mathbb{Q}_{0}$ of the limit problem (4.5), and
(4.8a) $\quad p^{\epsilon} \rightarrow p$ strongly in $L^{2}\left(0, T ; H^{1}\left(\Omega_{1}\right)\right) \times L^{2}\left(0, T ; H^{1}\left(\Omega_{2}\right)\right)$,

$$
\begin{gather*}
\mathbf{u}^{\epsilon} \rightarrow \mathbf{u} \text { strongly in } L^{2}\left(0, T ; \mathbf{L}^{2}\left(\Omega_{1}\right)\right) \times L^{2}\left(0, T ; \mathbf{L}^{2}\left(\Omega_{2}\right)\right),  \tag{4.8b}\\
\alpha^{1 / 2} \mathbf{u}^{\epsilon, 1} \cdot \mathbf{n} \rightarrow \alpha^{1 / 2} \mathbf{u}^{1} \cdot \mathbf{n} \text { strongly in } L^{2}\left(0, T ; L^{2}(\Gamma)\right) . \tag{4.8c}
\end{gather*}
$$

Proof. Test (4.2a) on $\mathbf{u}^{\epsilon}(t)$ and (4.2b) on $p^{\epsilon}(t)$, add and integrate to obtain
$\int_{0}^{T} \mathcal{A}^{\epsilon} \mathbf{u}^{\epsilon}(t)\left(\mathbf{u}^{\epsilon}(t)\right) d t+\frac{1}{2} \mathcal{C}^{\epsilon} p^{\epsilon}(T)\left(p^{\epsilon}(T)\right)=\int_{0}^{T} F^{\epsilon}(t)\left(p^{\epsilon}(t)\right) d t+\frac{1}{2} \mathcal{C}^{\epsilon} p_{0}\left(p_{0}\right)$.
Since $F^{\epsilon} \in L^{\infty}\left(0, T ; Q_{c}^{\prime}\right)$, we find that both of $\sup _{0 \leq t \leq T} \mathcal{C}^{\epsilon} p^{\epsilon}(t)\left(p^{\epsilon}(t)\right)$ and $\int_{0}^{T} \mathcal{A}^{\epsilon} \mathbf{u}^{\epsilon}(t)\left(\mathbf{u}^{\epsilon}(t)\right) d t$ are bounded, and this gives bounds on each of $\left\|\mathbf{u}^{\epsilon, 1}\right\|_{L^{2}\left(0, T ; L^{2}\left(\Omega_{1}\right)\right)}$ and $\left\|\epsilon \mathbf{u}^{\epsilon, 2}\right\|_{L^{2}\left(0, T ; L^{2}\left(\Omega_{2}\right)\right)}$ independent of $\epsilon>0$. From (4.4b), the uniform bound on $\frac{d}{d t} C^{\epsilon} p^{\epsilon}(t)$ in $L^{2}\left(0, T ; L^{2}\left(\Omega_{1}\right)\right)$, and the infsup condition on $\mathcal{B}^{\epsilon}$, we get $\left\|\nabla \cdot \mathbf{u}^{\epsilon, 1}\right\|_{L^{2}\left(0, T ; L^{2}\left(\Omega_{1}\right)\right)}$ and $\left\|\mathbf{u}^{\epsilon, 1} \cdot \mathbf{n}\right\|_{L^{2}\left(0, T ; L^{2}(\Gamma)\right)}$ bounded. The Darcy laws (4.4a) and (4.4f) imply that each of

$$
\left\|\nabla p^{\epsilon, 1}\right\|_{L^{2}\left(0, T ; L^{2}\left(\Omega_{1}\right)\right)},\left\|\widetilde{\nabla} p^{\epsilon, 2}\right\|_{L^{2}\left(0, T ; L^{2}\left(\Omega_{2}\right)\right)},\left\|\frac{1}{\epsilon} \partial_{z} p^{\epsilon}\right\|_{L^{2}\left(0, T ; L^{2}\left(\Omega_{2}\right)\right)},
$$

is bounded. Finally, we use Poincaré inequality with (4.4c) and (4.4d) to bound

$$
\left\|p^{\epsilon, 1}\right\|_{L^{2}\left(0, T ; H^{1}\left(\Omega_{1}\right)\right)},\left\|p^{\epsilon, 2}\right\|_{L^{2}\left(0, T ; H^{1}\left(\Omega_{2}\right)\right)}
$$

