

# SINGLE PHASE FLOW IN PARTIALLY FISSURED MEDIA

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ABSTRACT. Totally fissured media in which the individual cells are isolated by the fissure system are effectively described by double porosity models with microstructure. Such models contain the geometry of the individual cells in the medium and the flux across their interface with the fissure system which surrounds them. We extend these results to a dual-permeability model which accounts for the secondary flux arising from direct cell-to-cell diffusion within the solid matrix. Homogenization techniques are used to construct a new macroscopic model for the flow of a single phase compressible fluid through a partially fissured medium from an exact but highly singular microscopic model, and it is shown that this macroscopic model is mathematically well posed. Preliminary numerical experiments illustrate differences in the behaviour of solutions to the partially fissured from that of the totally fissured case.

## 1. Introduction.

The bulk characteristics of laminar flow through porous media are determined in the homogeneous case by two essential parameters, the *porosity* and the *permeability* of the medium [10]. A more detailed description of flow in naturally fractured porous media was initiated by necessity in the petroleum industry during the 1940's, where the high rate of recovery in the initial stages of reservoir production in fractured media often led to substantial overestimates of well production and capacity. In fact, the storage capacity of naturally fractured reservoirs varies extensively and depends largely on the degree of fracturing and the consequential range and distribution of the values of porosity and permeability. An extensive list of references on flow in fractured rocks is available in [34]. Any theory of flow through fractured media must account for this range of size in the pores and interstitial openings. The primary pores are the smallest, but they account for about 30% of the volume, while the relatively widely spaced and highly permeable fractures constitute only about 2% of the volume. This leads to the basic characteristics of fractured media, namely, that most storage can occur in the pore system while the fractures are responsible for most transport. The wide range in values of porosity and permeability in these two regions together with their volume distribution and geometric arrangement greatly complicate the development of models for flow in naturally fractured media. The objective is to accurately characterize the pressure changes and depletion history of the medium, and much effort over decades has been devoted to reproducing the transient response of the fluid exchange between fractures and matrix blocks.

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Any attempt to exactly model the flow through such highly inhomogeneous media leads to very singular problems of partial differential equations with rapidly oscillating coefficients. As an alternative, many methods of averaging have been developed, and these lead to various models of *dual-porosity* and *dual-permeability* types. The development of such dual models began with [8] where the fractured medium is represented by two independent overlapping flow fields, one representing the porous matrix and the other representing the system of fissures. These are coupled together to form a system of two (possibly degenerate) parabolic equations over the flow domain, one for the density field in each component of the medium, and these can be specialized further to reflect the assumptions incorporated in the corresponding model. The two components are treated symmetrically in the resulting system of two parabolic partial differential equations; such models are thus said to be of *parallel flow* type. In particular, this type of dual-porosity model for the idealized case of a *totally fissured medium* is developed in [8]: there is no flow in the porous matrix but only through the system of fissures, because the matrix is assumed to be composed of individual blocks which are isolated from each other by the very well developed system of fissures. In the more general dual-permeability case for which the fissure system is less developed and there is some flow permitted within the porous matrix, we call this a *partially fissured medium*. This more general model should prove useful in describing the variety of features which occur in naturally fractured media. These parallel flow models of dual-porosity or dual-permeability type have been developed substantially for a variety of problems; see [33, 14, 1, 7, 16, 21, 28].

Essential limitations of the parallel flow models include the suppression of the geometry of the small matrix blocks and their corresponding interfaces on which the coupling occurs as well as the lack of any distinction between the space and time scales of the two components of the medium. These deficiencies motivated the class of models of *distributed microstructure* type. Such models are known in many cases to be the limit (by homogenization) as the scale of the inhomogeneity tends to zero, and they provide a means not only to justify rigorously the model but also to represent it as a continuous distribution of blocks with prescribed geometry. Here we shall develop a distributed microstructure model for the flow of a single phase, slightly compressible fluid in a *partially fissured medium*, hereafter denoted by *PFM*. This is defined to be a porous medium in  $\mathbf{R}^3$  composed of two interwoven and connected components, the first being a matrix of porous blocks and the second being a system of fissures, so it exhibits both dual-porosity and dual-permeability characteristics. Note that it is impossible to satisfy these geometric constraints in  $\mathbf{R}^2$ . Limiting cases of the geometry arise when one of the two components of the medium becomes disconnected. In the special case of disjoint porous blocks which are separated by the system of fissures, it is called a *totally fissured medium* and denoted hereafter by *TFM*. Single phase flow, as well as more complicated flows, in a *TFM* have been investigated by several authors; see [2,5,6,11,20,31]. The recent book [19] contains a survey of these and other results on distributed microstructure models. Below, we develop such a model of single phase flow in the general case of a *PFM* which in the limit (as the ratio of the volume of space occupied by the connecting portion of the matrix to the bulk volume of the matrix tends to zero) reduces to the corresponding model for a *TFM*.

The common characteristics of fissured media are that the matrix of porous blocks occupies a much larger volume than the fissures and that it is relatively much more resistant to fluid flow than is the fissure system. As a consequence, most of the flow passes through the system of fissures, while bulk storage of fluid takes place primarily inside the porous

matrix formed by the blocks. In a *TFM* the flow in the blocks is induced only by the exchange of fluid which takes place on the block–fissure interfaces, and any interaction between the blocks is possible only via the neighboring system of fissures, which separate the blocks. The proper description of flow in a fissured medium requires both global and local characteristics; it is not possible to capture the duality between macro– and micro–structure by means of standard models for flow in porous media (see [11]).

In partially fissured media, blocks are connected to neighboring blocks, so that some part of the flow passes through the block interconnections. While the primary flow will continue to be that from blocks into fissures followed by flow within the fissures, the flow in the porous matrix has more than only a local character, as in the case of a *TFM*. In a *PFM*, it is possible that the behavior in nearby blocks can influence directly the behavior in each, not just indirectly via the system of fissures. In many situations this effect is less prominent than the bulk flow in the fractures, but in others where the matrix has a moderately higher permeability and the interconnections between the blocks are sufficiently large it can have a noticeable effect .

Exact microscopic models of flow in a fissured medium customarily treat the fissures and the matrix systems as two Darcy media with different physical parameters. The discontinuities in the parameter values across the matrix–fissure interfaces are severe, with the ratios of their values in the fissures and blocks usually being of some orders of magnitude; moreover, the characteristic width of the fissures will be very small in comparison with the size of the blocks. Consequently, the exact microscopic model, written as a classical interface problem, is numerically and analytically intractable. The common technique used to overcome this difficulty is to construct models which describe the flow on two scales, macroscopic and microscopic (see [2,5,6,11,20,31]). At the macroscopic scale of the reservoir the whole domain of flow is seen as occupied by a pseudo–porous medium with the “impermeable” solid part being replaced by the matrix of permeable blocks and the pores representing the fissures. In these models, the microscopic scale appears through the necessity to define the flow on matrix blocks. The flow in the two scales is related through interface conditions on the faces of the blocks that conserve mass and momentum (and, in the case of some more complicated fluids, additional quantities); these interface conditions present themselves as boundary conditions on the blocks and as distributed source terms in the macroscopic equations.

Derivations of these two–scale models of distributed microstructure type have been carried out for the case of totally fissured media, and they are based on an averaging over the exact geometry of the region (see [2,3]) or by the construction of a continuous distribution of blocks over the region as in [31] or by assuming some periodic structure for the domain that permits the use of the homogenization technique (see [20] or [23] for a review). The general modeling framework has also been applied to derive models for multiphase, multicomponent, and nonisothermal flows in a *TFM*, for which some analytical as well as numerical results exist (see [4,11,24,25, 19]).

