PARABOLIC PDE WITH HYSTERESIS

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Abstract. Our objective here is to describe some results on the well-posedness of initial-boundary-value problems for a nonlinear parabolic partial differential equation with memory effects and with general boundary conditions. The system consists of a general porous medium equation together with a family of ordinary differential equations subject to constraints. Such a system includes a very general class of hysteresis functionals known as Preisach models of hysteresis.

1. Introduction

We begin by describing a system consisting of a parabolic partial differential equation and an ordinary differential equation which are coupled naturally by terms which depend on the difference of the unknowns. This system takes the form

$$\text{(1.1.a)} \quad \frac{\partial}{\partial t}a(u(x,t)) - \Delta u(x,t) - c(v(x,t)-u(x,t)) \ni f(x,t)\ , \quad x \in \Omega\ ,\ t \in (0,T]\ ,$$

$$\text{(1.1.b)} \quad \frac{\partial}{\partial t}b(v(x,t)) + c(v(x,t)-u(x,t)) \ni g(x,t)\ ,$$

$$\text{(1.1.c)} \quad -\frac{\partial}{\partial \nu}u(s,t) \in d(u(s,t))\ , \quad s \in \partial\Omega\ ,$$

in which $u = u(x,t)$ and $v = v(x,t)$ are functions defined on a bounded domain Ω in Euclidean space $\mathbb{R}^n$, and $T > 0$ denotes the length of the time interval. Note that (1.1) contains a generalized *porous medium equation*, and we make no assumptions of strict monotonicity of $a(\cdot)$. In particular, we allow the degenerate case $a(\cdot) \equiv 0$, and this reduces (1.1) to a *pseudoparabolic* equation [7].

If each of $a(\cdot)$, $b(\cdot)$, $c(\cdot)$ and $d(\cdot)$ were a monotone (non-decreasing) function, then the inclusion symbols, $\ni$, would be replaced by the corresponding equality symbol. Such systems arise in many contexts, for example, in the diffusion of chemicals through a saturated porous medium in which (1.1.b) models the local storeage or absorbtion in immobile nondiffusive sites. In that case, u is the concentration of a chemical species in the fluid which occupies the pores and v is the concentration on the surface of the medium. These are commonly called *first order kinetic models*, and they arise in many applications to describe diffusion through an absorbing

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medium. These systems can be regarded as a degenerate case of a corresponding parabolic system which contains an additional term $-\Delta v(x,t)$ in (1.1.b). These in turn arise as *parallel models* of flow through a heterogeneous medium consisting of two components with different diffusivities and an exchange flux driven by the difference in concentration between the two components. In (1.1.b) this diffusion term has been deleted because of the immobility of the concentration in the absorbtion sites.

For our purposes it will be necessary to permit $a(\cdot), b(\cdot)$, and especially $c(\cdot)$ to be multi-valued. That is, we shall consider the case where these are *maximal monotone graphs* in $\mathbb{R} \times \mathbb{R}$ [5]. In particular, if $b(0) = \mathbb{R}$, hence, b is the inverse of the zero graph, then the system (1.1) reduces to (1.1.a) with $v = 0$. Likewise, if $c(0) = \mathbb{R}$, then $u = v$, and the system reduces to

$$\frac{\partial}{\partial t}(a(u(x,t)) + b(u(x,t))) - \Delta u(x,t) = f(x,t) + g(x,t) \qquad x \in \Omega\ ,\ t \in (0,T]$$

together with the boundary condition (1.1.c). These are merely additive perturbations of the porous medium equation.

