Flow in Porous Media

A porous medium Ω in $\mathbb{R}^3$ is filled with a fluid, and this fluid is driven from locations of higher pressure to those of lower pressure. In order to model this situation, let $p(x,t)$ denote the *pressure* of fluid at the point $x \in \Omega$ and time $t > 0$, and denote the corresponding *density* of fluid by $\rho(x,t)$. The quantity of fluid in each small element of volume V is $\int_V \phi(x)\rho(x,t)\,dx$, where the *porosity* $\phi(x)$ of the medium at the point x is the volume fraction of the medium that is accessible to the fluid. The *fluid flux* is the mass flow rate $\mathbf{q}(x,t)$, so the rate at which fluid moves across a surface element S with normal $\mathbf{n}$ is given by $\int_S \mathbf{q}(x,t)\cdot\mathbf{n}\,dS$. Then the *conservation of fluid mass* takes the integral form

$$\frac{\partial}{\partial t}\int_B \phi(x)\rho(x,t)\,dx + \int_{\partial B} \mathbf{q}\cdot\mathbf{n}\,dS = \int_B F(x,t)\,dx\ , \qquad B \subset \Omega\ ,$$

in which $F(x,t)$ denotes any volume distributed *source density*. When the flux and density are differentiable, we can write this *conservation law* in the differential form

$$\frac{\partial}{\partial t}\phi(x)\rho(x,t) + \boldsymbol{\nabla}\cdot\mathbf{q}(x,t) = F(x,t)\ , \qquad x \in \Omega\ . \tag{0.1a}$$

The fluid flux is given by $\mathbf{q}(x,t) = \rho(x,t)\mathbf{v}(x,t)$, where the *volumetric flow rate* of fluid or *Darcy velocity* $\mathbf{v}(x,t)$ depends on the pressure gradient through the constitutive relationship

$$\mathbf{v}(x,t) = -\frac{k(x)}{\mu}\left(\boldsymbol{\nabla}p(x,t) - \rho(x,t)\mathbf{g}(x)\right)\ . \tag{0.1b}$$

This is *Darcy's law* for an *isotropic* medium. The constant μ is the *viscosity* of the fluid, and this equation defines the *permeability* $k(x)$ of the porous medium. The value of k is a measure of the volume-averaged velocity of fluid flow through the medium generated by a given pressure gradient. That is, μ/k is the *resistance* of the medium to flow. The vector $\mathbf{g}$ is the gravitational force, usually taken as $-g\mathbf{e}_3$. Finally, the type of fluid considered is described by an *equation of state*, $\rho = s(p)$. The function $s(\cdot)$ which relates the pressure and density is monotone, in fact, it is usually chosen to be strictly increasing. By substituting the appropriate quantities above we obtain the nonlinear parabolic equation

$$\frac{\partial}{\partial t}\phi(x)\rho(x,t) - \boldsymbol{\nabla}\cdot\frac{k(x)}{\mu}\big(\rho(x,t)\boldsymbol{\nabla}p(x,t) - \rho^2(x,t)\mathbf{g}(x)\big) = F(x,t)\ ,\ x\in\Omega,\ t>0\ . \tag{0.2}$$

The simplest situation for the description of *fluid flow* is that of a *slightly compressible* fluid. Here the equation of state has the form $s(p) = \rho_0 \exp c_0\, p$ where $c_0 > 0$ is the *compressibility* of the fluid. Thus, the compressibility is constant: $c_0 = \frac{1}{\rho}\frac{d\rho}{dp}$. Then we approximate $\rho^2 \approx \rho_0^2 + 2\rho_0(\rho - \rho_0) = \rho_0(2\rho - \rho_0)$ so that (0.2) simplifies to the linear parabolic equation for density

$$\frac{\partial}{\partial t}\phi(x)\rho(x,t) - \boldsymbol{\nabla}\cdot\frac{k(x)}{c_0\mu}\left(\boldsymbol{\nabla}\rho(x,t) - \mathbf{g}(x)(\rho_0(2\rho-\rho_0))\right) = F(x,t)\ . \tag{0.3}$$

Alternatively, if we linearize the state equation by $\rho \approx \rho_0(1 + c_0 p)$ and $\rho \approx \rho_0$, we obtain the linear parabolic equation for pressure

$$\frac{\partial}{\partial t}\phi(x)\rho_0 c_0 p(x,t) - \boldsymbol{\nabla} \cdot \frac{k(x)}{\mu} \big(\rho_0 \boldsymbol{\nabla} p(x,t) - \mathbf{g}(x)(\rho_0^2 + 2\rho_0 c_0 p(x,t))\big) = F(x,t) \; . \tag{0.4}$$