In this paper we shall construct by means of homogenization a model for the simplest type of flow, that of a single phase, compressible fluid, in a partially fissured medium. We shall apply general ideas of homogenization (see [9, 29] and the specific framework introduced in [5]) for modeling of flows in fissured media. The plan is as follows. In §2 we review the construction of a model for single phase flow in a *TFM*. In §3 we develop an exact  $\epsilon$ –model for diffusion in a *PFM* which provides the basis for the homogenization construction; §4 contains technical calculations which lead to the limiting model composed

of macroscopic and microscopic equations. In §5 we summarize the limiting model and comment on its well-posedness. The concluding §6 consists of some remarks on the observed relative behavior of the various models in some preliminary numerical experiments. These indicate that the qualitative differences in behavior of solutions of the *PFM* model from those of the *TFM* are sufficiently large to be observable. Realistic numerical models constructed from typical data will be developed elsewhere.

## 2. A Homogenized Model for Single Phase Flow in a *TFM*.

Here we review the derivation by homogenization of a model for single phase flow in totally fissured media following [6,11]. The notation below closely follows that of these papers as well.

We begin with the microscopic model of single phase flow in a fissured domain  $\Omega$ , a bounded open subset of  $R^3$ , over the time interval  $I = (0, T)$ ,  $T > 0$ . The fissure and matrix components of the domain are denoted by  $\Omega_f$  and  $\Omega_m$ , respectively. Their boundaries are denoted by  $\partial\Omega_f$  and  $\partial\Omega_m$ . The fissure-matrix interface is given by  $I_{fm} = \partial\Omega_f \cap \partial\Omega_m$ . The domain is assumed to have a periodic structure, with the cell of the period being taken to be  $Y = (0, 1)^3$  for simplicity (see [11,12,13] for different choices in the shape of the period); hence,  $\Omega$  consists of a lattice of copies of  $Y$ . The cell  $Y$  retains the double component structure of the fissured domain and  $\bar{Y} = \bar{Y}_f \cup \bar{Y}_m$ , with  $Y_f$  and  $Y_m$  denoting the fissure and matrix parts of the cell. Let  $\Gamma_{fm}$  be the part of  $I_{fm}$  contained in  $Y$ , and let  $\Gamma_{ff}$  and  $\Gamma_{mm}$  denote the respective intersections of  $\partial Y$  with  $\Omega_f$  and  $\Omega_m$ . We note that, in the totally fissured case, the block interconnection  $\Gamma_{mm}$  is empty (see Figure 1). By  $\eta_m$ , we denote the normal unit vector to  $\Gamma_{fm}$  which points in the direction out of  $Y_m$  and by  $\eta_f$  its counterpart out of  $Y_f$ .

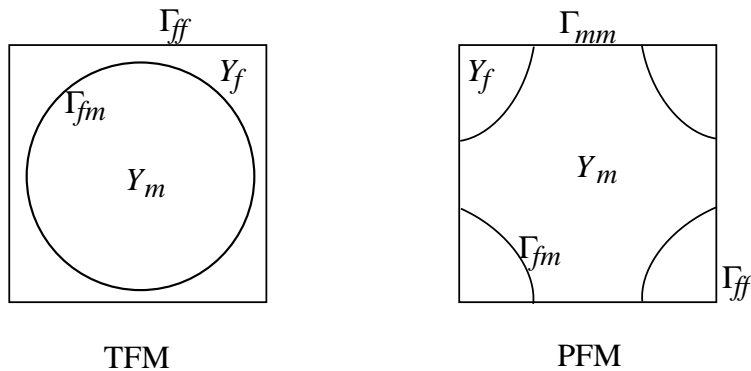


FIGURE 1. Fissured Media

In addition to the assumption of periodicity of the geometry, we assume that the physical parameters of the problem have  $Y$ -periodic character, which implies that the solutions to the differential problem also exhibit certain periodic behavior. They have, however, also some macroscopic (non-periodic) behavior which is seen on the scale of the whole reservoir. We are interested in capturing and possibly decoupling both of these solution modes, the “global” (macroscopic) mode and the “local” (microscopic-periodic) mode; this will be achieved by the technique of homogenization. To this aim, we shall investigate the asymptotics of solutions as  $\varepsilon \rightarrow 0$  to a family of properly scaled problems posed on domains  $\Omega^\varepsilon$  formed by unions of copies of cells  $\varepsilon Y$ . Below, we use  $\varepsilon$  as a superscript or

subscript on coefficients or variables to denote objects periodic with respect to  $\varepsilon Y$ ; we omit this notation when  $\varepsilon = 1$ .

In order to define the  $\varepsilon$ -model, we first recall the model of flow of a slightly compressible, viscous fluid of density  $\rho$ , viscosity  $\mu$ , and compressibility  $c$  in an ordinary porous medium of porosity  $\phi$  and permeability  $k$ . The equation of state relating the pressure to the density is given by

$$(2.1) \quad d\rho = c\rho dp.$$

Conservation of momentum is expressed by Darcy's law, which together with the continuity equation (conservation of mass) leads to the equation (see [10, 11])

$$(2.2) \quad \phi \frac{\partial \rho}{\partial t} - \nabla \cdot (\lambda \nabla \rho) = 0,$$

in which the mobility  $\lambda$  is defined by

$$(2.3) \quad \lambda = \frac{k}{\mu c}.$$

Let the system of fissures and matrix blocks in  $\Omega^\varepsilon$  be denoted by  $\Omega_f^\varepsilon$  and  $\Omega_m^\varepsilon$ , respectively. The  $\varepsilon$ -model on  $\Omega^\varepsilon$  consists of differential equations on each of the subdomains  $\Omega_f^\varepsilon$  and  $\Omega_m^\varepsilon$  for the density, which will be denoted by  $\rho_{f,\varepsilon}$  on  $\Omega_f^\varepsilon$  and by  $\gamma_\varepsilon$  on  $\Omega_m^\varepsilon$ , respectively, plus two interface conditions on  $\Gamma_{fm}^\varepsilon$  to insure conservation of mass and momentum across  $\Gamma_{fm}^\varepsilon$ . An exterior boundary condition and an initial condition must also be specified, but they do not enter into the derivation of the limit model. In the totally fissured case, it has been shown that, to preserve the magnitude of the flux crossing the interfaces contained within a fixed volume of the medium as  $\varepsilon \rightarrow 0$ , it is necessary to scale the mobility in the blocks by the factor  $\varepsilon^2$  (see [5]). Thus, the  $\varepsilon$ -model of diffusion in a *TFM* has the form

$$(2.4) \quad \varphi_f \frac{\partial \rho_{f,\varepsilon}}{\partial t} - \nabla \cdot (\lambda_f \nabla \rho_{f,\varepsilon}) = 0, \quad x \in \Omega_f^\varepsilon, \quad t \in I,$$

$$(2.5) \quad \varphi_m \frac{\partial \gamma_\varepsilon}{\partial t} - \nabla \cdot (\varepsilon^2 \lambda_m \nabla \gamma_\varepsilon) = 0, \quad x \in \Omega_m^\varepsilon, \quad t \in I,$$

$$(2.6) \quad \lambda_f \nabla \rho_{f,\varepsilon} \cdot \eta_m = \varepsilon^2 \lambda_m \nabla \gamma_\varepsilon \cdot \eta_m, \quad x \in \Gamma_{fm}^\varepsilon, \quad t \in I,$$

$$(2.7) \quad \gamma_\varepsilon = \rho_{f,\varepsilon}, \quad x \in \Gamma_{fm}^\varepsilon, \quad t \in I.$$