This generalization to multi-valued graphs will permit a very elegant treatment of a class of parabolic problems with *hysteresis*, and we shall describe this below. These are of the form

$$\frac{\partial}{\partial t}\left(a(u) + \mathcal{H}(u)\right) - \Delta u = f \tag{1.2}$$

in which $\mathcal{H}$ denotes a *hysteresis* functional, that is, its value depends not only on the current value of the input, u, but also on the *history* of the input in a very nonlinear way. Due to the complex nature of the operators customarily used to represent hysteresis [12], their addition to systems of differential equations leads to substantial technical problems for the development of a good theory. An excellent introduction to hysteresis is the monograph [20], and one should consult the recent survey [18]. The new book [28] is an excellent source for history and recent developments of mathematical models of hysteresis as well as their addition to partial differential equations, especially those of parabolic type. See also [11], [27]. The forthcoming monograph [13] concerns quasilinear wave equations with elasto-plastic hysteretic constitutive laws arising in mechanics.

Our plan is as follows. In Section 2 we show how the ordinary differential equation (1.1.b) provides a mechanism for modeling the class of hysteresis functionals known as *generalized play.* The main point is that the maximal monotone graphs permit the consideration of differential equations with *constraints.* This construction extends naturally to include the very general class of hysteresis functionals known as *Preisach type* [15], [17]. See [27] for a similar method developed earlier and reported in [28]. In Section 3 we describe an example in which the form (1.2) arises naturally, a free-boundary problem which generalizes the Stefan problem to permit super-cooled water or super-heated ice [29], [14]. In Section 4 we describe a related problem in which the hysteresis occurs on the *boundary* [10], [11]. Because of the generality attained via our use of maximal monotone graphs, this class includes boundary conditions of all the usual types, including Dirichlet, Neumann, Robin, and the fourth type, i.e., the *dynamic* boundary conditions [7], [24]. Additional papers that deal with problems closely related to those of the present paper are [1], [25], and [26], where parabolic problems with a hysteresis source term are studied. Section 5 contains some remarks on extensions and related problems.

2. Hysteresis Models in PDE

We first consider an elementary but fundamental example of hysteresis. The example depends on three parameters, α, β, and ϵ, with $0 < \epsilon$, $\alpha < \beta$. Denote by $[x]_+$ and $[x]_-$, respectively, the positive and negative parts of the real number x. Let's look at the response $w(\cdot)$ that arises from a given input, $u(\cdot)$. The output $w = \mathcal{H}_{\alpha,\beta,\epsilon}(u)$ varies according to the following: if $u > \beta + \epsilon$, then $w = 1$; if $u < \alpha$, then $w = 0$; if $\alpha < u < \beta + \epsilon$, then $0 \le w \le 1$ and

$$w'(t) = \begin{cases} \left[\dfrac{u'(t)}{\epsilon}\right]_+ & \text{if } w = \dfrac{u-\beta}{\epsilon}, \\ 0 \text{ if } \dfrac{u-\beta}{\epsilon} < w < \dfrac{u-\alpha}{\epsilon}, & \\ \left[\dfrac{u'(t)}{\epsilon}\right]_- & \text{if } w = \dfrac{u-\alpha}{\epsilon}. \end{cases}$$

For example, suppose we start with $u(0) = 0$ and $w(0) = 0$. As long as $u(t)$ remains between α and β, the output w stays constant at the value $w(t) = 0$. If $u(t)$ increases to β and then beyond $\beta + \epsilon$, then $w(t)$ increases to $+1$ where it remains until $u(t)$ gets down to $\alpha + \epsilon$. If $u(t)$ decreases below α, then $w(t)$ will drop to 0 and remain there until $u(t)$ again reaches β, and so on. It is clear that the output $w(t)$ depends not only on the present value but also on the history of the input $u(s)$ for previous times, $0 < s < t$. The limiting case obtained from $\epsilon \to 0$ is the *relay* hysteresis functional $\mathcal{H}_{\alpha,\beta}$ that is basic to the Preisach representation of a very general class of hysteresis functionals. This class will be included in our theory below. When $\alpha = \beta = 0$ this reduces further to the *Heaviside graph* $H(\cdot)$, which is defined by $H(u) = \{0\}$ if $u < 0$, $H(0) = [0, 1]$, and $H(u) = \{1\}$ if $u > 0$.

