THE STOKES SYSTEM

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Contents

1. Stokes' System

The motion of a (possibly compressible) homogeneous fluid is described by its density $\rho(x,t)$, pressure $p(x,t)$ and velocity $\mathbf{v}(x,t)$. Assume that the fluid is *barotropic*, i.e., the density and pressure are related by a *state equation*

$$\rho = s(p)$$

in which the constitutive function $s(\cdot)$ is non-decreasing and characterizes the type of fluid. The *conservation of mass* of fluid is expressed by

$$\dot{\rho} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0\,,$$

and the *conservation of momentum* has the form

$$\rho \dot{\mathbf{v}} - (\lambda_1 + \mu_1)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{v}) - \mu_1 \Delta \mathbf{v} + \boldsymbol{\nabla} p = \rho\, \mathbf{f}(x) \text{ in } \Omega.$$

Here $\mathbf{f}(x)$ is the mass-distributed force density over Ω. These three nonlinear equations comprise the system for a general *compressible fluid.*

In order to obtain a linear model to approximate the solutions of this system, we consider small oscillations about a *rest state* at which $\mathbf{v} = \mathbf{0}$ and, hence, the pair $\rho_0(x)$, $p_0(x)$ satisfies

$$\boldsymbol{\nabla} p_0 = \rho_0\, \mathbf{f}(x)\,, \quad \rho_0 = s(p_0)\,.$$

Let $c_0(x) \equiv \frac{1}{\rho_0(x)} \frac{\partial \rho}{\partial p}(p_0(x))$ denote the *compressibility* of the fluid at the rest state. Using the chain rule with the state equation yields

$$\boldsymbol{\nabla} \rho_0 = \frac{\partial \rho}{\partial p} \boldsymbol{\nabla} p_0 = \rho_0 c_0(x) \rho_0 \mathbf{f}(x)\,.$$

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Introduce a small parameter $\varepsilon > 0$ to characterize the size of the oscillations and the deviation from the rest state, and consider the corresponding asymptotic expansions

$$\begin{aligned}\rho &= \rho_0(x) + \varepsilon\rho^1(x,t) + \mathcal{O}(\varepsilon^2)\,,\\ p &= p_0(x) + \varepsilon p^1(x,t) + \mathcal{O}(\varepsilon^2)\,,\\ \mathbf{v} &= \varepsilon\mathbf{v}^1(x,t) + \mathcal{O}(\varepsilon^2)\,.\end{aligned}$$

Again from the chain rule we obtain

$$\rho^1(x,t) = c_0(x)\rho_0(x)p^1(x,t)\,.$$

Conservation of mass implies to first order in ε that

$$\dot\rho^1 + \boldsymbol{\nabla}\cdot(\rho_0\mathbf{v}^1) = 0\,.$$

From conservation of momentum we get

$$\varepsilon(\rho_0+\varepsilon\rho^1)\dot{\mathbf{v}}^1 - \varepsilon(\lambda_1+\mu_1)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}^1) - \varepsilon\mu_1\Delta\mathbf{v}^1 + \boldsymbol{\nabla}(p_0+\varepsilon p^1) = (\rho_0+\varepsilon\rho^1)\mathbf{f}(x)\,,$$

and then the definitions of rest state and compressibility give the linear system

$$\begin{gathered}\rho_0(x)\dot{\mathbf{v}}^1 - (\lambda_1+\mu_1)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}^1) - \mu_1\Delta\mathbf{v}^1 + \boldsymbol{\nabla}p^1 = c_0(x)\rho_0(x)\mathbf{f}(x)\,p^1\,,\\ c_0(x)\dot p^1 + \boldsymbol{\nabla}\cdot\mathbf{v}^1 + c_0(x)\rho_0(x)\mathbf{f}(x)\cdot\mathbf{v}^1 = 0\,,\\ \mathbf{v}^1 = \mathbf{0} \text{ on } \Gamma_0\,,\ \lambda_1(\boldsymbol{\nabla}\cdot\mathbf{v}^1)\mathbf{n} + 2\mu_1\,\varepsilon(\mathbf{v}^1,\mathbf{n}) - p\,\mathbf{n} = 0 \text{ on } \Gamma_1\end{gathered}$$

for small variations of a *compressible fluid.* Here the two sets $\Gamma_0,\ \Gamma_1$ comprise a partition of the boundary $\Gamma = \partial\Omega$. In the incompressible case, $c_0(x) = 0$, we obtain the *Stokes' system*,

$$\begin{gathered}\rho_0(x)\dot{\mathbf{v}}^1 - \mu_1\Delta\mathbf{v}^1 + \boldsymbol{\nabla}p^1 = \mathbf{0}\,,\qquad \boldsymbol{\nabla}\cdot\mathbf{v}^1 = 0\,,\\ \mathbf{v}^1 = \mathbf{0} \text{ on } \Gamma_0\,,\ 2\mu_1\,\varepsilon(\mathbf{v}^1,\mathbf{n}) - p\,\mathbf{n} = 0 \text{ on } \Gamma_1\,.\end{gathered}$$

Navier-Stokes System. The *material derivative* of velocity has been approximated here by the acceleration. For the calculation of the acceleration of a fluid element, the displacement of that element along with the points must be considered. The momentum of the small subdomain $B \subset \Omega$ travelling with the fluid is $\int_B \rho\mathbf{v}(x+\mathbf{u}(x,t),t)\,dx$, and its derivative is given by the Chain rule as

$$\int_B \rho\left(\frac{\partial\mathbf{v}(x+\mathbf{u}(x,t),t)}{\partial t} + \partial_j\mathbf{v}(x+\mathbf{u}(x,t),t)\,v_j(x+\mathbf{u}(x,t),t)\right)dx\,.$$

Thus, the momentum equation for the fluid includes the additional term $(\mathbf{v}\cdot\nabla)\mathbf{v} = v_j\partial_j\mathbf{v}$, and the corresponding system is the *Navier-Stokes system*

$$\begin{gathered}\rho\dot{\mathbf{v}} - \mu_1\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \boldsymbol{\nabla}p = \mathbf{f}\,,\quad \boldsymbol{\nabla}\cdot\mathbf{v} = 0 \text{ in } \Omega\,,\\ \mathbf{v} = 0 \text{ on } \Gamma_0\,,\quad -p\,\mathbf{n} + 2\mu_1\,\varepsilon(\mathbf{v},\mathbf{n}) = \mathbf{g} \text{ on } \Gamma_1\,,\end{gathered}$$

for a viscous incompressible fluid. Note that the quadratic nonlinearity arises from the *geometry* of the motion, and it is *not* based on any independent assumptions.