From the preceding estimates, it follows there are weakly convergent subsequences $\left[\mathbf{u}^{\epsilon, 1}, \epsilon \mathbf{u}^{\epsilon, 2}\right] \stackrel{w}{\sim} \mathbf{u}$ and $p^{\epsilon} \stackrel{w}{\longrightarrow} p$ in the respective spaces

$$
\mathbb{V}=L^{2}(0, T ; \mathbf{V}), \quad \mathbb{Q}=L^{2}(0, T ; Q)
$$

and $\partial_{z} p^{\epsilon} \rightarrow 0$ strongly in $L^{2}\left(0, T ; L^{2}\left(\Omega_{2}\right)\right)$.
Choose $\varphi \in C_{0}^{\infty}\left(\Omega_{2}\right)$, apply (4.2b) to $q=\epsilon \varphi$ and take the limit as $\epsilon \rightarrow 0$ to get $\partial_{z} u_{N}^{2}=0$. Then apply it to $\epsilon q^{2}$ for a general $q \in Q$ to get $u_{N}^{2}=0$ as before. Next apply (4.2a) to $\mathbf{v}=\left(\frac{1}{\epsilon} \tilde{\mathbf{v}}, 0\right)$ where $\tilde{\mathbf{v}} \in$ $C_{0}^{\infty}\left(\Omega_{2}\right)$. Assume (2.11) and take the limit as $\epsilon \rightarrow 0$ to get $\tilde{\mathbf{u}}^{2}=\tilde{\mathbf{u}}^{2}(\tilde{\mathbf{x}}, t)$ independent of $z$.

The above shows and that the limits $\mathbf{u}, p$ belong to the corresponding subspaces

$$
\mathbb{V}_{0}=L^{2}\left(0, T ; \mathbf{V}_{0}\right), \quad \mathbb{Q}_{0}=L^{2}\left(0, T ; Q_{0}\right)
$$

Since $\mathbf{u}^{\epsilon}, p^{\epsilon}$ are solutions of (4.4), they satisfy

$$
\begin{gather*}
\int_{0}^{T} \mathcal{A}^{\epsilon} \mathbf{u}^{\epsilon}(t)(\mathbf{v}(t)) d t+\int_{0}^{T} \mathcal{B}^{\epsilon^{\prime}} p^{\epsilon}(t)(\mathbf{v}(t)) d t-\int_{0}^{T} \mathcal{B}^{\epsilon} \mathbf{u}^{\epsilon}(t)(q(t)) d t  \tag{4.9}\\
\quad-\int_{0}^{T} \mathcal{C}^{\epsilon} p^{\epsilon}(t)\left(\frac{d q(t)}{d t}\right) d t=\int_{0}^{T} f^{\epsilon}(t)(q(t)) d t+\mathcal{C}^{\epsilon} p_{0}(q(0))
\end{gather*}
$$

$$
\text { for all } \mathbf{v} \in \mathbb{V}, q \in \mathbb{Q} \text { with } q \in H^{1}\left(0, T ; Q_{c}\right) \text { and } q(T)=0
$$

Let $\mathbf{v} \in \mathbb{V}_{0}$ and $q \in \mathbb{Q}_{0}$ be given with $q \in H^{1}\left(0, T ; Q_{c}^{1}\right)$ and $q(T)=0$. Apply (4.9) to $\left[\mathbf{v}^{1}, \frac{1}{\epsilon} \mathbf{v}^{2}\right] \in \mathbb{V}$ and $q$, then pass to the limit as $\epsilon \rightarrow 0$ to obtain

$$
\begin{gather*}
\int_{0}^{T} \mathcal{A}^{0} \mathbf{u}(t)(\mathbf{v}(t)) d t+\int_{0}^{T} \mathcal{B}^{0^{\prime}} p(t)(\mathbf{v}(t)) d t-\int_{0}^{T} \mathcal{B}^{0} \mathbf{u}(t)(q(t)) d t  \tag{4.10}\\
\quad-\int_{0}^{T} \mathcal{C}^{0} p(t)\left(\frac{d q(t)}{d t}\right) d t=\int_{0}^{T} f^{0}(t)(q(t)) d t+\mathcal{C}^{0} p_{0}(q(0))
\end{gather*}
$$

$$
\text { for all } \mathbf{v} \in \mathbb{V}_{0}, q \in \mathbb{Q}_{0} \text { with } q \in H^{1}\left(0, T ; Q_{c}^{1}\right) \text { and } q(T)=0
$$

This is an equivalent weak formulation of the limit evolution system (4.5). (See Section III. 3 of [15].) We shall establish uniqueness for (4.10), and that implies the original sequences $\left[\mathbf{u}^{\epsilon, 1}, \epsilon \mathbf{u}^{\epsilon, 2}\right], p^{\epsilon}$ converge weakly to $\mathbf{u}, p$ as indicated above.

Define the continuous linear $\mathcal{L}: Q_{0} \rightarrow Q_{0}^{\prime}$ by $\mathcal{L}(p)=-\mathcal{B}^{0} \mathbf{u} \in Q_{0}^{\prime}$ where $\mathbf{u} \in \mathbf{V}_{0}, p \in Q_{0}, \mathcal{A}^{0} \mathbf{u}+\mathcal{B}^{0}{ }^{\prime} p=0$ in $\mathbf{V}_{0}^{\prime}$. The system (4.10) is equivalent to the linear degenerate Cauchy problem

$$
\begin{equation*}
\frac{d}{d t} \mathcal{C}^{0} p(t)+\mathcal{L} p(t)=f^{0}(t), 0<t<T, \mathcal{C}^{0} p(0)=\mathcal{C}^{0} p_{0} \tag{4.11}
\end{equation*}
$$