In order to see how an ordinary differential equation with constraint can produce such a hysteresis functional, we consider a somewhat more general situation. Let a maximal monotone graph $b(\cdot)$ be given; this hysteresis model will be of the type *generalized play* described by horizontal translates of $w \in b(u)$. The hysteresis functional $\mathcal{H}_{\alpha,\beta,\epsilon}$ described above is given by the choice $b = H_\epsilon$, where

$$H_\epsilon(r) = \begin{cases} 1 & \text{if } r \ge \epsilon \\ \frac{r}{\epsilon} & \text{if } o < r < \epsilon \\ 0 & \text{if } r \le 0\ . \end{cases}$$

Thus, we introduce a new variable, v, in order to represent the *phase* constraints:

$$w \in b(v), \quad u - \beta \le v \le u - \alpha\ .$$

We use the *sign function* (or graph) sgn to realize these constraints. Recall that it is defined by $\operatorname{sgn}(x) = 1$ if $x > 0$, $\operatorname{sgn}(x) = -1$ if $x < 0$, and $\operatorname{sgn}(0) = [-1, 1]$. We shall use its scaled version, $\mathrm{sgn}_{\alpha,\beta}$, defined by $\mathrm{sgn}_{\alpha,\beta}(x) \equiv \beta$ if $x > 0$, $\mathrm{sgn}_{\alpha,\beta}(x) \equiv \alpha$ if $x < 0$, and $\mathrm{sgn}_{\alpha,\beta}(0) = [\alpha, \beta]$. Thus, let $u(t)$ be a time-dependent input to this generalized play model, and let $w(t)$ be the corresponding output or response. There is at each time t a corresponding phase variable $v(t)$ which is related to $w(t)$ and $u(t)$ as above, and so it is required that $w(t)$ be non-decreasing when $v(t) = u(t) - \beta$, non-increasing when $v(t) = u(t) - \alpha$, and stationary ($w'(t) = 0$) in the interior region where $u - \beta < v < u - \alpha$. This is equivalent to requiring that $w(t), v(t)$ satisfy

$$w(t) \in b(v(t)) \quad , \quad w' + \mathrm{sgn}^{-1}_{-\beta,-\alpha}(v(t) - u(t)) \ni 0 \ .$$

Thus, we are led to ordinary differential equations of the form

$$w(t) \in b(v(t)) \quad , \quad w'(t) + c(v(t) - u(t)) \ni 0,$$

with maximal monotone graphs $b(\cdot)$ and $c(\cdot)$, as models of hysteresis in which the output is the solution $w(t)$ with input $u(t)$. This is just what appears in (1.1.b).

We have described above the relay hysteresis functional which is determined by the sign graph and the two parameters $\alpha < \beta$. These translation parameters are the switching positions of the relay, and we denote the output of this relay by $w_{\alpha,\beta} = \mathcal{H}_{\alpha,\beta}(u)$. This operator can be represented by a rectangular loop which is the input-output graph of the relay which switches up to the value $w = 1$ at $u = \beta$ and down to the value $w = -1$ at $u = \alpha$. In addition to the family of relay functionals $\mathcal{H}_{\alpha,\beta}$, let there be prescribed a family of weights, $\mu(\alpha, \beta)$ on $S = \{y = (\alpha, \beta) : \alpha < \beta\}$. Then we obtain a hysteresis functional depending on the measure space (S, μ), and its output is given by

$$\mathcal{H}(u)(t) = \int_S w(y,t)\, d\mu(y) \ ,$$

where w is the solution $w(y,t) = w_{\alpha,\beta}(t)$ of

$$w(y,t) \in b(v(y,t)) \ , \quad \frac{\partial}{\partial t} w(y,t) + \mathrm{sgn}_y^{-1}(v(y,t) - u(t)) \ni 0, \quad y \in S \ .$$

