The Periodic Boundary-Value Problem

Denote the unit cube in $\mathbb{R}^N$ by $Q \equiv (0,1)^N$. Let $a(\cdot) \in L^\infty(Q)$ be uniformly positive: $a(x) \geq a_0 > 0,\ x \in Q$.
Consider the *periodic boundary-value problem*

$$
\begin{aligned}
-\nabla\cdot a(x)\nabla u(x) &= F(x), \quad x \in Q, && \text{(1a)}\\
u|_{x_j=0} &= u|_{x_j=1} \text{ and} && \text{(1b)}\\
(a\frac{\partial u}{\partial x_j})|_{x_j=0} &= (a\frac{\partial u}{\partial x_j})|_{x_j=1} \text{ on } \partial Q,\ 1 \leq j \leq N. && \text{(1c)}
\end{aligned}
$$

These are precisely the boundary conditions that imply that the solution u and the flux $a\nabla u \cdot \mathbf{n}$ are continuous when u is Q-periodically extended to $\mathbb{R}^N$.

The Weak Solution

Set $V = \{v \in H^1(Q) :\ v|_{x_j=0} = v|_{x_j=1} \text{ on } \partial Q,\ 1 \leq j \leq N\}$.
If u is a solution of (1), then $u \in V$ and for each $v \in V$ we have

$$
\int_Q F(x)v(x)\,dx = \int_Q a(x)\nabla u(x)\cdot\nabla v(x)\,dx - \int_{\partial Q} a\nabla u\cdot\mathbf{n}v\,dS
$$

$$
= \int_Q a(x)\nabla u(x)\cdot\nabla v(x)\,dx - \sum_{j=1}^{j=N}\int_{\partial Q_j}\left((a\frac{\partial u}{\partial x_j}\,v)|_{x_j=1} - (a\frac{\partial u}{\partial x_j}\,v)|_{x_j=0}\right)\,dS\,,
$$

where $\partial Q_j = \{s = (s_1, ..., s_N) \in \partial Q :\ \ s_j = 0\}$. This last sum is zero because of the boundary conditions, so we obtain

$$
u \in V :\quad \int_Q a(x)\nabla u(x)\cdot\nabla v(x)\,dx = \int_Q F(x)v(x)\,dx \text{ for all } v \in V. \quad (2)
$$

Conversely, we can show that any appropriately smooth solution of (2) is a solution of (1).

Notes

- Any two solutions of (2) differ by a constant in V, so we have uniqueness only up to constants.
- By taking $v(x) = 1$ in (2) we find a *necessary* condition for existence of a solution:

$$\int_Q F(x)\,dx = 0\,. \tag{3}$$

It is clear that the constant functions in V play a prominent role here. Uniqueness holds up to them, and the data $F(\cdot)$ must be L^2-orthogonal to them.

Hereafter we assume that (3) holds. We define the subspace $V_0 = \{v \in V : \int_Q v(x)\,dx = 0\}$. These are the functions of V with mean-value equal to zero. Then (2) is equivalent to

$$\tilde{u} \in V_0 : \quad \int_Q a(x)\nabla\tilde{u}(x)\cdot\nabla v(x)\,dx = \int_Q F(x)v(x)\,dx \text{ for all } v \in V_0. \tag{4}$$

where $\tilde{u}(x) = u(x) - \int_Q u(y)\,dy$. Thus we obtain an alternative weak formulation for which we have uniqueness. What remains is to show that the scalar product $\int_Q \nabla u(x)\cdot\nabla v(x)\,dx$ is equivalent to the $H^1(Q)$-scalar product on V_0.

The Estimate

We want to show that the gradient and the mean-value of a function provide a bound on the mean-square of the function.

Let $v \in H^1(Q)$ and x, $y \in Q$. By integrating along a path piecewise parallel to the axes, we obtain

$$v(x) - v(y) = \sum_{j=1}^{j=N} \left(v(y_1, ..., y_{j-1}, y_j, x_{j+1}, ...x_N) - v(y_1, ..., y_{j-1}, x_j, x_{j+1}, ...x_N)\right) = \sum_{j=1}^{j=N} \int_{x_j}^{y_j} \partial_j v(y_1, ..., y_{j-1}, s, x_{j+1}, ...x_N)\,ds\,.$$

Square both sides to get the estimate

$$v^2(x) + v^2(y) - 2v(x)v(y) \le N \sum_{j=1}^{j=N} \int_0^1 (\partial_j v)^2(y_1, ..., y_{j-1}, s, x_{j+1}, ...x_N)\, ds\,,$$

and integrate this with respect to x and then to y to obtain

$$2\|v\|^2_{L^2(Q)} - 2\left(\int_Q v(x)\, dx\right)^2 \le N \sum_{j=1}^{j=N} \|\partial_j v\|^2_{L^2(Q)}\,,$$

that is,

$$\|v\|^2_{L^2(Q)} \le \left(\int_Q v(x)\, dx\right)^2 + \frac{N}{2} \|\nabla v\|^2_{L^2(Q)}\,.$$

This shows that $\left(\|\nabla v\|^2_{L^2(Q)} + (\int_Q v)^2\right)^{1/2}$ is equivalent to the usual norm on $H^1(Q)$.

Summary

- The equations (1) are the strong form and (2) is equivalent to (4), the weak form of the periodic boundary-value problem.
- The scalar product $(\nabla u,\, \nabla v)_{L^2(Q)}$ is equivalent to the $H^1(Q)$ scalar product on V_0.

Theorem 0.1 *Assume $a(\cdot) \in L^\infty(Q)$ is uniformly positive, $a(x) \ge a_0 > 0$, $x \in Q$ and that $\int_Q F(x)\, dx = 0$. Then the periodic boundary-value problem (4) has a unique solution. That is, there exists a solution of (2), and any two solutions of (2) differ by a constant.*