PDE-Models with Hysteresis on the Boundary

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Abstract

A general parabolic equation of the form of the porous media equation is considered with nonlinear boundary conditions that model hysteresis phenomena. Conditions of this type describe certain adsorption processes in porous media. Several examples are given and numerical results are shown.

Contents

1 The Problem

Let a bounded domain Ω in $\mathbf{R}^n$ with smooth boundary Γ be given. We assume that $\Gamma = \Gamma_D \cup \Gamma_H$ and denote the outer normal vector on Γ_H by ν. Then we study the

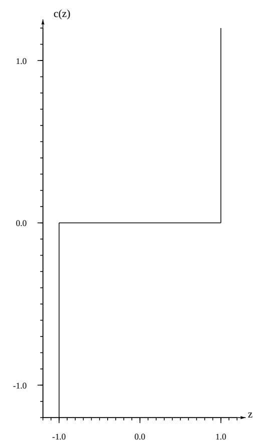

Figure 1: Graph of c

following initial boundary value problem for functions u on Ω and v, w on Γ_H.

$$\left\{\begin{array}{lll} \partial_t a(u) - \Delta u \ni f & t>0,\ x\in\Omega & (a) \\ \partial_t w + \partial_\nu u = g,\ w\in b(v) & t>0,\ x\in\Gamma_H & (b) \\ \partial_\nu u \in c(v-u) & t>0,\ x\in\Gamma_H & (c) \\ u = u_D & t>0,\ x\in\Gamma_D & (d) \\ a(u) = a_I & t=0,\ x\in\Omega & \\ w = w_I & t=0,\ x\in\Gamma_H & \end{array}\right. \tag{1}$$

Here, the following functions are given: f on $[0,T]\times\Omega$, g on $[0,T]\times\Gamma_H$, u_D on $[0,T]\times\Gamma_D$, a_I on Ω, and w_I on Γ_H.

Each of $a(.)$, $b(.)$, and $c(.)$ is a maximal monotone graph in $\mathbf{R}^2$, see [5]. The special interest in problem (1) is motivated by the fact that the two conditions (1b,c) on Γ_H may model *hysteresis* phenomena on the boundary. Specifically, consider the maximal monotone graph in $\mathbf{R}^2$ given by

$$\mathrm{sgn}\ (z) = \left\{\begin{array}{ll} \{-1\}, & z<0 \\ [-1,1], & z=0 \\ \{1\}, & z>0 \end{array}\right.$$

and its inverse

$$\mathrm{sgn}^{-1}(z) = \left\{\begin{array}{ll} (-\infty,0], & z=-1 \\ \{0\}, & -1<z<1 \\ [0,\infty), & z=1 \end{array}\right. .$$

If we choose $c(z) = \mathrm{sgn}^{-1}(\frac{z}{\delta})$ (figure 1 shows c with $\delta=1$), then the condition (1c) becomes a *constraint* for v, namely

$$u-\delta \le v \le u+\delta, \tag{2}$$

and also for $\partial_\nu u$, namely

$$\partial_\nu u \quad \begin{cases} \leq 0, & v = u - \delta \\ = 0, & u - \delta < v < u + \delta \\ \geq 0, & v = u + \delta \end{cases} .$$

If b and b^{-1} are functions, one can rewrite this as

$$\partial_\nu u \quad \begin{cases} \leq 0, & w = b(u - \delta) \\ = 0, & b(u - \delta) < w < b(u + \delta) \\ \geq 0, & w = b(u + \delta) \end{cases} .$$

The condition (1b) is an ordinary differential equation for w; for the choice $g \equiv 0$ the variable w remains constant as long as the constraint (2) is strictly satisfied. If the constraint is *active*, then $\partial_\nu u$ - the *control* - and w are selected such that via condition (1b) the corresponding equality in (2) is maintained. Therefore, one gets

$$b(u - \delta) \leq w \leq b(u + \delta). \tag{3}$$

Thus the relationship between u and $w \in b(v)$ is an example of a *generalized play*, see [10] [11] (figure 4). In this case, the function b prescribes the general shape of a loop in the $u - w$-plane, and the number δ its width the u-direction. On the right hand boundary of the loop the path is always directed upwards, and on the left hand boundary the path is always directed downwards. As soon as the variation of u leads into the interior of the loop, the value of w remains constant until the boundary is reached again.