If  $\rho_{f,\varepsilon}$  and  $\gamma_\varepsilon$  are expanded in powers of  $\varepsilon$  and the formal analysis of these expansions is carried out (see [5]), it can be seen that the leading terms for the densities in the fractures and matrix blocks satisfy the following system of equations:

$$(2.8) \quad \varphi_f \frac{|Y_f|}{|Y|} \frac{\partial \rho_{f,0}}{\partial t}(x, t) - \nabla_x \cdot (\Lambda_f \nabla_x \rho_{f,0}) = q_{mf}(x, t), \quad x \in \Omega, \quad t \in I,$$

$$(2.9) \quad q_{mf}(x, t) = -\frac{1}{|Y|} \int_{\Gamma_{fm}} \lambda_m \nabla_y \gamma_0 \cdot \eta_m d\Gamma, \quad x \in \Omega, \quad t \in I,$$

$$(2.10) \quad \varphi_m \frac{\partial \gamma_0}{\partial t}(x, y, t) - \nabla_y \cdot (\lambda_m \nabla_y \gamma_0) = 0, \quad y \in Y_m(x), \quad x \in \Omega, \quad t \in I,$$

$$(2.11) \quad \gamma_0 = \rho_{f,0}, \quad y \in \partial Y_m(x), \quad x \in \Omega, \quad t \in I,$$

where  $|Y|$  and  $|Y_f|$  denote the volumes of the reference sets  $Y$  and  $Y_f$ , respectively. The effective mobility tensor  $\Lambda_f$  is given by

$$(2.12) \quad (\Lambda_f)_{ij} = \frac{1}{|Y|} \int_{Y_f} \lambda_f \left( \delta_{i,j}|Y_f| + \frac{\partial \omega_i}{\partial y_j} \right) dy,$$

with the auxiliary functions  $\omega_k$ ,  $k = 1, 2, 3$ , being  $Y$ -periodic solutions (modulo a constant) of

$$(2.13) \quad \nabla_y^2 \omega_k = 0, \quad y \in Y_f,$$

$$(2.14) \quad \nabla_y \omega_k \cdot \eta_f = -e_k \cdot \eta_f, \quad y \in \Gamma_{fm},$$

where  $e_k$  is the unit vector in the direction of the  $k$ -axis.

Equation (2.8) is to be solved in  $\Omega$  for the macroscopic density,  $\rho_{f,0}$ . The right-hand side of this equation contains the distributed source term,  $q_{mf}$ , which evaluates the flux across the boundary of the block  $Y_m(x)$  topologically attached to the point  $x \in \Omega$  in the two-sheeted covering of  $\Omega$ . Blocks over different points in  $\Omega$  are disconnected; thus, no flow can take place directly from one such block to another. It is this feature that identifies this distributed microstructure or two-scale model as being a dual porosity model for flow in a totally fissured medium.

If the scaling of the permeability in the blocks had been omitted, then the limit process would have led to a single porosity macroscopic system that fails to represent the delay that is inherent in the flux entering the fractures from the blocks. It is precisely this delay that led three decades ago to the introduction by Barenblatt, Zheltov, and Kochina [8] and Warren and Root [33] of simpler parallel flow models, which were limited by the computational capacities then available, in order to match observed reservoir behavior better. For further discussion, see [14, 11,5, 17,18] and the references therein.

### 3. Single Phase Flow in a *PFM*: the $\varepsilon$ -Model.

In this section we develop an  $\varepsilon$ -model for single phase flow in a partially fissured medium. In the next section we apply homogenization to the  $\varepsilon$ -model to derive the limiting, macroscopic model for this type of flow.

Let us first discuss what would happen if we were to change only the geometry of the *TFM*. This seems to provide a possible model for a *PFM*, since we did not explicitly use the assumption that the matrix blocks be disconnected in the construction of the  $\varepsilon$ -model, nor did it seem to be used when passing formally to the limit as  $\varepsilon \rightarrow 0$ . However, the scaling of the permeability in the blocks and the form of the interface conditions implicitly contain the assumption of local disconnectivity; nowhere was there a provision for global flow to take place totally within the matrix. This lack is clearly apparent in the auxiliary problems (2.13)–(2.14) whose solutions are used to close the homogenization process and to evaluate the permeability tensor in the macroscopic treatment of the fractures. Hence, no macroscopic model can result from the *TFM*  $\varepsilon$ -model that can successfully model flow having two global parts, as is intuitively inherent in the case of a *PFM*. Thus, it is necessary to redesign the  $\varepsilon$ -model to account for the connectivity of the blocks, while still accounting for the local interaction between the fracture and block structures. In particular, it is necessary to provide for the existence of a globally defined density in the matrix, in addition to the local description of the density in a block; i.e., both the rapidly

varying and the slowly varying components of the density in the matrix must enter into the model. Thus, we are led heuristically to introduce two scalings of the permeability in the matrix, but only one in the fractures. (The porosity, the viscosity, and the compressibility do not scale.)

As in the  $\varepsilon$ -model for diffusion in a *TFM*, we use  $\rho_{f,\varepsilon}$  to describe the density in the fissures; but, in order to describe the density in the matrix, instead of one variable we use two variables. The first,  $\rho_{m,\varepsilon}$ , leads to the global description of the density in the matrix, while the second,  $\gamma_\varepsilon$ , will provide the required information about the local behavior of the density as restricted to a single cell. We specify two coefficients,  $\alpha$  and  $\beta$ , which determine the “proportion” between the slow and rapid (global and local) phases of the “total” density in the matrix as measured on the interface  $\Gamma_{fm}$ . Note that  $\alpha + \beta = 1$ ,  $\beta > 0$ ,  $\alpha \geq 0$ .

The  $\varepsilon$ -model is as follows:

$$\begin{aligned}
(3.1) \quad & \varphi_f \frac{\partial \rho_{f,\varepsilon}}{\partial t} - \nabla \cdot (\lambda_f \nabla \rho_{f,\varepsilon}) = 0 && \text{in } \Omega_f^\varepsilon \times I, \\
(3.2) \quad & \varphi_m \frac{\partial \rho_{m,\varepsilon}}{\partial t} - \nabla \cdot (\lambda_m \nabla \rho_{m,\varepsilon}) = 0 && \text{in } \Omega_m^\varepsilon \times I, \\
(3.3) \quad & \varphi_m \frac{\partial \gamma_\varepsilon}{\partial t} - \nabla \cdot (\varepsilon^2 \lambda_m \nabla \gamma_\varepsilon) = 0 && \text{in } \cup Y_m^\varepsilon \times I, \\
(3.4) \quad & \beta \lambda_f \nabla \rho_{f,\varepsilon} \cdot \eta_f + \varepsilon^2 \lambda_m \nabla \gamma_\varepsilon \cdot \eta_m = 0 && \text{on } \Gamma_{fm}^\varepsilon \times I, \\
(3.5) \quad & \alpha \lambda_f \nabla \rho_{f,\varepsilon} \cdot \eta_f + \lambda_m \nabla \rho_{m,\varepsilon} \cdot \eta_m = 0 && \text{on } \Gamma_{fm}^\varepsilon \times I, \\
(3.6) \quad & \rho_{f,\varepsilon} = \alpha \rho_{m,\varepsilon} + \beta \gamma_\varepsilon && \text{on } \Gamma_{fm}^\varepsilon \times I, \\
(3.7) \quad & \gamma_\varepsilon = \kappa \rho_{m,\varepsilon} && \text{on } \Gamma_{mm}^\varepsilon \times I, \\
(3.8) \quad & \lambda_m \nabla \rho_{m,\varepsilon} \cdot \eta_m + \kappa \varepsilon^2 \lambda_m \nabla \gamma_\varepsilon \cdot \eta_m = 0 && \text{on } \Gamma_{mm}^\varepsilon \times I.
\end{aligned}$$

The first three equations describe “fast”, “moderate”, and “very slow” flow, which are defined in the fissures, the matrix, and individual blocks, respectively. In order to stress the difference between the definitions of  $\rho_{m,\varepsilon}$  and  $\gamma_\varepsilon$ , note the different spatial domains on which the equations (3.2) and (3.3) are to be solved. The equation (3.3) is to be solved in the set of interiors of what are now artificially disconnected individual blocks, while (3.2) is to be solved in the whole of  $\Omega_m^\varepsilon$ , which includes all of the blocks and the interfaces between adjacent blocks to form a globally connected set. The conditions (3.4) and (3.5) conserve mass across the interfaces  $\Gamma_{fm}^\varepsilon$  between the density in fissures and the “total” density in the matrix, with prescribed proportions between the two components of the total density in the matrix; (3.5) is an ordinary interface condition, while (3.4) is typical for a fissured medium interface condition with the permeability coefficient scaled to preserve the magnitude of the flux across the union of all interfaces contained in a fixed volume as  $\varepsilon \rightarrow 0$ . As a consequence of those two relations, the fluxes described by the two density variables in the matrix satisfy the equation

$$(3.9) \quad \lambda_m \nabla \rho_{m,\varepsilon} \cdot \eta_m = \frac{\alpha}{\beta} \varepsilon^2 \lambda_m \nabla \gamma_\varepsilon \cdot \eta_m \quad \text{on } \Gamma_{fm}^\varepsilon.$$

The condition (3.6) expresses conservation of momentum between the fissures and the matrix, with prescribed proportions between the two phases (global and local) in the matrix. We note that the condition (3.6) is, in a mathematical sense, dual to the conditions

(3.4) and (3.5) (see below notes on the well-posedness of the problem). The system is complemented by a pair of conservation equations (3.7) and (3.8) (momentum and mass) on (the artificial interface)  $\Gamma_{mm}^\varepsilon$ . The constant  $\kappa$  appearing in these (pairwise dual) equations gives the option of imposing another proportion between global and local phases of the density in the matrix to hold on  $\Gamma_{mm}^\varepsilon$ . We require that  $0 \leq \kappa \leq 1$ .