2. The Weak Solution of the Stokes System

The motion of an incompressible homogeneous fluid is described by its pressure $p(x,t)$ and velocity $\mathbf{v}(x,t)$. The (evolutionary) *Stokes system* is to find such a pair of functions on the smoothly bounded region Ω in $\mathbb{R}^n$ for $t > 0$ which satisfy the initial-boundary-value problem

$$\begin{gathered} \rho_0(x)\dot{\mathbf{v}} - \mu\Delta\mathbf{v} + \boldsymbol{\nabla} p = \mathbf{f}\,, \qquad \boldsymbol{\nabla}\cdot\mathbf{v} = 0 \text{ in } \Omega\times\mathbb{R}^+\,, \\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma\times\mathbb{R}^+\,, \\ \mathbf{v}(0) = \mathbf{v}_0 \text{ in } \Omega\,, \end{gathered}$$

where $\Gamma = \partial\Omega$ is the boundary of Ω. If we separate variables, i.e., look for a solution in the form $\mathbf{v}(x,t) = e^{\lambda t}\mathbf{u}(x)$ for some number λ, we are led to the *stationary Stokes system*

$$\rho_0(x)\lambda\mathbf{u} - \mu\Delta\mathbf{u} + \boldsymbol{\nabla} p = \mathbf{f}\,, \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0 \text{ in } \Omega\,, \tag{2.1a}$$

$$\mathbf{u} = \mathbf{0} \text{ on } \Gamma\,, \tag{2.1b}$$

for the pair $\mathbf{u}(x)$, $p(x)$. We focus on this system, but the results apply as well to the evolutionary system.

Remark 2.1. *For a pair of functions* $\mathbf{u}\in\mathbf{H}^1(\Omega)$, $q\in H^1(\Omega)$, *we have*

$$\int_\Omega \boldsymbol{\nabla}\cdot\mathbf{u}\, q\, dx = -\int_\Omega \mathbf{u}\cdot\boldsymbol{\nabla} q\, dx + \int_\Gamma \mathbf{u}\cdot\mathbf{n}\, q\, ds$$

Formally, *this shows that* $\mathrm{Ker}(\boldsymbol{\nabla}\cdot) = \mathrm{Rg}(\boldsymbol{\nabla})^\perp$ *so we expect* $\mathrm{Rg}(\boldsymbol{\nabla}) = \mathrm{Ker}(\boldsymbol{\nabla}\cdot)^\perp$ *up to closure in appropriate spaces. Eventually, we will need to construct carefully realizations of the gradient and divergence operators.*

Now, if the pair $\mathbf{u}(x)$, $p(x)$ is a solution to the stationary Stokes system (2.1), then for every $\mathbf{w}\in\mathbf{H}^1(\Omega)$ we have

$$\begin{aligned} \lambda\int_\Omega \rho_0(x)\mathbf{u}\cdot\mathbf{w}\, dx + \mu\int_\Omega \boldsymbol{\nabla} u_i\cdot\boldsymbol{\nabla} w_i\, dx - \int_\Gamma (\boldsymbol{\nabla} u_i\cdot\mathbf{n}) w_i\, ds \\ - \int_\Omega (\boldsymbol{\nabla}\cdot\mathbf{w}) p\, dx + \int_\Gamma (\mathbf{w}\cdot\mathbf{n}) p\, ds = \int_\Omega \mathbf{f}\cdot\mathbf{w}\, dx\,. \end{aligned}$$

We define the space $\mathbf{V}_0 = \{\mathbf{w}\in\mathbf{H}^1(\Omega):\ \boldsymbol{\nabla}\cdot\mathbf{w} = 0 \text{ in } \Omega,\ \mathbf{w} = \mathbf{0} \text{ on } \Gamma\}$. The weak form of the problem is now to find

$$\mathbf{u}\in\mathbf{V}_0 :\ \lambda\int_\Omega \rho_0(x)\mathbf{u}\cdot\mathbf{w}\, dx + \mu\int_\Omega \boldsymbol{\nabla} u_i\cdot\boldsymbol{\nabla} w_i\, dx = \int_\Omega \mathbf{f}\cdot\mathbf{w}\, dx \text{ for all } \mathbf{w}\in\mathbf{V}_0\,. \tag{2.2}$$

Define a continuous bilinear form $a(\cdot,\cdot)$ on $\mathbf{V}_0$ by

$$a(\mathbf{u},\mathbf{w}) = \int_\Omega (\lambda\rho_0(x)\mathbf{u}\cdot\mathbf{w} + \mu\boldsymbol{\nabla} u_i\cdot\boldsymbol{\nabla} w_i)\, dx\,, \quad \mathbf{u},\ \mathbf{w}\in\mathbf{V}_0,$$

and the linear functional $f(\cdot)$ on $\mathbf{V}_0$ by

$$f(\mathbf{w}) = \int_\Omega \mathbf{f}\cdot\mathbf{w}\, dx\,, \quad \mathbf{w}\in\mathbf{V}_0,$$

where $\mathbf{f} \in \mathbf{L}^2(\Omega)$ is given. Note that the principle part of $a(\cdot,\cdot)$ is a double sum

$$\boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} w_i = \sum_{i,j=1}^{n} \frac{\partial u_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}$$

which contains *all* first-order derivatives. Thus, the space $\mathbf{V}_0$ is appropriate, and we have the estimate

$$a(\mathbf{w},\mathbf{w}) \geq \int_\Omega \sum_{i,j=1}^{n} \left(\frac{\partial w_i}{\partial x_j}\right)^2 dx\,.$$