For uniqueness of the solution it suffices by Proposition III.3.3 of [15] to show that $\mathcal{L}$ is symmetric and $Q_{0}$-coercive. The symmetry
follows as before, so it remains to verify the coercivity. Denote by $\mathbf{W}_{0}$ the space $\mathbf{V}_{0}$ with the scalar product $\mathcal{A}^{0}(\cdot, \cdot)$ given by (3.3a). Then the imbeddings $\mathbf{V}_{0} \rightarrow \mathbf{W}_{0}$ and $\mathbf{W}_{0}^{\prime} \rightarrow \mathbf{V}_{0}^{\prime}$ are bounded and $\mathcal{A}^{0}: \mathbf{W}_{0} \rightarrow \mathbf{W}_{0}^{\prime}$ is the Riesz isomorphism. If $\mathcal{L}(p)=-\mathcal{B}^{0} \mathbf{u}$, then

$$
\begin{align*}
\mathcal{L} p(p)=\mathcal{A}^{0} \mathbf{u}(\mathbf{u})= & \|\mathbf{u}\|_{\mathbf{W}_{0}}^{2}=\left\|\mathcal{A}^{0} \mathbf{u}\right\|_{\mathbf{W}_{0}^{\prime}}^{2}  \tag{4.12}\\
& \geq c_{a}\left\|\mathcal{A}^{0} \mathbf{u}\right\|_{\mathbf{V}_{0}^{\prime}}^{2}=c_{a}\left\|\mathcal{B}^{0^{\prime}} p\right\|_{\mathbf{V}_{0}^{\prime}}^{2} \geq c_{a} c\|p\|_{\mathcal{Q}_{0}}^{2}
\end{align*}
$$

where the last follows since $\mathcal{B}^{0^{\prime}}$ is bounding. This shows that $\mathcal{L}$ is $Q_{0}$-coercive.

It remains to verify the strong convergence statements in (4.8). For $\mathbf{v} \in \mathbf{V}$ denote the (weaker) norm

$$
\begin{equation*}
\|\mathbf{v}\|^{2}=\int_{\Omega_{1}} a_{1}(x)\left|\mathbf{v}^{1}\right|^{2} d x+\int_{\Omega_{2}} a_{2}(x)\left|\mathbf{v}^{2}\right|^{2} d x+\int_{\Gamma} \alpha\left|\mathbf{v}^{1} \cdot \mathbf{n}\right|^{2} d S . \tag{4.13}
\end{equation*}
$$

Then we have

$$
\begin{aligned}
& \left\|\mathbf{u}^{\epsilon}\right\|^{2}=\int_{0}^{T} \mathcal{A}^{\epsilon} \mathbf{u}^{\epsilon}\left(\mathbf{u}^{\epsilon}\right) d t= \\
& \quad-\frac{1}{2} \mathcal{C}^{\epsilon} p^{\epsilon}(T)\left(p^{\epsilon}(T)\right)+\frac{1}{2} \mathcal{C}^{\epsilon} p_{0}\left(p_{0}\right)+\int_{0}^{T} f^{\epsilon}(t) p^{\epsilon}(t) d t
\end{aligned}
$$

and by weak lower-semicontinuity,

$$
\begin{aligned}
& \limsup _{\epsilon \rightarrow 0}\left\|\mathbf{u}^{\epsilon}\right\|^{2} \leq \\
& \qquad \begin{array}{l}
-\frac{1}{2} \mathcal{C}^{0} p(T)(p(T))+\frac{1}{2} \mathcal{C}^{0} p_{0}\left(p_{0}\right)+\int_{0}^{T} f^{0}(t) p(t) d t \\
\\
=\int_{0}^{T} \mathcal{A}^{0} \mathbf{u}(\mathbf{u}) d t \leq \liminf _{\epsilon \rightarrow 0}\left\|\mathbf{u}^{\epsilon}\right\|^{2} .
\end{array}
\end{aligned}
$$

Thus we have $\lim _{\epsilon \rightarrow 0}\left\|\mathbf{u}^{\epsilon}\right\|^{2}=\|\mathbf{u}\|^{2}$, and with weak convergence in the norm (4.13) (weaker than $\|\cdot\|_{\mathrm{V}}$ ), we obtain strong convergence in the norm (4.13).

Finally, from the flux convergence (4.8b) and the Darcy laws (4.4a) and (4.4f) we have $\left\{\boldsymbol{\nabla} p^{\epsilon}\right\}$ strongly convergent in $\mathbf{L}^{2}\left(0, T ; \mathbf{L}^{2}(\Omega)\right)$. From the boundary condition (4.4c) and the Poincaré inequality we get the sequence $\left\{p^{\epsilon, 1}\right\}$ strongly convergent in $L^{2}\left(0, T ; H^{1}\left(\Omega_{1}\right)\right)$. Then from (4.8c) and the transmission condition (4.4d) we get (the trace of) $\left\{p^{\epsilon, 2}\right\}$ strongly convergent in $L^{2}\left(0, T ; L^{2}(\Gamma)\right)$, hence, in $L^{2}\left(0, T ; H^{1}\left(\Omega_{2}\right)\right)$.

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