Such a construction is known as a *Preisach* model of hysteresis, and this represents a very general class of hysteresis functionals. This representation is a continuous weighted sum of the outputs of a family of parallel connected relay switches, parametrized by their switching values and independently responding to the common input, $u(t)$. Thus, the Preisach model is a superposition of the simplest type of hysteresis functionals, the relay. When this hysteresis class is combined with the nonlinear diffusion equation as above, we have the system of the form

$$\begin{aligned} &\text{(2.1.a)} && \frac{\partial}{\partial t} a(u(x,t)) + \frac{\partial}{\partial t} \int_S b(v(x,y,t))\, d\mu(y) - \Delta u(x,t) \\ & && \qquad = f(x,t), \quad x \in \Omega \ , \ t \in (0,T] \ , \\ &\text{(2.1.b)} && \frac{\partial}{\partial t} b(v(x,y,t)) + \mathrm{sgn}_y^{-1}(v(x,y,t) - u(x,t)) \ni 0, \quad y \in S \ , \\ &\text{(2.1.c)} && -\frac{\partial}{\partial \nu} u(s,t) \in d(u(s,t)), \quad s \in \partial\Omega \ . \end{aligned}$$

This is a degenerate parabolic system of coupled equations with Neumann type boundary conditions for which the initial conditions $a(u(x,0))$ and $b(v(x,y,0))$ are to be specified, and it realizes the parabolic equation (1.2) with the indicated Preisach hysteresis.

We have shown in [15], [17] that the dynamics of problem (2.1) is determined by a nonlinear semigroup of contractions on the Banach space $L^1(\Omega) \times L^1(\Omega \times S)$. The negative of the generator of this contraction semigroup is constructed as an operator $\mathbb{C}$ which realizes the system (2.1) as a Cauchy problem

$$w'(t) + \mathbb{C}(w(t)) \ni f(t), \quad 0 < t, \quad w(0) = w_0$$

in the Banach space $L^1(\Omega) \times L^1(\Omega \times S)$. Then one shows that $\mathbb{C}$ is an *m-accretive* operator on this Banach space.

We recall this notion briefly. A (possibly multi-valued) operator or relation $\mathbb{C}$ in a Banach space X is a collection of related pairs $[x,y] \in X \times X$ denoted by $y \in \mathbb{C}(x)$; the range $Rg(\mathbb{C})$ consists of all such y. The operator $\mathbb{C}$ is called *accretive* if for all $y_1 \in \mathbb{C}(x_1)$, $y_2 \in \mathbb{C}(x_2)$ and $\varepsilon > 0$

$$\|x_1 - x_2\| \le \|x_1 - x_2 + \varepsilon(y_1 - y_2)\| \ .$$

This is equivalent to requiring that $(I + \varepsilon\mathbb{C})^{-1}$ is a contraction on $Rg(I + \varepsilon\mathbb{C})$ for every $\varepsilon > 0$. If, in addition, $Rg(I + \varepsilon\mathbb{C}) = X$ for some (equivalently, for all) $\varepsilon > 0$, then $\mathbb{C}$ is called *m-accretive*. For such an operator, one can approximate the derivative in the evolution equation by a backward-difference quotient of step size $h > 0$ and the function $f(t)$ by the step function $f^h(t)$ $(= f_k^h$ for $kh \le t < (k+1)h)$ and get a unique solution $\{w_k^h : 1 \le k\}$ of

$$\frac{w_k^h - w_{k-1}^h}{h} + \mathbb{C}(w_k^h) \ni f_k^h \ , \qquad k = 1, 2, \dots \ ,$$

with $w_0^h = w_0$. Since $\mathbb{C}$ is m-accretive, this scheme is uniquely solved recursively to obtain w_k^h and, hence, the piecewise-constant approximate solution $w^h(t)$ $(= w_k^h$ for $kh \le t < (k+1)h)$ of the Cauchy problem. The fundamental result is the following.