This gives the *ellipticity* condition

$$a(\mathbf{w},\mathbf{w}) \geq c_0\|\mathbf{w}\|^2, \qquad \text{for all } \mathbf{w} \in \mathbf{V}\,, \tag{2.3}$$

for some constant $c_0 > 0$. We noted already that $a(\cdot,\cdot)$ is continuous in both variables with respect to the $\mathbf{V}_0$ norm, and likewise the functional $f(\cdot)$ is continuous. We have the following result immediately.

Theorem 2.1. *There is exactly one weak solution of the stationary Stokes system*

$$\mathbf{u} \in \mathbf{V}_0 : \quad a(\mathbf{u},\mathbf{w}) = f(\mathbf{w}) \textit{ for all } \mathbf{w} \in \mathbf{V}_0\,. \tag{2.4}$$

There remains the issue of the sense in which the weak solution of (2.2) satisfies the strong formulation (2.1).

3. The Strong Solution

With the space $\mathbf{V}_0 = \{\mathbf{w} \in \mathbf{H}^1(\Omega) : \ \boldsymbol{\nabla}\cdot\mathbf{w} = 0 \text{ in } \Omega,\ \mathbf{w} = \mathbf{0} \text{ on } \Gamma\}$, the weak form of the stationary Stokes problem is given by

$$\mathbf{u} \in \mathbf{V}_0 : \ \lambda \int_\Omega \rho_0(x)\mathbf{u}\cdot\mathbf{w}\,dx + \mu\int_\Omega \boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} w_i\,dx = \int_\Omega \mathbf{f}\cdot\mathbf{w}\,dx \text{ for all } \mathbf{w} \in \mathbf{V}_0\,.$$

We need to characterize the annihilator of the space $\mathbf{V}_0$, the kernel of the divergence operator in $\mathbf{H}_0^1(\Omega)$. This will be used to show the equivalence of the weak form of the Stokes equation with the strong formulation, and it is related to the characterization of the range of the *gradient* operator in $\mathbf{H}^{-1}(\Omega)$.

For background, we begin with the following profound result on the annihilator of

$$\mathbb{V} \equiv \{\mathbf{v} \in C_0^\infty(\Omega) = \mathcal{D}(\Omega) : \ \boldsymbol{\nabla}\cdot\mathbf{v} = 0\}.$$

Theorem 3.1 (DeRham)**.** *Let* Ω *be a domain in* $\mathbb{R}^n$*. Then* $\mathbf{f} = \boldsymbol{\nabla} p$ *in* $\mathcal{D}'(\Omega)$ *if and only if* $\mathbf{f}(\mathbf{v}) = 0$ *for all* $\mathbf{v} \in \mathbb{V}$*.*

A related result is the following.

Theorem 3.2. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with Lipschitz boundary. If* $p \in \mathcal{D}'(\Omega)$ *satisfies* $\boldsymbol{\nabla} p \in \mathbf{H}^{-1}(\Omega)$*, then* $p \in L^2(\Omega)$*, and we have the estimate*

$$\|p\|_{L^2(\Omega)/\mathbb{R}} \leq C_\Omega \|\boldsymbol{\nabla} p\|_{\mathbf{H}^{-1}(\Omega)}\,. \tag{3.5}$$

Theorem 3.2 was proved by Magenes & Stampacchia if Ω has a C^1 boundary and by Neĉas for the case of a Lipschitz boundary. Related results were obtained by Deny & Lions.

Recall that the space $L^2(\Omega)/\mathbb{R}$ is just the quotient of $L^2(\Omega)$ with the constant functions. An estimate equivalent to (3.5) is

$$\|p\|_{L^2(\Omega)} \leq C_\Omega \left(| \int_\Omega p(x)\, dx | + \|\boldsymbol{\nabla} p\|_{\mathbf{H}^{-1}(\Omega)} \right) . \tag{3.6}$$

Furthermore, these estimates are equivalent to the statement that the gradient operator $\boldsymbol{\nabla} :\ L^2(\Omega) \to \mathbf{H}^{-1}(\Omega)$ has *closed range.*

Now we define the space $\mathbf{V}_1$ to be the *closure* in $\mathbf{H}_0^1(\Omega)$ of the space $\mathbb{V}$.

Theorem 3.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. Then $\mathbf{V}_1 = \mathbf{V}_0$.*

Proof. First we check that $\mathbf{V}_1 \subset \mathbf{V}_0$: if $\mathbf{u} = \lim \mathbf{u}_m$ in $\mathbf{H}_0^1(\Omega)$, with $\mathbf{u}_m \in \mathbb{V}$, then we have $\boldsymbol{\nabla} \cdot \mathbf{u}_m \to \boldsymbol{\nabla} \cdot \mathbf{u} = 0$, so $\mathbf{u} \in \mathbf{V}_0$.