The combination of the three constants  $\alpha$ ,  $\beta$ , and  $\kappa$  determines the proportions between the different components of the total density in the matrix on the boundary of the blocks  $Y_m^\varepsilon$ . The relevant values in a particular application can be established by an experimental or numerical study. Some choices of the values of the parameters  $\{\alpha, \beta, \kappa\}$  have special interpretations, as discussed below. For example, the case of  $\alpha = 0$ ,  $\beta = 1$  is interpreted as follows: the interface  $\Gamma_{fm}^\varepsilon$  is “impervious” for the “global flow” in the matrix (described by the variable  $\rho_{m,\varepsilon}$ ) or, in other words, that the changes in  $\rho_{m,\varepsilon}$  arise only by interaction with  $\gamma^\varepsilon$ , which, in turn is “fed” by the flow in the fissures across  $\Gamma_{fm}$ . On the other hand, the choice of  $\kappa = 1$  for conditions on  $\Gamma_{mm}^\varepsilon$  leads to the interpretation that the fluxes associated with the “local” and “global” variable are mutually “reflected” from the (artificial) boundary. One might also see then (3.7) and (3.8) as a pair of “standard interface conditions” modified to indicate that both variables are considered on the same side of  $\Gamma_{mm}^\varepsilon$ , rather than on opposite sides as it is the case of classical interface conditions. Finally, if  $\alpha = 0$ ,  $\beta = 1$ ,  $\kappa = 1$ , and  $\Gamma_{mm}^\varepsilon = \emptyset$  (formally), then the model reduces to the  $\varepsilon$ -model for the *TFM* case. The model derived in the limiting process from this choice is equivalent to the model for *TFM*, as shown later. Other choices of  $\alpha$ ,  $\beta$ , and  $\kappa$  lead to different patterns of splitting between the two pseudo-phases.

Independent of the choice of  $\alpha$ ,  $\beta$ , and  $\kappa$ , one can prove that the system (3.1)–(3.8) is well-posed, when complemented by the appropriate initial and boundary conditions

$$(3.10) \quad \rho_{f,\varepsilon}(x, 0) = \rho_{f,init}, \quad x \in \Omega_f^\varepsilon,$$

$$(3.11) \quad \rho_{m,\varepsilon}(x, 0) = \rho_{m,init}(x), \quad x \in \Omega_m^\varepsilon,$$

$$(3.12) \quad \gamma(x, 0) = \gamma_{init}(x), \quad x \in \Omega_m^\varepsilon,$$

$$(3.13) \quad \lambda_f \nabla \rho_{f,\varepsilon} \cdot \eta_\Omega = 0, \quad x \in \partial\Omega \cap \partial\Omega_f^\varepsilon,$$

$$(3.14) \quad \lambda_m \nabla \rho_{m,\varepsilon} \cdot \eta_\Omega = 0, \quad x \in \partial\Omega \cap \partial\Omega_m^\varepsilon,$$

$$(3.15) \quad \varepsilon^2 \lambda_m \nabla \gamma_\varepsilon \cdot \eta_\Omega = 0, \quad x \in \partial\Omega \cap \partial\Omega_m^\varepsilon.$$

It is a system of linear parabolic equations coupled by interface conditions. The coupling on interfaces is the crucial element in the system, and it is the main source of difficulty in its analysis and approximation. One can see that the dynamics of the problem is governed by an analytic semigroup, in the general setting of the following well-known result (see e.g. [30]).

**Theorem 1.** *Assume that  $\mathcal{V}$  and  $\mathcal{H}$  are Hilbert spaces, with  $\mathcal{V}$  dense and continuously imbedded in  $\mathcal{H}$ . Let  $a(\cdot, \cdot)$  be a continuous bilinear form defined on  $\mathcal{V}$  such that the form  $a(\cdot, \cdot) + (\cdot, \cdot)_\mathcal{H}$  is  $\mathcal{V}$ -coercive; i.e., for some positive constant  $c$ ,  $a(u, u) + (u, u)_\mathcal{H} \geq c\|u\|_\mathcal{V}^2$ . Then, whenever  $f \in C^\nu([0, \infty), \mathcal{H})$ ,  $0 < \nu \leq 1$ , and  $u_0 \in \mathcal{H}$ , there exists a unique  $u \in C([0, \infty), \mathcal{H}) \cap C^1((0, \infty), \mathcal{H})$  such that  $u(t) \in \mathcal{V}$  for  $t > 0$  and*

$$(3.16) \quad \begin{aligned} (u'(t), v)_\mathcal{H} + a(u(t), v) &= (f, v)_\mathcal{H}, \quad \forall v \in \mathcal{V}, \\ u(0) &= u_0. \end{aligned}$$

To apply the theorem in order to prove well-posedness of (3.1)–(3.15) (or its more general form, with an external source term  $f$  as admitted by the theorem) we need to define an appropriate abstract setting for the problem. Let

$$\mathcal{H} = L^2(\Omega_f^\varepsilon) \times L^2(\Omega_m^\varepsilon) \times L^2(\Omega_m^\varepsilon).$$

Note that  $L^2(\Omega_m^\varepsilon) = L^2(\cup Y_m^\varepsilon)$ . The scalar product in  $\mathcal{H}$  is defined as

$$(u, v)_{\mathcal{H}} = \int_{\Omega_f^\varepsilon} \phi_f u_1 v_1 dx + \int_{\Omega_m^\varepsilon} \phi_m (u_2 v_2 + u_3 v_3) dx.$$

Next, set

$$\mathcal{V} = \{u \in H^1(\Omega_f^\varepsilon) \times H^1(\Omega_m^\varepsilon) \times H^1(\Omega_m^\varepsilon) : u_1 = \alpha u_2 + \beta u_3 \text{ on } \Gamma_{fm}, \kappa u_2 = u_3 \text{ on } \Gamma_{mm}\}$$

and define the bilinear form

$$a(u, v) \equiv \int_{\Omega_f^\varepsilon} (\lambda_f \nabla u_1 \cdot \nabla v_1) dx + \int_{\Omega_m^\varepsilon} (\lambda_m \nabla u_2 \cdot \nabla v_2 + \varepsilon^2 \lambda_m \nabla u_3 \cdot \nabla v_3) dx.$$

Under appropriate assumptions on the data of the problem (specified in Corollary 2 below), the required hypotheses of the theorem hold on  $\mathcal{V}$ ,  $\mathcal{H}$ ,  $a(\cdot, \cdot)$ . Hence, the Cauchy problem (3.16) has a unique solution.