Theorem (Crandall-Liggett). *Assume $\mathbb{C}$ is m-accretive, $w_0 \in \overline{D(\mathbb{C})}$, $f \in L^1([0,T],X)$ and that $f^h \to f$ in $L^1([0,T],X)$. Then $w^h \to w(\cdot)$ uniformly as $h \to 0$ and $w(\cdot) \in C([0,T],X)$.*

Thus, $w(\cdot)$ is an obvious candidate for a solution of the Cauchy problem. It can be uniquely characterized as an *integral solution*. Moreover, if $f_1, f_2 \in L^1([0,T],X)$ and w_1, w_2 are integral solutions of $w_j' + \mathbb{C}(w_j) \ni f_j \ , \quad 0 \le t, \quad j = 1,2$, then

$$\|w_1(t) - w_2(t)\| \le \|w_1(0) - w_2(0)\| + \int_0^t \|f_1(s) - f_2(s)\| \, ds \ , \quad 0 \le t.$$

However, this rather technical characterization does not even require any differentiability of the solution. For an introduction to the abstract Cauchy problem in Banach space and its applications to initial-boundary-value problems for partial differential equations, see [2], [3], [6] and their included references.

3. A Free-Boundary Problem

We describe now a model of the melting and refreezing of water/ice in a porous medium and the hysteresis effects that result from the assumption that the melting and the freezing temperatures are *different*. The ice melts at a slightly positive temperature and freezing occurs after a slightly negative temperature is reached. For this example of phase change with super-heating or super-cooling, existence of a solution was shown in [29] by compactness methods, and the full well-posedness of the problem in L^1 was described in [14] as an example of the system (1.1).

We begin with the description of the phase relation between energy and temperature. Let α and β be given numbers with $\alpha < 0 < \beta$. Begin with a unit volume of ice at temperature $u < \alpha$ and apply a uniform heat source of intensity F, so $e \equiv Ft$ is the accumulated *internal energy*. The temperature increases according to the relation $e = c(u)$ until it reaches the value $u = \beta > 0$. Then it remains at $u = \beta$ until L units of additional heat have been added; $L > 0$ is the *latent heat*. During this period there is a fraction w of water coexisting with the ice, and w increases at the constant rate F/L. The *water fraction* w, $0 \le w \le 1$, is the *phase variable*. After all the ice has melted, $w = 1$ and the temperature u begins to rise again according to $e = c(u) + L$. If the process is reversed by drawing heat out of the unit volume, the temperature falls according to $e = c(u) + L$ until it reaches $u = \alpha < 0$, then w decreases until it reaches $w = 0$, and thereafter the temperature u falls with $e = c(u)$. Note that the freezing took place at $u = \alpha$ and the melting at $u = \beta$. If $\alpha = \beta$ this is just the traditional Stefan problem. However the model presented above permits superheated ice and supercooled water, and it is here that hysteresis occurs. The relation between energy and temperature is given in terms of the Heaviside function by $e \in c(u) + LH(u - \beta)$ when u is increasing and by $e \in c(u) + LH(u - \alpha)$ when it is decreasing. The difference $e - c(u)$ is just L times the *simple relay*: $w \in \mathcal{H}_{\alpha,\beta}(u)$ means $w = H(u - \beta)$ if u is increasing from below α, and $w \in H(u - \alpha)$ if u is decreasing from above β. Also, w remains constant for $\alpha < u < \beta$, since there is no phase change until one of the threshold values is reached.

We shall formulate a free-boundary problem which describes heat conduction through a domain G in Euclidean space R^m subject to the constitutive assumptions above on the hysteresis phase relation between energy and temperature. This we call the *Super-Stefan problem*. Denote the boundary of G by ∂G and set $\Omega = G \times (0, \infty)$. The temperature at the point $x \in G$ and the time $t > 0$ is $u(x,t)$ and the smooth monotone functions $c(u)$, $k(u)$ are given with $c(0) = k(0) = 0$; their derivatives $c'(u)$, $k'(u)$ denote the *specific heat* and *conductivity*, respectively, of ice-water at temperature u. The phase change from water to ice occurs at $u = \alpha < 0$ and from ice to water at $u = \beta > 0$. The space-time region Ω is then separated into an always-ice region Ω_- where $u < \alpha$, an always-water region Ω_+ where $u > \beta$, and a region Ω_0 where $\alpha \le u \le \beta$ and in which the phase depends on it's preceding history.