Next we show that $\mathbf{V}_1 = \mathbf{V}_0$. Let $T \in \mathbf{V}_0'$ and assume further that the restriction to $\mathbf{V}_1$ vanishes. Since $\mathbf{V}_0$ is a closed subspace of $\mathbf{H}_0^1(\Omega) = H_0^1(\Omega)^n$, T has a continuous linear extension to all of $\mathbf{H}_0^1(\Omega)$, namely, $\mathbf{t} \in \mathbf{H}^{-1}(\Omega)$. But $\mathbf{t}(\mathbf{v}) = 0$ for all $\mathbf{v} \in \mathbb{V}$, so from Theorem 3.1 and Theorem 3.2 we obtain $\mathbf{t} = \boldsymbol{\nabla} p$ for some $p \in L^2(\Omega)$. Therefore, $T(\mathbf{v}) = \mathbf{t}(\mathbf{v}) = -(p, \boldsymbol{\nabla} \cdot \mathbf{v}) = 0$ for all $\mathbf{v} \in \mathbf{V}_0$, and this shows that $T = 0$. Since this is true for every $T \in \mathbf{V}_0'$ for which $T|_{\mathbf{V}_1} = 0$, we have $\mathbf{V}_1 = \mathbf{V}_0$ as desired.

□

Corollary 3.4. *The annihilator $\mathbf{V}_0^\perp$ is $\{\boldsymbol{\nabla} p :\ p \in L^2(\Omega)\}$ in $\mathbf{H}^{-1}(\Omega)$, and $\mathbb{V}$ is dense in V_0.*

Corollary 3.5. *If $\mathbf{u}$ is a solution of the Stokes problem, then $\mathbf{u} \in H^1(\Omega)$ and there is a $p \in L^2(\Omega)$ for which*

$$\rho_0(x)\lambda \mathbf{u} - \mu \Delta \mathbf{u} + \boldsymbol{\nabla} p = \mathbf{f}\,, \qquad \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \text{ in } \Omega\,,$$
$$\mathbf{u} = \mathbf{0} \text{ on } \Gamma\,.$$

Finally, we show that from Theorem 3.2 we can obtain a restricted form of Theorem 3.1 that is sufficient for our purposes above. Thus, our Theorem 3.3 is independent of Theorem 3.1.

Theorem 3.6 (Luc Tartar). *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. Then $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ and $\mathbf{f}(\mathbf{v}) = 0$ for all $\mathbf{v} \in \mathbb{V}$ if and only if $\mathbf{f} = \boldsymbol{\nabla} p,\ p \in L^2(\Omega)$.*

Proof. Let there be an increasing sequence of open sets for which $\bar{\Omega_m} \subset \Omega_{m+1}$ and $\cup \Omega_m = \Omega$. For each $m \geq 1$ the *gradient operator* $\boldsymbol{\nabla}_m \in \mathcal{L}(L^2(\Omega), \mathbf{H}^{-1}(\Omega))$ has closed range. Thus, its adjoint $\boldsymbol{\nabla}_m' \in \mathcal{L}(\mathbf{H}_0^1(\Omega), L^2(\Omega))$ is *divergence*, and its annihilator in $\mathbf{H}^{-1}(\Omega)$ is $\mathrm{Ker}(\boldsymbol{\nabla}'_m)^\perp = \mathrm{Rg}(\boldsymbol{\nabla}_m)$.

Let $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ and $\mathbf{f}(\mathbf{v}) = 0$ for all $\mathbf{v} \in \mathbb{V}$. Then for each $\mathbf{u} \in \mathrm{Ker}(\boldsymbol{\nabla}'_m)$, the zero-extension to Ω is denoted by $\tilde{\mathbf{u}}$, and it has compact support in Ω. By a regularization argument, $\tilde{\mathbf{u}}$ is the limit in $\mathbf{H}_0^1(\Omega)$ of functions from $\mathbb{V}$, so we have $\tilde{\mathbf{u}} \in V_0$ and $\mathbf{f}(\tilde{\mathbf{u}}) = 0$. Hence, the restriction of $\mathbf{f}$ to Ω_m is in $\mathrm{Ker}(\boldsymbol{\nabla}'_m)^\perp = \mathrm{Rg}(\boldsymbol{\nabla}_m)$, so $\mathbf{f} = \boldsymbol{\nabla} p_m$ on Ω_m for a

$p_m \in L^2(\Omega_m)$. Since $\Omega_m \subset \Omega_{m+1}$ we have $p_{m+1} - p_m = c$ on Ω_m, and we can modify p_{m+1} by a constant so that $p_{m+1} = p_m$ on Ω_m. Thus, $\mathbf{f} = \boldsymbol{\nabla} p$ with $p \in L^2_{loc}(\Omega)$, and then by Theorem 3.2 we obtain $p \in L^2(\Omega)$. ☐

4. The Normal Trace

Recall that the *trace map* $\gamma : H^1(\Omega) \to L^2(\Gamma)$ is the *restriction* to the boundary Γ of each function u in $H^1(\Omega)$. The *kernel* of this continuous and linear map is the subspace $H^1_0(\Omega)$ of those functions which vanish on the boundary, and it is characterized as the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. We denote the *range* of γ by $\mathbb{B}$, and it follows that this is a complete space with the *quotient norm*

$$\|\psi\|_{\mathbb{B}} \equiv inf\{\|w\|_{H^1} : \ w \in H^1(\Omega) \text{ with } \gamma(w) = \psi\}.$$

Note that $\mathbb{B}$ is the space of all *boundary values* of functions in $H^1(\Omega)$, and we can identify $L^2(\Gamma)$ with its dual and obtain $\mathbb{B} \subset L^2(\Gamma) \subset \mathbb{B}'$. The corresponding quotient map of $H^1(\Omega)/H^1_0(\Omega)$ onto $\mathbb{B}$ is an isomorphism, and so also is the dual map γ' of $\mathbb{B}'$ onto $H^1_0(\Omega)^\perp$, the indicated *annihilator* of $H^1_0(\Omega)$ in the dual $H^1(\Omega)'$.