Furthermore, if b is a multiple of the signum function

$$b = \gamma \text{ sgn },$$

then conditions (1b,c) model a *perfect relay*, see [10] [11] (figure 6). One may consider this case as a degenerate play. As in the previous case, in the interior of the loop all lines are horizontal, whereas on the boundary of the loop one may distinguish the following situations: on the upper and lower part w is constant and the normal flux is zero, i.e., u satisfies a unilateral boundary condition; on the right and left part u is constant, i.e., it satisfies a Dirichlet condition, and w is governed by an ordinary differential equation.

System (1) consists of a general *porous media equation* (1a) - which becomes degenerate whenever $a(u)$ vanishes - in the interior of Ω subject to a nonlinear dynamic *Neumann* condition (1b,c) on the boundary Γ_H and a *Dirichlet* condition (1d) on Γ_D. The variable w is the *internal state* of the hysteron, $v - u$ is the *order parameter*, and u is the external *input.*

A rather remarkable variety of boundary conditions can be obtained from (1b,c). For example, if $b \equiv 0$ we have an explicit *Neumann boundary condition*, and if $c \equiv 0$ it is homogeneous. (Clearly any general solvability results cannot allow simultaneously $c = b = 0$, for this forces $g = 0$.) If $b(0) = \mathbf{R}$ (i.e., $b^{-1} = 0$), then $v \equiv 0$ and we have a nonlinear Neumann constraint, and if in addition

$$c(z) = \begin{cases} 0, & z < 0 \\ [0, \infty), & z = 0 \end{cases},$$

it is the *Signorini condition*

$$u \geq 0, \quad \partial_\nu u \geq 0, \quad u \partial_\nu u = 0.$$

If $c(0) = \mathbf{R}$ we get $v = u$ on Γ and this satisfies a nonlinear *dynamic boundary condition*

$$\partial_t b(u) + \partial_\nu u \ni g$$

of Neumann type. If $b(0) = c(0) = \mathbf{R}$ we have the homogeneous *Dirichlet boundary condition.* If both b^{-1} and c are functions, one gets a *nonlinear adsorption condition* of the form

$$\partial_t w + \partial_\nu u = g, \quad \partial_\nu u = c(b^{-1}(w) - u).$$

For previous work on some of these various classes, we refer to [2] [4] [6].

Theoretical results for the problem (1) are given in the paper [8]. There well-posedness for the corresponding stationary and the evolution problem are proved using methods that had previously been developed in [12]. In this paper we concentrate on special examples and their numerical solution. Papers that deal with problems closely related to those of the present paper are [1] [9] [13] [14] [15] where parabolic problems with a hysteresis source term are studied.

Adsorption in porous media may be governed by conditions on the surfaces of the solid material that are of hysteresis type (see [3] pp 357 ff. where experimental evidence for hysteresis in adsorption processes is described). In that case u is the concentration of a chemical species that is dissolved in the fluid occupying the pores, and w is its concentration on the surfaces once it has been adsorbed. Usually, one assumes that on the pore scale the adsorption rate - given as the normal flux in the Neumann boundary condition for the diffusion-convection process in the fluid - is a prescribed function of u and/or w on the boundary. But here we make the assumption that adsoprtion takes place only if the concentration u exceeds certain thresholds and that the range of the concentration w is bounded. In this way, one gets hysteresis phenomenona of the kind discussed in this paper. In [7] this idea is applied to homogenization of reactive transport through porous media.