Now we demonstrate that this problem is a variational form of our differential problem (3.1)–(3.15) by the following calculation. Let  $v = (v_1, v_2, v_3) \in \mathcal{V}$ , multiply (3.1), (3.2), and (3.3) by the corresponding components of  $v$ , and integrate the resulting equations over  $\Omega_f$ ,  $\Omega_m$ , and  $\Omega_m$ . Integration over  $\Omega_m^\varepsilon$  means integration over individual blocks, followed by summation over all of the blocks. Application of Green's theorem and the boundary conditions on  $\partial\Omega$  leads to the relations

$$\begin{aligned} \int_{\Omega_f^\varepsilon} (\phi_f \frac{\partial \rho_{f,\varepsilon}}{\partial t} v_1 + \lambda_f \nabla \cdot \rho_{f,\varepsilon} \nabla v_1) dx &= \int_{\Gamma_{fm}} \lambda_f \nabla \rho_{f,\varepsilon} \cdot \eta_f v_1 d\Gamma, \\ \int_{\Omega_m^\varepsilon} (\phi_m \frac{\partial \rho_{m,\varepsilon}}{\partial t} v_2 + \lambda_m \nabla \rho_{m,\varepsilon} \cdot \nabla v_2) dx &= \int_{\Gamma_{fm}} \lambda_m \nabla \rho_{m,\varepsilon} \cdot \eta_m v_2 d\Gamma + \int_{\Gamma_{mm}} \lambda_m \nabla \rho_{m,\varepsilon} \cdot \eta_m v_2 d\Gamma, \\ \int_{\Omega_m^\varepsilon} (\phi_m \frac{\partial \gamma_\varepsilon}{\partial t} v_3 + \varepsilon^2 \lambda_m \nabla \gamma_\varepsilon \cdot \nabla v_3) dx &= \int_{\Gamma_{fm}} \varepsilon^2 \lambda_m \nabla \gamma_\varepsilon \cdot \eta_m v_3 d\Gamma + \int_{\Gamma_{mm}} \varepsilon^2 \lambda_m \nabla \gamma_\varepsilon \cdot \eta_m v_3 d\Gamma, \end{aligned}$$

where the integrals over  $\Gamma_{fm}$  and  $\Gamma_{mm}$  should be understood as a sum over all blocks of integrals on the interfaces restricted to the individual blocks. The flux conservation conditions (3.4) and (3.5) give

$$\begin{aligned} \int_{\Gamma_{fm}} \lambda_f \nabla \rho_{f,\varepsilon} \cdot \eta_f v_1 d\Gamma &= -\frac{1}{\beta} \int_{\Gamma_{fm}} \varepsilon^2 \lambda_m \nabla \gamma_\varepsilon \cdot \eta_m v_1 d\Gamma, \\ \int_{\Gamma_{fm}} \lambda_m \nabla \rho_{m,\varepsilon} \cdot \eta_m v_2 d\Gamma &= \frac{\alpha}{\beta} \int_{\Gamma_{fm}} \varepsilon^2 \lambda_m \nabla \gamma_\varepsilon \cdot \eta_m v_2 d\Gamma, \end{aligned}$$

while it follows from (3.8) that

$$\int_{\Gamma_{mm}} \lambda_m \nabla \rho_{m,\varepsilon} \cdot \eta_m v_2 d\Gamma = -\kappa \int_{\Gamma_{mm}} \varepsilon^2 \lambda_m \nabla \gamma_\varepsilon \cdot \eta_m v_2 d\Gamma.$$

Add the equations above, apply the relations  $v_1 = \alpha v_2 + \beta v_3$  on  $\Gamma_{fm}$  and  $v_3 = \kappa v_2$  on  $\Gamma_{mm}$ , and set  $u = (\rho_{f,\varepsilon}, \rho_{m,\varepsilon}, \gamma)$ . Together with (3.13)–(3.15) we then obtain

$$(3.17) \quad (u', v)_{\mathcal{H}} + a(u, v) = 0.$$

Conversely, these calculations can be reversed to show that a solution of (3.17) is a generalized solution of (3.1)–(3.8) and (3.13)–(3.15). This leads to the following corollary.

**Corollary 2.** *Let  $\lambda_f$  and  $\lambda_m$  be symmetric, positive-definite tensors, and let  $\phi_f$  and  $\phi_m$  be positive. Then, the system (3.1)–(3.15) with square-integrable initial values as specified in (3.13)–(3.15) is a well-posed Cauchy problem.*

#### 4. Single Phase Flow in a PFM: the Macroscopic Model.

We shall apply the method of matched asymptotic expansions to the  $\varepsilon$ -model of the previous section when the functions are expressed in terms of two spatial variables, the “slow” variable  $x$  and the “fast” variable  $y = x/\varepsilon$ , which represents the local behavior on the scale of the cell  $\varepsilon Y$ , as  $\varepsilon \rightarrow 0$ . The time variable will always belong to the interval  $I$ ; it will not be necessary to repeat this below.

We assume the following formal asymptotic expansions (see [9,29,32] for the general multiple scale expansion method and [19, 5] for applications to flows in fissured media):

$$(4.1) \quad \rho_{f,\varepsilon}(x) = \rho_{f,0}(x, y) + \varepsilon \rho_{f,1}(x, y) + \varepsilon^2 \rho_{f,2}(x, y) + \dots,$$

$$(4.2) \quad \rho_{m,\varepsilon}(x) = \rho_{m,0}(x, y) + \varepsilon \rho_{m,1}(x, y) + \varepsilon^2 \rho_{m,2}(x, y) + \dots,$$

$$(4.3) \quad \gamma_\varepsilon(x) = \gamma_0(x, y) + \varepsilon \gamma_1(x, y) + \varepsilon^2 \gamma_2(x, y) + \dots,$$

$$(4.4) \quad \nabla = \nabla_x + \varepsilon^{-1} \nabla_y;$$

in addition, we assume that the functions  $\rho_{f,i}$ ,  $i \geq 0$ , are periodic in the  $y$ -variable with period  $Y$ .

Now, insert (4.1)–(4.4) into (3.1)–(3.8) and compare like powers of  $\varepsilon$ . By (3.1,  $k$ ), we shall mean the equation for  $k^{\text{th}}$ -order terms in  $\varepsilon$  in the equation (3.1). Thus, the pair (3.1,-2) and (3.4,-1) (which is satisfied for  $0 < \beta \leq 1$ ) give the equations

$$(4.5) \quad \nabla_y \cdot (\lambda_f \nabla_y \rho_{f,0}) = 0, \quad y \in Y_f, \quad x \in \Omega,$$

$$(4.6) \quad \lambda_f \nabla_y \rho_{f,0} \cdot \eta_f = 0, \quad y \in \Gamma_{fm}, \quad x \in \Omega.$$

Equations (4.5) and (4.6), together with periodicity of  $\rho_{f,0}$  on  $\Gamma_{ff}$ , imply that

$$(4.7) \quad \rho_{f,0} = \rho_{f,0}(x)$$

is independent of the fast variable  $y$ , as it should be so that  $\rho_{f,0}$  can indicate just the smoothed, global behavior of  $\rho$  in the fissures.

Similarly, (3.2,-2) and (3.9,-1) (as a consequence of (3.4,-1) and (3.5,-1)), together with (3.8,-1) imply that

$$(4.8) \quad \rho_{m,0} = \rho_{m,0}(x),$$

so that  $\rho_{m,0}$  also describes global behavior, now in the matrix. Then, it is immediate from (3.6,0) and (3.7,0) that

$$(4.9) \quad \rho_{f,0}(x) = \alpha\rho_{m,0}(x) + \beta\gamma_0(x, y), \quad y \in \Gamma_{fm}^\varepsilon; \quad \gamma_0(x, y) = \kappa\rho_{m,0}(x), \quad y \in \Gamma_{mm}^\varepsilon.$$

Next, (3.1,-1) gives the relation

$$\nabla_x \cdot (\lambda_f \nabla_y \rho_{f,0}) + \nabla_y \cdot (\lambda_f \nabla_x \rho_{f,0}) + \nabla_y \cdot (\lambda_f \nabla_y \rho_{f,1}) = 0, \quad x \in \Omega, \quad y \in Y_f;$$

from (4.7), we see that

$$(4.10) \quad \nabla_y \cdot (\lambda_f \nabla_y \rho_{f,1}) = 0, \quad y \in Y_f.$$