Now we define the space $\mathbf{W}(\Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega) : \ \boldsymbol{\nabla} \cdot \mathbf{u} \in L^2(\Omega)\}$ with the graph norm

$$\|\mathbf{u}\|^2_{\mathbf{W}(\Omega)} = \|\mathbf{u}\|^2_{\mathbf{L}^2} + \|\boldsymbol{\nabla} \cdot \mathbf{u}\|^2_{L^2}$$

It follows easily that this space is complete, and it can be shown that the smooth functions are dense in this space. Furthermore $\mathbf{W}(\Omega)$ is much larger than the space $\mathbf{H}^1(\Omega) = H^1(\Omega)^n$ of vector-valued functions, and the injection $\mathbf{H}^1(\Omega) \to \mathbf{W}(\Omega)$ is continuous. The trace map has a natural meaning on $\mathbf{H}^1(\Omega)$, namely, componentwise, but we shall need to extend it somewhat to all of $\mathbf{W}(\Omega)$. This is the point of the following result.

Theorem 4.1. *There exists a unique map γ_n continuous and linear from $\mathbf{W}(\Omega)$ onto $\mathbb{B}'$ such that, for all functions $\mathbf{u} \in \mathbf{H}^1(\Omega)$, $\gamma_n(\mathbf{u}) = \gamma(\mathbf{u}) \cdot \mathbf{n}$ is the* normal trace*, i.e., the generalized restriction to the boundary Γ of $\mathbf{u} \cdot \mathbf{n}$, and this map satisfies the extended* Stokes formula

$$(\mathbf{u}, \boldsymbol{\nabla} w)_{\mathbf{L}^2} + (\boldsymbol{\nabla} \cdot \mathbf{u}, w)_{L^2} = \gamma_n(\mathbf{u})(\gamma(w)), \qquad \mathbf{u} \in \mathbf{W}(\Omega),\ w \in H^1(\Omega). \tag{4.7}$$

Proof. Let $\mathbf{u} \in \mathbf{W}(\Omega)$ and define $f \in H^1(\Omega)'$ by

$$f(w) = \int_\Omega (\boldsymbol{\nabla} \cdot \mathbf{u} w + \mathbf{u} \cdot \boldsymbol{\nabla} w)\, dx\,, \qquad w \in H^1(\Omega).$$

For each $w \in H^1_0(\Omega)$ we have $f(w) = 0$. (This follows first for $w \in C_0^\infty$ and then by continuity for all $w \in H^1_0(\Omega)$.) That is, $f \in H^1_0(\Omega)^\perp$, so from above we see there is a $g \in \mathbb{B}'$ for which $f = \gamma'(g) = g \circ \gamma$. It follows that for each $\psi \in \mathbb{B}$ and for each $w \in H^1(\Omega)$ with $\gamma(w) = \psi$, the value of

$$g(\psi) = \int_\Omega (\boldsymbol{\nabla} \cdot \mathbf{u} w + \mathbf{u} \cdot \boldsymbol{\nabla} w)\, dx \tag{4.8}$$

is independent of the choice of w, and we have $|g(\psi)| \le \|\mathbf{u}\|_{\mathbf{W}(\Omega)} \|w\|_{H^1(\Omega)}$. By taking the infimum over all such w we obtain $|g(\psi)| \le \|\mathbf{u}\|_{\mathbf{W}(\Omega)} \|\psi\|_{\mathbb{B}}$. It follows that this continuous linear functional on $\mathbb{B}$ satisfies $\|g\|_{\mathbb{B}'} \le \|\mathbf{u}\|_{\mathbf{W}(\Omega)}$. We denote the dependence of this g on $\mathbf{u}$ by setting $g = \gamma_n(\mathbf{u})$, and this defines $\gamma_n \in \mathcal{L}(\mathbf{W}(\Omega), \mathbb{B})$ as desired.

Finally, we note that for $\mathbf{u} \in \mathbf{H}^1(\Omega)$ and smooth w on Ω we have

$$\gamma_n(\mathbf{u})(\gamma(w)) = \int_\Omega \boldsymbol{\nabla} \cdot (\mathbf{u}w)\,dx = \int_\Gamma \gamma(\mathbf{u}) \cdot \mathbf{n}\gamma(w)\,dS\,,$$

and the smooth functions are dense in $H^1(\Omega)$, so we have $\gamma_n(\mathbf{u}) = \mathbf{u} \cdot \mathbf{n}$ in $L^2(\Gamma)$.

It remains to show that γ_n maps *onto* $\mathbb{B}'$. Let $h \in \mathbb{B}'$, and then solve the *Neumann problem* for a $p \in H^1(\Omega)$ with $p - \Delta p = 0$ in Ω and $\dfrac{\partial p}{\partial n} = h$ on Γ. That is,

$$p \in H^1(\Omega) : \int_\Omega (p\,q + \boldsymbol{\nabla} p \cdot \boldsymbol{\nabla} q)\,dx = h(\gamma(q)) \text{ for all } q \in H^1(\Omega)\,.$$

This has exactly one solution, and we then set $\mathbf{u} = \boldsymbol{\nabla} p \in \mathbf{L}^2(\Omega)$. It follows that $\boldsymbol{\nabla} \cdot \mathbf{u} = p \in L^2(\Omega)$, and so $\mathbf{u} \in \mathbf{W}(\Omega)$ with $h = \gamma_n(\mathbf{u})$ on Γ. ☐

Remark 4.1. *By solving an appropriate Dirichlet problem, we can show that the functional* (4.8) *satisfies* $\|g\|_{\mathbb{B}'} = \|\mathbf{u}\|_{\mathbf{W}(\Omega)}$.

Remark 4.2. *Note that it is only the* normal component *of the boundary trace that retains any meaning for functions from* $\mathbf{W}(\Omega)$.