Let $w(x,t)$ be the fraction of water at $(x,t) \in \Omega$, and note that according to our constitutive assumptions above we have $w \in \mathcal{H}_{\alpha,\beta}(u)$. The energy is given by $e = c(u) + Lw$. Let S_- be the boundary of Ω_- in Ω and S_+ the boundary of Ω_+ in Ω. The *unit normal* $N = (N_1, \ldots, N_m, N_t)$ on $S_- \cup S_+$ is oriented out of Ω_-

and Ω_+, and hence into Ω_0. We shall denote by $[g]$ the *saltus* or jump in values of the function g across the boundaries, S_- and S_+, in the direction of N : for $(x,t) \in S_- \cup S_+$

$$\big[g(x,t)\big] = \lim_{h\to 0^+} \Big\{ g\big((x,t)+hN\big) - g\big((x,t)-hN\big) \Big\} .$$

The strong form of the *Super-Stefan problem* is to find a pair of functions u and w on Ω for which

$$\frac{\partial}{\partial t} c(u) - \Delta k(u) = 0 \text{ in } \Omega_- \cup \Omega_0 \cup \Omega_+ , \tag{3.1.a}$$

$$w \in \mathcal{H}_{\alpha,\beta}(u) \text{ in } \Omega , \tag{3.1.b}$$

$$\big[\nabla k(u)\big] \cdot (N_1, \dots , N_m) = L N_t [w] \text{ on } S_- \cup S_+ , \tag{3.1.c}$$

$$u(x,0) = u_0 , \quad x \in G , \tag{3.2.a}$$

$$w(x,0) = w_0(x) \in [0,1] , \text{ where } \alpha \le u_0(x) \le \beta , \tag{3.2.b}$$

$$u(s,t) = 0 , \quad s \in \partial G , \quad t > 0 . \tag{3.3}$$

For the moment one should assume $w_0(x)$ is either identically zero or identically one; this is only to avoid introducing a mushy region. The classical (possibly nonlinear) heat equation (3.1.a) determines the temperature where $u \ne \alpha$ and $u \ne \beta$. The water fraction is given by the hysteresis functional (3.1.b) which was described above, so we have $w = 0$ in Ω_-, $w = 1$ in Ω_+, and w is either 0 or 1 in Ω_0 according to whether the temperature was last below α or above β, respectively. Let n be the unit vector in the direction $(N_1, \dots , N_m)$, and let V be the *velocity* of S_- or S_+ at time t in the direction of n. By dividing (3.1.c) by $(N_1^2 + \cdots + N_m^2)^{1/2}$, we obtain

$$\left[\frac{\partial}{\partial n} k(u)\right] + LV[w] = 0 \text{ on } S_- \cup S_+,$$

and this is equivalent to (3.1.c). It means the difference in heat flux across the *free boundary* S_+ determines the velocity V of that boundary by melting the fraction of ice $1-w = -[w]$ with latent heat L, and similarly the velocity of S_- is determined by the freezing of the fraction of water $w = [w]$. The Dirichlet boundary condition (3.3) is used here for simplicity, but any of the usual types can just as easily be attained. Finally, one can show by a computation of (3.1.a) and (3.1.c) as distributions on $\Omega \times (0T]$ that the *weak form* of the Super-Stefan problem is to find a pair $u \in L^1(\Omega)$, $w \in L^\infty(\Omega)$ for which

$$\frac{\partial}{\partial t} \left(c(u) + Lw\right) - \Delta k(u) = 0 \text{ in } \mathcal{D}'(\Omega) , \tag{3.4.a}$$

$$w \in \mathcal{H}_{\alpha,\beta}(u) \text{ in } \Omega , \tag{3.4.b}$$

and $k(u) \in L^2(0,T; H_0^1(G))$. With the appropriate change of variable, this is just the equation (1.2) with the simple relay hysteresis. In the presence of a distributed source, there may also arise regions consisting of a mixture of ice and water. Such *mushy regions* may then persist into Ω_0.