Then, (3.4,0) implies that

$$(4.11) \quad (\lambda_f \nabla_y \rho_{f,1}) \cdot \eta_f = -(\lambda_f \nabla_x \rho_{f,0}) \cdot \eta_f, \quad y \in \Gamma_{fm}^\varepsilon.$$

As in the derivation of the model for a *TFM*, we let  $\omega_{f,k}$ ,  $k = 1, 2, 3$ , denote  $Y$ -periodic solutions (up to a constant) of

$$(4.12) \quad \nabla_y^2 \omega_{f,k} = 0 \quad \text{in } Y_f,$$

$$(4.13) \quad \nabla_y \omega_{f,k} \cdot \eta_f = -e_k \cdot \eta_f \quad \text{on } \Gamma_{fm},$$

where, as before,  $e_k$  is the unit vector in the direction of the  $k$ -axis; the mobility  $\lambda_f$  has been assumed to be a diagonal tensor in (4.13), though it is a simple extension to allow it to have a more general form. Then,  $\rho_{f,1}$  can be represented in the form

$$(4.14) \quad \rho_{f,1}(x, y) = \sum_{j=1}^3 \omega_{f,j}(y) \frac{\partial \rho_{f,0}}{\partial x_j}(x) + c(x), \quad x \in \Omega, \quad y \in Y_f.$$

Next, the equation generated by (3.1,0) reads as follows:

$$(4.15) \quad \begin{aligned} \varphi_f \frac{\partial \rho_{f,0}}{\partial t} - \nabla_x \cdot (\lambda_f \nabla_x \rho_{f,0}) - \nabla_y \cdot (\lambda_f \nabla_x \rho_{f,1}) \\ - \nabla_x \cdot (\lambda_f \nabla_y \rho_{f,1}) - \nabla_y \cdot (\lambda_f \nabla_y \rho_{f,2}) = 0, \quad x \in \Omega, \quad y \in Y_f. \end{aligned}$$

Integrate (4.15) over  $Y_f$ , use (4.14) for the fourth term, use (4.7), and divide the result by  $|Y|$ :

$$(4.16) \quad \begin{aligned} \varphi_f \frac{|Y_f|}{|Y|} \frac{\partial \rho_{f,0}}{\partial t} - \frac{1}{|Y|} \nabla_x \cdot \left( \int_{Y_f} \lambda_f \nabla_x \rho_{f,0} dy \right) \\ - \frac{1}{|Y|} \int_{Y_f} \nabla_y \cdot (\lambda_f (\nabla_x \rho_{f,1} + \nabla_y \rho_{f,2})) dy \\ - \frac{1}{|Y|} \nabla_x \cdot \left( \sum_{i,j=1}^3 \int_{Y_f} \lambda_f \frac{\partial \omega_{f,i}(y)}{\partial y_i} \frac{\partial \rho_{f,0}}{\partial x_j} dy \right) = 0. \end{aligned}$$

Let

$$(4.17) \quad (\Lambda_f)_{ij} = \frac{1}{|Y|} \int_{Y_f} \lambda_f (|Y_f| \delta_{ij} + \frac{\partial \omega_{f,i}}{\partial y_j}) dy,$$

where  $\delta_{ij}$  is the Kronecker symbol, and set

$$(4.18) \quad q_{fm}(x, t) = \frac{1}{|Y|} \int_{\Gamma_{fm}} \lambda_m \nabla_y \gamma_0 \cdot \eta_m d\Gamma.$$

It follows from (3.4,1) that

$$(4.19) \quad \lambda_f (\nabla_x \rho_{f,1} + \nabla_y \rho_{f,2}) \cdot \eta_f = -\frac{1}{\beta} \lambda_m \nabla_y \gamma_0 \cdot \eta_m.$$

Now, observe that

$$\int_{\partial Y_f} p \cdot \eta_f d\Gamma = \int_{\Gamma_{ff}} p \cdot \eta_f d\Gamma + \int_{\Gamma_{fm}} p \cdot \eta_f d\Gamma = \int_{\Gamma_{fm}} p \cdot \eta_f d\Gamma = - \int_{\Gamma_{fm}} p \cdot \eta_m d\Gamma$$

for any  $Y$ -periodic  $p$ .

Since  $\rho_{f,1}$  and  $\rho_{f,2}$  are  $Y$ -periodic, it follows from the divergence theorem, the observation above, (4.19), and (4.18) that

$$(4.20) \quad \begin{aligned} \frac{1}{|Y|} \int_{Y_f} \nabla_y \cdot (\lambda_f (\nabla_x \rho_{f,1} + \nabla_y \rho_{f,2})) dy &= \frac{1}{|Y|} \int_{\partial Y_f} \lambda_f (\nabla_x \rho_{f,1} + \nabla_y \rho_{f,2}) \cdot \eta_f d\Gamma \\ &= \frac{1}{|Y|} \int_{\Gamma_{fm}} \lambda_f (\nabla_x \rho_{f,1} + \nabla_y \rho_{f,2}) \cdot \eta_f d\Gamma \\ &= -\frac{1}{\beta |Y|} \int_{\Gamma_{fm}} \lambda_m \nabla_y \gamma_0 \cdot \eta_m d\Gamma = -\frac{1}{\beta} q_{fm}(x, t). \end{aligned}$$

Thus, using (4.17) and (4.20), we can rewrite (4.16) in the form

$$(4.21) \quad \Phi_f \frac{\partial \rho_{f,0}}{\partial t}(x, t) - \nabla_x \cdot (\Lambda_f \nabla_x \rho_{f,0}) = -\frac{1}{\beta} q_{fm}(x, t),$$

where it is convenient, here and below, to set

$$\Phi_f = \varphi_f |Y_f| / |Y| \quad \text{and} \quad \Phi_m = \varphi_m |Y_m| / |Y|.$$

A similar construction can be given in order to determine the equation satisfied by  $\rho_{m,0}$ . Define auxiliary functions  $\omega_{m,k}$ ,  $k = 1, 2, 3$ , by replacing the subscript  $f$  everywhere it appears in (4.12) and (4.13) by the subscript  $m$ . Analogously, define an effective mobility tensor  $\Lambda_m$  by replacing  $f$  by  $m$  in (4.17). The argument above can be repeated to derive the following macroscopic equation for  $\rho_{m,0}$ :

$$(4.22) \quad \Phi_m \frac{\partial \rho_{m,0}}{\partial t} - \nabla_x \cdot (\Lambda_m \nabla_x \rho_{m,0}) = \frac{\alpha}{\beta} q_{fm}(x, t) - \kappa q_{mm}(x, t)$$

with

$$q_{mm}(x, t) = \frac{1}{|Y|} \int_{\Gamma_{mm}} \lambda_m \nabla_y \gamma_0 \cdot \eta_m \, d\Gamma.$$

The difference in the right hand side comes from the fact that, in calculations over  $\partial\Omega_m$ , we do not have to change the outer normal from  $\eta_f$  to  $\eta_m$  and that

$$\int_{\partial Y_m} p \cdot \eta_m \, d\Gamma = \int_{\Gamma_{mm}} p \cdot \eta_m \, d\Gamma + \int_{\Gamma_{fm}} p \cdot \eta_m \, d\Gamma,$$

where the integral over  $\Gamma_{mm}$  need not vanish for non-periodic  $p$ .

Physically, the total flux of  $\gamma_0$  calculated on the cell  $Y$  is comprised by  $q_{fm}$  and  $q_{mm}$ , the latter being a source in the matrix equation while  $q_{fm}$  splits to be a source in the fissure equation and a sink in the matrix equation.