5. The Mixed Problem

We shall consider the *stationary Stokes system*

$$\lambda\mathbf{u} - \mu\Delta\mathbf{u} + \boldsymbol{\nabla} p = \mathbf{f}\,, \qquad \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \text{ in } \Omega\,,$$
$$\mathbf{u} = \mathbf{0} \text{ on } \Gamma_0\,, \quad \mu\frac{\partial \mathbf{u}}{\partial n} - p\,\mathbf{n} = \mathbf{g} \text{ on } \Gamma_1\,,$$

for the pair $\mathbf{u}(x)$, $p(x)$. Here the two sets Γ_0, Γ_1 comprise a partition of the boundary $\Gamma = \partial\Omega$, and we are given the functions $\mathbf{f} \in L^2(\Omega)$ and $\mathbf{g} \in L^2(\Gamma_1)$.

Now, if the pair $\mathbf{u}(x)$, $p(x)$ is a solution to the stationary Stokes system, then for every $\mathbf{w} \in \mathbf{H}^1(\Omega)$ we have

$$\int_\Omega \lambda\mathbf{u} \cdot \mathbf{w}\,dx + \mu \int_\Omega \boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} w_i\,dx - \mu \int_\Gamma (\boldsymbol{\nabla} u_i \cdot \mathbf{n}) w_i\,ds$$
$$- \int_\Omega p\,(\boldsymbol{\nabla} \cdot \mathbf{w})\,dx + \int_\Gamma (\mathbf{w} \cdot \mathbf{n}) p\,ds = \int_\Omega \mathbf{f} \cdot \mathbf{w}\,dx\,.$$

We define the space $\mathbf{V} = \{\mathbf{w} \in \mathbf{H}^1(\Omega) : \ \boldsymbol{\nabla} \cdot \mathbf{w} = 0 \text{ in } \Omega,\ \mathbf{w} = \mathbf{0} \text{ on } \Gamma_0\}$. Then our solution is chosen from this space, and if we also choose our test function $\mathbf{w} \in \mathbf{V}$ above, it follows that

$$\int_\Omega \lambda\mathbf{u} \cdot \mathbf{w}\,dx + \mu \int_\Omega \boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} w_i\,dx - \mu \int_{\Gamma_1} (\boldsymbol{\nabla} u_i \cdot \mathbf{n}) w_i\,ds$$
$$+ \int_{\Gamma_1} (\mathbf{w} \cdot \mathbf{n}) p\,ds = \int_\Omega \mathbf{f} \cdot \mathbf{w}\,dx\,.$$

Thus, the weak form of the problem is now to find

$$\mathbf{u} \in \mathbf{V} : \int_\Omega (\lambda\mathbf{u} \cdot \mathbf{w} + \mu\boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} w_i)\,dx = \int_\Omega \mathbf{f} \cdot \mathbf{w}\,dx + \int_{\Gamma_1} \mathbf{g} \cdot \mathbf{w}\,ds \text{ for all } \mathbf{w} \in \mathbf{V}\,.$$

As before, we obtain the following result.

Theorem 5.1. *There is exactly one weak solution* $\mathbf{u}$ *to the stationary Stokes system.*

Note also that the pressure p is determined up to a constant.

Conversely, let's assume that $\mathbf{u}$ is a solution of the weak form of the stationary Stokes system. Then from the inclusion $\mathbf{u} \in \mathbf{V}$ we obtain $\mathbf{u} \in \mathbf{H}^1(\Omega)$, $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ in Ω, and $\mathbf{u} = \mathbf{0}$ on Γ_0 in the sense of boundary trace. Furthermore, by taking test functions $\mathbf{w} \in \mathbf{V}_0 = \mathbf{V} \cap \mathbf{H}_0^1(\Omega)$, we find that

$$\lambda \mathbf{u} - \mu \Delta \mathbf{u} - \mathbf{f} \in \mathbf{V}_0^{\perp}\,,$$

the indicated annihilator of $\mathbf{V}_0$ in $\mathbf{H}^{-1}(\Omega)$, and so it is a *gradient*, and we have

$$\lambda \mathbf{u} - \mu \Delta \mathbf{u} - \mathbf{f} = -\boldsymbol{\nabla} p\,, \quad p \in L^2(\Omega)\,.$$

It follows that for each $i = 1, 2, ..., n$ we have $\mu \boldsymbol{\nabla} u_i - p\,\mathbf{e}_i \in \mathbf{L}^2(\Omega)$ and $\boldsymbol{\nabla} \cdot (\mu \boldsymbol{\nabla} u_i - p\,\mathbf{e}_i)$ is the i-th component of $\mu \Delta \mathbf{u} - \boldsymbol{\nabla} p = \lambda \mathbf{u} - \mathbf{f} \in \mathbf{L}^2(\Omega)$. This shows that we have each $\mu \boldsymbol{\nabla} u_i - p\,\mathbf{e}_i \in \mathbf{W}(\Omega)$ and so there is a well-defined *normal trace* on the boundary. We use this equation to substitute for $\mathbf{f}$ in the weak form to obtain

$$\int_\Omega (\mu \boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} w_i + (\mu \Delta \mathbf{u} - \boldsymbol{\nabla} p) \cdot \mathbf{w})\, dx = \int_{\Gamma_1} \mathbf{g} \cdot \mathbf{w}\, ds \text{ for all } \mathbf{w} \in \mathbf{V}\,.$$