4. Dynamic Boundary Conditions

Next we describe a problem with the same formal structure, a degenerate-parabolic initial boundary value problem for $t > 0$

$$\frac{\partial}{\partial t} a(u) - \Delta u \ni f \ , \qquad x \in \Omega \ , \tag{4.1.a}$$

$$\frac{\partial}{\partial t} b(v) + \frac{\partial u}{\partial \nu} \ni g \quad \text{and} \tag{4.1.b}$$

$$\frac{\partial u}{\partial \nu} \in c(v - u) \ , \qquad s \in \Gamma \tag{4.1.c}$$

with initial values specified at $t = 0$ for $a(u)$ and $b(v)$. At each $t > 0$, $u(t)$ is a function on the bounded domain Ω in $\mathbb{R}^n$ with smooth boundary Γ, and $v(t)$ is a function on Γ. Each of $a(\cdot), b(\cdot), c(\cdot)$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ [5]. Thus, the system (4.1) consists of a generalized porous medium equation in the interior of Ω subject to a nonlinear dynamic constraint on the boundary.

A rather remarkable variety of boundary conditions are obtained in (4.1). For example, if $b \equiv 0$ we have an explicit *Neumann boundary condition*, and if $c \equiv 0$ it is homogeneous. If $b(0) = \mathbb{R}$ (i.e., $b^{-1} = 0$), then $v \equiv 0$ and we have a nonlinear Robin constraint, and if $c(0) = \mathbb{R}$ we get $v = u$ on Γ, and this satisfies a nonlinear *dynamic boundary condition*

$$\frac{\partial}{\partial t} b(u) + \frac{\partial u}{\partial \nu} \ni g. \tag{4.1.b'}$$

If $b(0) = c(0) = \mathbb{R}$ we have the homogeneous *Dirichlet boundary condition*. Our interest in (4.1) arises primarily from the fact that (4.1.b) together with (4.1.c) can represent *boundary hysteresis*. One can include a superposition of such generalized plays as in (2.1) and, thereby, a *Preisach hysteresis* on the boundary.

We have shown in [10] that the dynamics of the problem (4.1) is given by a nonlinear semigroup of contractions on the Banach space $L^1(\Omega) \times L^1(\Gamma)$. This was announced in [11] with some numerical examples. Although the hysteresis effects obtained from the pair of graphs $b(\cdot)$, $c(\cdot)$ were our primary motivation, we were able to include the third graph $a(\cdot)$ with no essential additional difficulty. This is merely a reflection of the power of the method which was developed in [23]; this method permits the extension to gradient nonlinearities of p-Laplacean type in (4.1.a) as well as corresponding elliptic Laplace-Beltrami operators in (4.1.b) for the manifold Γ. See [19] for a treatment of the degenerate case $a(\cdot) = 0$ corresponding to a Stefan problem on the boundary Γ. Adsorption in porous media may be governed by conditions on the surfaces of the solid material that are of hysteresis type. If one assumes that the process is governed by certain thresholds, the adsorption rate shows a hysteresis phenomenon of the kind discussed here. In [9] this idea is applied to homogenization of reactive transport through porous media. See also [8], [23].