The local problem for the density on the block at the point  $x$  results from (3.3,0), (3.7,0) (together with the consequence of (3.6,0) derived above):

$$(4.23) \quad \varphi_m \frac{\partial \gamma_0}{\partial t} - \nabla_y \cdot (\lambda_m \nabla_y \gamma_0) = 0, \quad y \in Y_m,$$

$$(4.24) \quad \gamma_0(x, y) = \kappa \rho_{m,0}(x), \quad y \in \Gamma_{mm},$$

$$(4.25) \quad \gamma(x, y) = \frac{1}{\beta} \rho_f(x) - \frac{\alpha}{\beta} \rho_m(x), \quad y \in \Gamma_{fm}.$$

## 5. The Limit Model.

In this section we summarize the limit two-scale *PFM*-model and complement it with suitable conditions on the external boundary  $\partial\Omega$  of  $\Omega$  for  $t \in I$ , as well as initial conditions for  $x \in \Omega$  and  $t = 0$ . Then, we discuss the model, including its relation to the *TFM*-model, and address its well-posedness.

We shall rewrite the equations derived in the previous section for  $\rho_{f,0}, \rho_{m,0}, \gamma_0$  (and drop the subscript zero). In summary, the model consists of the following system of equations,

holding for  $x \in \Omega$  and  $t \in I$ :

$$(5.1) \quad \Phi_f \frac{\partial \rho_f}{\partial t} - \nabla \cdot (\Lambda_f \nabla \rho_f) = -\frac{1}{\beta} q_{fm},$$

$$(5.2) \quad \Phi_m \frac{\partial \rho_m}{\partial t} - \nabla \cdot (\Lambda_m \nabla \rho_m) = \frac{\alpha}{\beta} q_{fm} - \kappa q_{mm},$$

$$(5.3) \quad q_{fm}(x, t) = \frac{1}{|Y|} \int_{\Gamma_{fm}} \lambda_m \nabla_y \gamma \cdot \eta_m d\Gamma,$$

$$(5.4) \quad q_{mm}(x, t) = \frac{1}{|Y|} \int_{\Gamma_{mm}} \lambda_m \nabla_y \gamma \cdot \eta_m d\Gamma,$$

$$(5.5) \quad \varphi_m \frac{\partial \gamma}{\partial t} - \nabla_y \cdot (\lambda_m \nabla_y \gamma) = 0, \quad y \in Y_m(x),$$

$$(5.6) \quad \gamma(x, y, t) = \kappa \rho_m(x, t), \quad y \in \Gamma_{mm},$$

$$(5.7) \quad \gamma(x, y, t) = \frac{1}{\beta} \rho_f(x, t) - \frac{\alpha}{\beta} \rho_m(x, t), \quad y \in \Gamma_{fm},$$

$$(5.8) \quad \rho_f(x, 0) = \rho_{f, \text{init}}(x), \quad t = 0,$$

$$(5.9) \quad \rho_m(x, 0) = \rho_{m, \text{init}}(x), \quad t = 0,$$

$$(5.10) \quad (\Lambda_f \nabla_x \rho_f(x, t)) \cdot \eta_\Omega = 0, \quad x \in \partial\Omega,$$

$$(5.11) \quad (\Lambda_m \nabla_x \rho_m(x, t)) \cdot \eta_\Omega = 0, \quad x \in \partial\Omega,$$

$$(5.12) \quad \gamma(x, y, 0) = \gamma_{\text{init}}(x, y), \quad y \in Y_m(x).$$

Note that the variables  $\rho_f$  and  $\rho_m$  depend on the global space variable  $x$  and the time  $t$ , but not on the local space variable  $y$ , and the coefficients  $\Phi_f$ ,  $\Phi_m$ ,  $\varphi_m$ ,  $\Lambda_f$ ,  $\Lambda_m$ , and  $\lambda_m$  are functions of  $x$ , while  $\gamma$  depends on all three variables; consequently,  $q_{fm}$  and  $q_{mm}$  depend on  $x$  and  $t$ . The initial values  $\rho_f(\cdot, 0)$  and  $\rho_m(\cdot, 0)$  also must depend on  $x$  alone; if the simulation begins from an undisturbed state, the values for  $\gamma(x, y, 0)$  should be consistent with the initial values for  $\rho_f$  and  $\rho_m$ ; i.e., (5.6) and (5.7) should be satisfied at the initial time, as well as later. Except under unusual circumstances, the boundary values for  $\rho_f$  and  $\rho_m$  should be equal.

The *PFM*-model (5.1)–(5.12) can be characterized as a two-sheeted model, as was the model for a *TFM* discussed in [2,6,11]. Here, we shall call the sheet on which the global equations are defined the *macrosheet*; on it reside the equations (5.1) and (5.2) and the associated boundary and initial conditions (5.8)–(5.11) for the two globally defined densities. The topology on this sheet is the standard Euclidean topology on  $\mathbf{R}^3$ . Since in the *TFM*-model there was only one global function, the density in the fissures or fractures, the macrosheet was called the fracture sheet in that model. The second sheet, which we call the *microsheet* in this model and was called the block sheet in the *TFM*-model, is more complicated. It consists of the product space  $\Omega \times Y$ , with the discrete topology on  $\Omega$  and the usual Euclidean topology on  $Y$ ; i.e., the blocks are topologically disconnected. Equation (5.5) defines the local density function,  $\gamma$ , subject to the boundary conditions (5.6) and (5.7), which impose consistency in the momentum between the macrosheet and the microsheet, and the initial condition (5.12). Consistency (conservation) in the mass both on the macrosheet and between the two sheets is expressed by the flux conditions (5.3)–(5.4).

For the particular choice  $\alpha = 0$ ,  $\beta = 1$ , and  $\kappa = 1$ , the source in the fissure equation is equal to  $-q_{fm}$ , the term which represents the interaction of the fissure system with the blocks. The source in the matrix equation equals  $-q_{mm}$ , which accounts for the balance of the two components of the “total density” in the matrix. Further, if  $\Gamma_{mm} \rightarrow \emptyset$ , then  $q_{mm} \rightarrow 0$ , so that the equation (5.2) is decoupled from the system, and its solution remains constant in time. Then, the total flux of the “local variable” goes through  $\Gamma_{fm}$ , and the system is reduced to the *TFM* model, as expected.

The system (5.1)–(5.12) can be considered as a pair of parabolic equations coupled through an integro–differential relation dependent upon the solution of an infinite system of parabolic equations in diagonal form. The essential feature distinguishing flow in fissured reservoirs from flow in unfissured media, that of the delay caused by the slower flow in the matrix blocks, is indicated by the integral terms in the equations (5.1)–(5.2). In the case of *TFM* obtained from the above system upon setting  $\alpha = 0$ ,  $\beta = 1$ , and  $\Gamma_{mm} = \emptyset$ , those integral terms can be represented as convolution integrals with kernels describing the fading memory effects, as discussed in [2,20,23,24]. The integro–differential system for the *TFM*–model reduces to the single equation

$$(5.13) \quad \Phi_f \frac{\partial \rho_f}{\partial t} - \nabla \cdot (\Lambda_f \nabla \rho_f) = -\tau * \frac{\partial \rho_f}{\partial t}, \quad x \in \Omega, \quad t \in I,$$

with a positive, monotone decreasing kernel  $\tau$ , singular at the origin. The discrete equivalent of *TFM*–model was first shown to be mathematically well–posed by Arbogast [2]. Later, Hornung and Showalter [20] and Pezżyńska [23,26], using techniques related to strongly positive kernels (see [15,22], offered other analyses as well as numerical techniques (see[27]) for *TFM* by studying the integro–differential equation (5.13).

To our knowledge, there exist no analytical nor numerical results for systems analogous to the *PFM*–model, described by (5.1)–(5.12). Below we show that the system is well–posed. A numerical method, so as to be generalizable to more complex flows in a *PFM*, will be developed elsewhere.