Since $\boldsymbol{\nabla} \cdot \mathbf{w} = 0$, the i-th component of the integrand on the left side is given by $\boldsymbol{\nabla} \cdot [(\mu \boldsymbol{\nabla} \mathbf{u}_i - p \mathbf{e}_i) w_i]$. The generalized Stokes theorem implies that its integral over Ω is given by

$$\int_{\Gamma_1} \gamma_{\mathbf{n}}(\mu \boldsymbol{\nabla} u_i - p \mathbf{e}_i)\, w_i\, ds\,,$$

and so from above we obtain

$$\int_{\Gamma_1} \left(\mu \frac{\partial \mathbf{u}}{\partial n} - p\,\mathbf{n} \right) \cdot \mathbf{w}\, dx = \int_{\Gamma_1} \mathbf{g} \cdot \mathbf{w}\, ds \text{ for all } \mathbf{w} \in \mathbf{V}\,.$$

It is in this sense that we have

$$\mu \frac{\partial \mathbf{u}}{\partial n} - p\,\mathbf{n} = \mathbf{g} \text{ on } \Gamma_1\,.$$

Remark 5.1. *Because of the special properties of the space* $\mathbb{B}$*, the condition* $\gamma_n(\mathbf{w}) = 0$ *on* Γ_1 *is different from the condition* $\gamma_n(\mathbf{w}) = 0$ *for all* $w \in \mathbf{V}$.

6. THE NAVIER-STOKES SYSTEM

We shall consider the *stationary Navier-Stokes system*

$$\lambda \mathbf{u} - \mu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \boldsymbol{\nabla} p = \mathbf{f}\,, \quad \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \text{ in } \Omega\,, \quad \mathbf{u} = \mathbf{0} \text{ on } \Gamma\,, \tag{6.9}$$

for the pair $\mathbf{u} \in \mathbf{H}^1(\Omega)$, $p \in L^2(\Omega)$. The boundary condition is meant in the sense of trace on $\Gamma = \partial\Omega$, and we are given the function $\mathbf{f} \in L^2(\Omega)$. This is distinguished from the Stokes system by the nonlinear momentum term $(\mathbf{u} \cdot \nabla)\mathbf{u} = u_j \partial_j \mathbf{u}$, and the corresponding problem is the *Navier-Stokes system.*

Now, if the pair $\mathbf{u}(x)$, $p(x)$ is a solution to the Navier-Stokes system, then for every $\mathbf{w} \in \mathbf{H}^1(\Omega)$ we have

$$\int_\Omega \lambda \mathbf{u}\cdot\mathbf{w}\,dx + \mu\int_\Omega \boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} w_i\,dx - \mu\int_\Gamma (\boldsymbol{\nabla} u_i\cdot\mathbf{n}) w_i\,ds$$
$$+\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{w}\,dx - \int_\Omega p\,(\boldsymbol{\nabla}\cdot\mathbf{w})\,dx + \int_\Gamma (\mathbf{w}\cdot\mathbf{n})p\,ds = \int_\Omega \mathbf{f}\cdot\mathbf{w}\,dx\,.$$

We define the space $\mathbf{V}_0 = \{\mathbf{w}\in\mathbf{H}^1(\Omega) : \ \boldsymbol{\nabla}\cdot\mathbf{w} = 0 \text{ in } \Omega,\ \mathbf{w} = \mathbf{0} \text{ on } \Gamma\}$. Then our solution belongs to this space, and if we also choose our test function $\mathbf{w}\in\mathbf{V}_0$ above, it follows that

$$\int_\Omega \lambda\mathbf{u}\cdot\mathbf{w}\,dx + \mu\int_\Omega \boldsymbol{\nabla} u_i\cdot\boldsymbol{\nabla} w_i\,dx + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{w}\,dx = \int_\Omega \mathbf{f}\cdot\mathbf{w}\,dx\,.$$

Thus, the weak form of the problem is now to find

$$\mathbf{u}\in\mathbf{V}_0 : \int_\Omega (\lambda\mathbf{u}\cdot\mathbf{w} + \mu\boldsymbol{\nabla} u_i\cdot\boldsymbol{\nabla} w_i)\,dx + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{w}\,dx = \int_\Omega \mathbf{f}\cdot\mathbf{w}\,dx \text{ for all } \mathbf{w}\in\mathbf{V}_0\,. \tag{6.10}$$

Next we confirm that the additional momentum terms above are meaningful, so we define the function

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\,dx \tag{6.11}$$

for appropriate $\mathbf{u},\mathbf{v},\mathbf{w}$. To this end, we obtain the following consequences of the Sobolev imbedding Theorem 4.3 from our textbook.

For each $u\in H_0^1$, we have

$$\|u\|_{L^q} \le C(\Omega)\|u\|_{H^1}, \quad 1\le q<\infty, \text{ if } n=2,$$
$$\|u\|_{L^6} \le C(\Omega)\|u\|_{H^1}, \text{ if } n=3,$$
$$\|u\|_{L^4} \le C(\Omega)\|u\|_{H^1}, \text{ if } n=4.$$

These lead to the following.

Lemma 6.1. *If* $2\le n\le 4$, *then the trilinear form* $b(\mathbf{u},\mathbf{v},\mathbf{w})$ *is bounded on* $\mathbf{H}_0^1(\Omega)^3$.

Proof. We use Hölder's inequality repeatedly.
For $n=2$:

$$\left|\int_\Omega u_j\,\partial_j v_i\,w_i\,dx\right| \le C\|u_j\|_{L^4}\|\partial_j v_i\|_{L^2}\|w_i\|_{L^4}\,.$$

For $n\ge 3$:

$$\left|\int_\Omega u_j\,\partial_j v_i\,w_i\,dx\right| \le C\|u_j\|_{L^{2n/(n-2)}}\|\partial_j v_i\|_{L^2}\|w_i\|_{L^n}\,,$$

since $\dfrac{n-2}{2n}+\dfrac{1}{n}=\dfrac{1}{2}$. ☐

Conversely, let's assume that $\mathbf{u}$ is a solution of the weak Navier-Stokes system (6.10). Then from the inclusion $\mathbf{u} \in \mathbf{V}_0$ we obtain $\mathbf{u} \in \mathbf{H}^1(\Omega)$, $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ in Ω, and $\mathbf{u} = \mathbf{0}$ on Γ in the sense of boundary trace. Furthermore, by taking test functions $\mathbf{w} \in \mathbf{V}_0$, we find that

$$\lambda \mathbf{u} - \mu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \mathbf{f} \in \mathbf{V}_0^{\perp}\,,$$

the indicated annihilator of $\mathbf{V}_0$ in $\mathbf{H}^{-1}(\Omega)$, and so it is a *gradient*, and we have

$$\lambda \mathbf{u} - \mu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \mathbf{f} = -\boldsymbol{\nabla} p\,, \quad p \in L^2(\Omega)\,.$$

Thus, $\mathbf{u}$ is a solution of the original system (6.9), and so these two formulations are equivalent.