2 Numerical Examples

We consider a γ-multiple of the signum function

$$b = \gamma \ \mathrm{sgn}$$

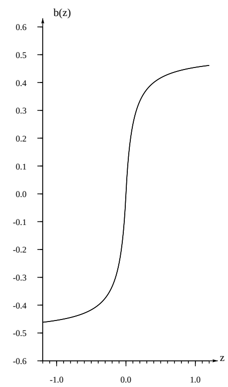

Figure 2: Graph of b_ε with $\varepsilon = 0.1$

($\varepsilon = 0$) or a smooth approximation thereof, namely

$$b_\varepsilon(z) = \gamma \frac{z}{\varepsilon + |z|}$$

(figure 2 shows b_ε with $\gamma = \frac{1}{2}$ and $\varepsilon = 0.1$), and the inverse of the scaled signum function

$$c(z) = \operatorname{sgn}^{-1}(\frac{z}{\delta})$$

(see figure 1). For the following examples we simplify by using $a(u) = u$, and $f, g = 0$. We are going to use the function

$$h(t) = \alpha 2^{-\frac{t}{\beta}} \sin(2\pi\omega t)$$

several times. For the examples we have chosen $\gamma = \frac{1}{2}$ and $\delta = 1$. The initial values are all zero in the examples. As a numerical method we have used the standard time-explicit difference scheme with constant step-sizes in x and t.

1. As a one-dimensional example, let $\Omega = (0, 1)$, $\Gamma_H = \{0\}$, $\Gamma_D = \{1\}$. We assume $u_D(t) = h(t)$ with $\alpha = 4$, $\beta = 10$, and $\omega = \frac{1}{5}$. We have used step sizes $\Delta x = 0.02$ and $\Delta t = 8 \cdot 10^{-6}$. Figures 3 and 5 show u and w at $x = 1$ as functions of time with $\varepsilon = 0.1, 0.0$, resp.; the dotted line is the function h. Figures 4 and 6 show w versus u; the oblique lines that cut the corners are due to the discretization of time. Figure 4 shows the typical form of a *play*, whereas figure 6 has the form of a *perfect relay*. The fact that the boundary of the loops differ in both cases can be seen in figures 4 and 6 and also in figures 3 and 5. Not only has w upper and lower bounds, but the change between the regime $-\delta < w < \delta$ and $w = -\delta$ or $w = \delta$ is abrupt for $\varepsilon = 0$. Therefore

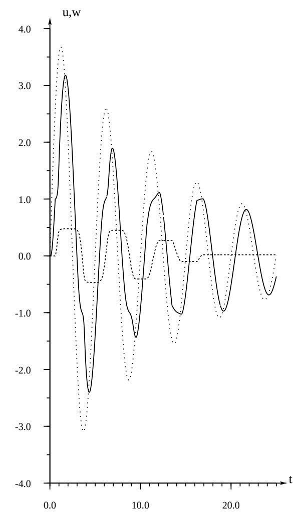

Figure 3: u and w as functions of t at $x = 1$ for example 1 with $\varepsilon = 0.1$. u at $x = 1$: solid line, u at $x = 0$: dotted line, w at $x = 1$: dashed line.

the curves in figure 5 have sharp corners. In this sense, a relay is a degenerate case. On the other hand, a comparison of figures 3 and 5 shows that the relay ($\varepsilon = 0$) can be approximated by a play (ε small) quite well.

2. The following is an example in 2D. We take $\Omega = \{(x_1, x_2) : \ 0 < x_1, x_2 < 1\}$ as the unit square in $\mathbf{R}^2$. Here we assume $\Gamma_D = \{(x_1, x_2) : \ x_1 = 0 \text{ or } x_1 = 1\}$ and $\Gamma_H = \{(x_1, x_2) : \ x_2 = 0 \text{ or } x_2 = 1\}$. We assume

$$u_D(t, x) = \begin{cases} h(t), & x_1 = 0 \\ -h(t), & x_1 = 1 \end{cases}$$

for $x \in \Gamma_D$ with $\alpha = 4$, $\beta = 2$, and $\omega = 1$. We have used step sizes $\Delta x = 0.03125$ and $\Delta t = 5 \cdot 10^{-5}$. Figure 7 shows the profile of the solution u at time $t = 1.25$ with $\varepsilon = 0$.

3. For another example in 2D we take again $\Omega = \{(x_1, x_2) : \ 0 < x_1, x_2 < 1\}$. Now we assume $\Gamma_D = \{(x_1, x_2) : \ x_1 = 0\}$ and $\Gamma_H = \partial\Omega \setminus \Gamma_D$ as in the previous example. Again we assume $u_D(t) = h(t)$ for $x \in \Gamma_D$ with the same parameters as in example 2. Figure 8 shows the profile of the solution u at time $t = 1.25$ with $\varepsilon = 0.1$. As for the one-dimensional examples, one sees easily that there is a similarity and a difference between the two cases $\varepsilon > 0$ and $\varepsilon = 0$: Obvoiusly, the first case is an approximation of the latter. But for $\varepsilon = 0$ there is a range of points on the boundary where u is constant; here u equals to the threshold values $\pm\delta$. Such a phenomenon does not occur for $\varepsilon > 0$.