**Theorem 3.** *Let the assumptions of the Corollary 2 from §3 hold. Then, the system (5.1)–(5.12) is well–posed.*

*Proof.*

This is a consequence of the general result recalled in Theorem 1. We define appropriate spaces

$$\mathcal{H} = L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega \times Y_m),$$

with the scalar product

$$(u, v)_{\mathcal{H}} = \int_{\Omega} (\Phi_f u_1 v_1 + \Phi_m u_2 v_2 + \frac{1}{|Y|} \int_{Y_m} \phi_m u_3 v_3 dy) dx,$$

and

$$\mathcal{V} = \{v = (v_1, v_2, v_3) \in H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega, H^1(Y_m)), \\ v_1 = \alpha v_2 + \beta v_3 \text{ on } \Gamma_{fm}, \quad v_3 = \kappa v_2 \text{ on } \Gamma_{mm}\}.$$

Note that  $\mathcal{V}$  and  $\mathcal{H}$  satisfy the assumptions of Theorem 1. Next, we define the form

$$a(u, v) = \int_{\Omega} (\Lambda_f \nabla u_1 \cdot \nabla v_1 + \Lambda_m \nabla u_2 \cdot \nabla v_2 + \frac{1}{|Y|} \int_{Y_m} \lambda_m \nabla_y u_3 \cdot \nabla_y v_3 dy) dx$$

and then verify that the sum,  $a(\cdot, \cdot) + (\cdot, \cdot)_{\mathcal{H}}$  is  $\mathcal{V}$ -coercive. This involves showing that the “macroscopic” permeability tensors  $\Lambda_f$  and  $\Lambda_m$  are positive definite. This has been done, for example, in [6] for a *TFM*. Then, using a calculation to be shown below, we conclude that the variational form of the Cauchy problem (5.1)–(5.12) is (3.16) with  $u = (\rho_f, \rho_m, \gamma)$  and  $u_0 = (\rho_{f,init}, \rho_{m,init}, \gamma_{init})$ . Hence, the well-posedness of the problem follows.

It remains to check that (3.16) is our variational problem. First, take  $v = (v_1, v_2, v_3) \in \mathcal{V}$ , multiply (5.5) by  $v_3$ , integrate the equation over  $Y_m$  and  $\Omega$ , and apply Green’s theorem to get

$$\begin{aligned} & \int_{\Omega} \int_{Y_m} (\phi_m \gamma' v_3 + \lambda_m \nabla_y \gamma \nabla_y v_3) dy dx = \int_{\Omega} \left( \int_{\Gamma_{fm}} \lambda_m \nabla_y \gamma \cdot \eta_m v_3 d\Gamma + \int_{\Gamma_{mm}} \lambda_m \nabla_y \gamma \cdot \eta_m v_3 d\Gamma \right) dx \\ & = \int_{\Omega} \left( \frac{1}{\beta} \int_{\Gamma_{fm}} \lambda_m \nabla_y \gamma \cdot \eta_m v_1 d\Gamma - \frac{\alpha}{\beta} \int_{\Gamma_{fm}} \lambda_m \nabla_y \gamma \cdot \eta_m v_2 d\Gamma + \kappa \int_{\Gamma_{mm}} \lambda_m \nabla_y \gamma \cdot \eta_m v_2 d\Gamma \right) dx, \end{aligned}$$

where we have noted that  $v_3 = (v_1 - \alpha v_2)/\beta$  on  $\Gamma_{fm}$  and  $v_3 = \kappa v_2$  on  $\Gamma_{mm}$ . Since  $v_1$  and  $v_2$  are independent of  $y$ , (5.3) and (5.4) imply that the right hand side of the last identity is equal to

$$\int_{\Omega} \left( |Y| q_{fm} \left( \frac{1}{\beta} v_1 - \frac{\alpha}{\beta} v_2 \right) + \kappa v_2 |Y| q_{mm} \right) dx.$$

Now, multiply (5.1) and (5.2) by  $v_1$  and  $v_2$ , respectively, integrate over  $\Omega$ , sum the two equations, and apply the result above, scaled by the factor  $|Y|^{-1}$ , to get

$$\begin{aligned} & \int_{\Omega} \left( \Phi_f \rho'_f v_1 + \Lambda_f \nabla \rho_f \cdot \nabla v_1 + \Phi_m \rho'_m v_2 + \Lambda_m \nabla \rho_m \cdot \nabla v_2 + \frac{1}{|Y|} \int_{Y_m} (\phi_m \gamma' v_3 + \lambda \nabla_y \gamma \cdot \nabla_y v_3 dy) \right) \\ & = \int_{\Omega} \left( -\frac{1}{\beta} q_{fm} v_1 + \frac{\alpha}{\beta} q_{fm} v_2 - \kappa q_{mm} v_2 \right) dx + \frac{1}{|Y|} \int_{\Omega} \left( |Y| q_{fm} \left( v_1 \frac{1}{\beta} - v_2 \frac{\alpha}{\beta} \right) + v_2 \kappa |Y| q_{mm} \right) dx. \end{aligned}$$

Since the terms on the right hand side cancel, the last equation is the desired relation (5.14).

## 6. Concluding Remarks.

The essential objective above was the derivation by means of the homogenization method of the indicated *PFM* model with dual-permeability. In the geological setting of rocks and fissures in which the diffusion parameters of the two media are so extremely different, the additional global flow through the matrix can frequently be ignored; that is, the dual-porosity model of *TFM* is sufficient. In the more general situation of porous media with two components, the other extreme would consist of a pair of component media whose parameters are of similar order in magnitude; then the flow in each of the components is a substantial contribution. The model of *PFM* with dual-permeability as developed here contains both of these extreme cases.

Preliminary numerical experiments were undertaken to investigate the *qualitative* differences in the behavior of solutions to three different models of flow in a fissured medium: the classical *single porosity model*, the distributed microstructure or two-scale model of a *totally fissured medium*, and the corresponding model that was introduced above for a *partially fissured medium*. A sink was introduced in the fissure system, active only for an interval of time, after which the system began to move to a new equilibrium. The values of the responding densities as functions of time at a nearby point were recorded. We chose

$\alpha = 0$  and  $\kappa = 1$  in the *partially fissured medium* in order to maximize the separation of  $\rho_f$  and  $\rho_m$  by the blocks.

The first observation was that in all models the pressure dropped downward from the initial equilibrium value until the time at which the sink was deactivated. It was seen that, of these three models, the fissure density of the *partially fissured medium* had the largest initial response rate, and that of the *totally fissured medium* had the largest total value in response to the sink. The drop in the fissure density in the *totally fissured medium* was significantly more pronounced than the corresponding drop of density in the *single porosity model*. This is a reflection of the fact that the fast flow in the fissure system of the *totally fissured medium* provides a more efficient way to deliver fluid to the sink than the single porosity system. Moreover, the fissure system in the *partially fissured medium* initially responded even more dramatically, but the fissure density there quickly leveled off at a somewhat higher level. This apparent stability results from the weaker coupling to the blocks and the corresponding slower drop of block pressure in a *partially fissured medium* than in a *totally fissured medium*. The fissures are being supplied by these blocks during this period. The blocks in a *totally fissured medium* are coupled exclusively to the fissures, so they respond more quickly to the drop in fissure density than those of the *partially fissured medium* model where they are coupled to both the fissures and to the slower matrix flow. The wide variations in block density and the correspondingly slow flow within the blocks causes a substantial delay in the time that the system needs to stabilize. It was precisely this last effect, observed long ago, that provided the motivation for the development of such coupled models of flow through fractured reservoirs. Finally, the matrix density in the *partially fissured medium* dropped very slowly during the period in which the sink was active. It is this flow in the matrix which represents the *connectivity* between the blocks. It drains slowly into the system of blocks, and this contributes to the relative stabilization of the block system in the *partially fissured medium* in comparison with that of the *totally fissured medium*.

We note that the *totally fissured medium* is obtained formally by setting  $\kappa = 0$ . Moreover, by adjusting the parameters  $\alpha$ ,  $\beta$ , and  $\kappa$ , one can calibrate the model to obtain a very wide variety of response curves. If we are given a real multi-porosity reservoir, then, as previously referenced evidence in the literature shows, and our current simple results confirmed, it is important to construct a fully coupled model describing the effects of the inhomogeneous nature of the reservoir. Such two-scale models give us then the information about the local (microscopic) distribution of the fluid. However, this information can have a significantly different character in the dual-permeability case of interconnected blocks from the one for isolated blocks. In reality we can expect that some part of the *global* flow will occur within the interconnected system of the blocks, and this component of the total flow is described by this model for a *partially fissured medium*.

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