We develop some additional properties of the trilinear form $b(\mathbf{u}, \mathbf{v}, \mathbf{w})$.

Lemma 6.2. *For* $\mathbf{u} \in \mathbf{V}_0$ *and* $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ *we have* $b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0$.

Proof.

$$\int_\Omega u_j\,(\partial_j v_i)\, v_i\, dx = \frac{1}{2} \int_\Omega u_j\, \partial_j (v_i)^2\, dx = -\frac{1}{2} \int_\Omega (\partial_j u_j)(v_i)^2\, dx$$

so it follows by summing on all these that

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = -\frac{1}{2} \int_\Omega (\boldsymbol{\nabla} \cdot \mathbf{u}) |\mathbf{v}|_{\mathbb{R}^n}\, dx = 0\,.$$

□

Corollary 6.3. *For* $\mathbf{u} \in \mathbf{V}_0$ *and* $\mathbf{v},\ \mathbf{w} \in \mathbf{H}_0^1(\Omega)$ *we have* $b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v})$.

Proof.

$$0 = b(\mathbf{u}, \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w}) = b(\mathbf{u}, \mathbf{v}, \mathbf{w}) + b(\mathbf{u}, \mathbf{w}, \mathbf{v})$$

□

We shall prove there is a weak solution of the Navier-Stokes system (6.10) as a direct application of our basic existence Theorem 2.1 of our textbook. To do so, we begin by defining an operator $\mathcal{A} : \mathbf{V}_0 \to \mathbf{V}_0'$ by

$$\mathcal{A}(\mathbf{u})(\mathbf{w}) = \int_\Omega \lambda \mathbf{u} \cdot \mathbf{w}\, dx + \mu \int_\Omega \boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} w_i\, dx + b(\mathbf{u}, \mathbf{u}, \mathbf{w}), \quad \mathbf{u},\ \mathbf{w} \in \mathbf{V}_0\,.$$

We have shown in Lemma 6.1 that $\mathcal{A}(\cdot)$ is bounded, and we check from above that it is $\mathbf{V}_0$-coercive: by Corollary 6.3 we have

$$\mathcal{A}(\mathbf{u})(\mathbf{u}) \geq \mu \int_\Omega \boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} u_i\, dx + b(\mathbf{u}, \mathbf{u}, \mathbf{u}) \geq c_0 \|\mathbf{u}\|^2, \quad \mathbf{w} \in \mathbf{V}_0\,,$$

for some constant $c_0 > 0$. It remains to establish some form of continuity of $\mathcal{A}(\cdot)$.

Lemma 6.4. *The map* $\mathbf{u} \mapsto b(\mathbf{u}, \mathbf{u}, \cdot)$ *is weakly continuous from* $\mathbf{V}_0$ *to* $\mathbf{V}_0'$*, that is, if* $\mathbf{u}_m \overset{w}{\rightharpoonup} \mathbf{u}$ *in* $\mathbf{V}_0$ *then* $b(\mathbf{u}_m, \mathbf{u}_m, \mathbf{w}) \to b(\mathbf{u}, \mathbf{u}, \mathbf{w})$ *for each* $\mathbf{w} \in \mathbf{V}_0$.

Proof. By the Rellich-Kandorochov Theorem 4.2 of the text, we have strong convergence $\mathbf{u}_m \to \mathbf{u}$ in $\mathbf{L}^2(\Omega)$, and so for each $\mathbf{w} \in \mathbb{V}$ we obtain

$$b(\mathbf{u}_m, \mathbf{u}_m, \mathbf{w}) = -b(\mathbf{u}_m, \mathbf{w}, \mathbf{u}_m) = -\int_\Omega u_{m,j}\,(\partial_j w_i)\, u_{m,i}\, dx \to -b(\mathbf{u}, \mathbf{w}, \mathbf{u}) = b(\mathbf{u}, \mathbf{u}, \mathbf{w})\,.$$

Since the sequence $\{b(\mathbf{u}_m, \mathbf{u}_m, \cdot)\}$ is bounded in $\mathbf{V}_0'$, the same holds for each $\mathbf{w} \in \mathbf{V}_0$. □

Note that the preceding result depended on the observation that for each $\mathbf{w} \in \mathbb{V}$ the bilinear function $b(\cdot, \mathbf{w}, \cdot)$ is continuous on $\mathbf{L}^2(\Omega)$.

Corollary 6.5. *The operator* $\mathcal{A} : \mathbf{V}_0 \to \mathbf{V}_0'$ *is weakly continuous.*

Theorem 6.6. *There exists a weak solution* $\mathbf{u}$ *to the stationary Navier Stokes system* (6.10).

Proof. We have shown that $\mathcal{A} : \mathbf{V}_0 \to \mathbf{V}_0'$ is weakly continuous, hence, of type M, bounded, and coercive, so it is *onto.* ☐

Finally we note that the operator $\mathcal{A}$ is *not monotone.* We actually used the more general form of our Theorem 2.1. Furthermore, we have not said anything about *uniqueness* of the solution.

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