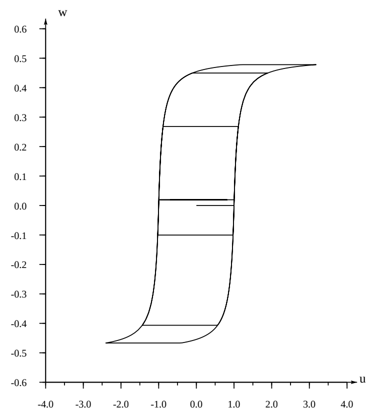

Figure 4: *Play:* w versus u at $x = 1$ for example 1 with $\varepsilon = 0.1$

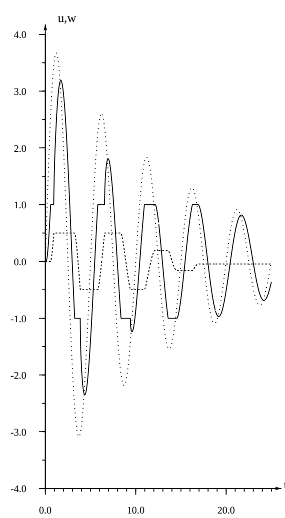

Figure 5: u and w as functions of t at $x = 1$ for example 1 with $\varepsilon = 0$. u at $x = 1$: solid line, u at $x = 0$: dotted line, w at $x = 1$: dashed line.

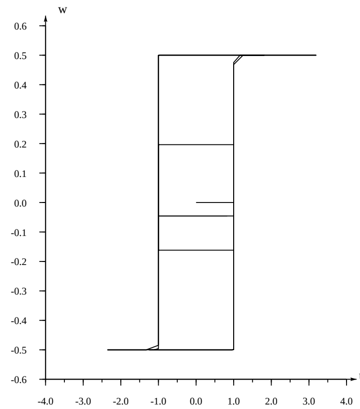

Figure 6: *Relay:* w versus u at $x = 1$ for example 1 with $\varepsilon = 0$

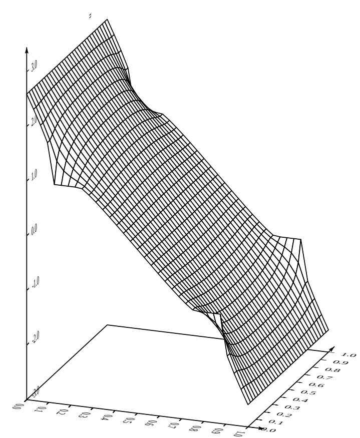

Figure 7: Profile at $t = 1.25$ for example 2 with $\varepsilon = 0$

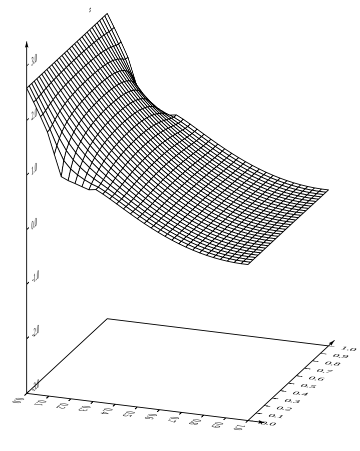

Figure 8: Profile at $t = 1.25$ for example 3 with $\varepsilon = 0.1$

It is obvious from the examples and their plots that parabolic equations with boundary conditions of hysteresis type can be treated numerically in a natural way that is similar to those with the usual Dirichlet or Neumann conditions. On the other hand the solutions of initial boundary value problems may have properties that are known for phase change problems. In special cases the boundary condition of hysteresis type results in switching back and forth from Dirichlet to Neumann or unilateral conditions on the boundary. The numerical experiments presented in this paper demonstrate these special features of the solutions of initial boundary value problems for partial differential equation.

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