Finally we would like to indicate the types of estimates that are involved for the problem (4.1), and we will do this for simplicity in the (much simpler) special case of *functions* $a(\cdot)$, $b(\cdot)$, $c(\cdot)$. The (negative of the) generator of the desired contraction semigroup is (the closure of) an operator $\mathbb{C}$ for which the *resolvent equation*, $(I+\varepsilon\mathbb{C})([a,b]) \ni [f,g]$ with $\varepsilon > 0$, takes the form

$$a(u) - \varepsilon\Delta u \ni f \ , \qquad x \in \Omega \ , \tag{4.2.a}$$

$$b(v) + \varepsilon\frac{\partial u}{\partial\nu} \ni g \ , \quad \text{and} \tag{4.2.b}$$

$$\frac{\partial u}{\partial\nu} \in c(v-u) \ , \qquad s \in \Gamma \ , \tag{4.2.c}$$

in the state space $L^1(\Omega)\times L^1(\Gamma)$. In order to show how one obtains the essential estimates that are needed, multiply the respective equations by appropriate functions φ on Ω and ψ on Γ and integrate to obtain

$$\int_\Omega \big(a(u)\varphi + \varepsilon\vec{\nabla}u\cdot\vec{\nabla}\varphi\big)\,dx + \int_\Gamma \big(b(v)\psi + \varepsilon c(v-u)(\psi-\varphi)\big)\,ds = \int_\Omega f\varphi\,dx + \int_\Gamma g\psi\,ds \ . \tag{4.3}$$

This suggests the variational formulation of (4.2) and leads to the essential a-priori estimates. For example, if we choose $\varphi = \operatorname{sgn}(u)$, $\psi = \operatorname{sgn}(v)$ and can obtain simultaneously $\varphi = \operatorname{sgn}(a(u))$, $\psi = \operatorname{sgn}(b(v))$, then we (formally) obtain the stability estimate

$$\|a(u)\|_{L^1(\Omega)} + \|b(v)\|_{L^1(\Gamma)} \le \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\Gamma)} \ . \tag{4.4}$$

By estimating similarly the *differences* of solutions, we establish that the *resolvent* map $[f,g] \mapsto [u,v] \to [a(u),b(v)]$ is a contraction, and this is the *accretiveness* of the operator $\mathbb{C}$. Under additional conditions on the monotone graphs $a(\cdot)$, $b(\cdot)$, and $c(\cdot)$, we find that $\mathbb{C}$ is m-accretive as desired.

5. Remarks and Extensions

We mention a useful modification of the enthalpy functional (3.1.b). The simple relay $\mathcal{H}_{\alpha,\beta}(u)$ described above is an idealization. Specifically, during the phase change the temperature u does not likely remain exactly constant but increases at a very small rate. Thus, it is reasonable to replace the Heaviside relation by a (single-valued) monotone function which closely approximates it. Also, if one could manage to force temperatures past either of the phase-change temperatures, the phase w, would not be expected to respond instantly, but only at a very high rate. We can easily accomodate both of these modifications, and the latter corresponds to a *dynamic model* of hysteresis [20], [21] which is both useful for numerical simulation and suggests a technique for dealing with models of *vector hysteresis*.

There are many generalizations of hysteresis to operators on *vector valued* functions, but it quickly becomes clear that the inclusion of such operators in *systems* of $PDE's$ is difficult. Even for the simplest case of a diagonal Laplace operator, one can apply the techniques above to systems of the form (1.2) essentially only when the hysteresis operator is also diagonal, so the system can be coupled only in strictly lower order terms. Moreover, for even the simplest isotropic diagonal hysteresis operators, the techniques described above will fail for (1.2) with the highly degenerate *double curl* operator (in place of Laplace) that arises in the special quasistatic parabolic case of the problem of conduction in an Ohmic ferromagnetic material. By using certain *dynamic models* mentioned above, one can combine approximate Preisach hysteresis operators with systems of $PDE's$, not only of the *parabolic* type (1.2) arising in the quasistatic case above, but also in the original *hyperbolic* system of Maxwell equations [22].

On the contrary, other classes of hysteresis operators can be paired with rather general nonlinear parabolic or hyperbolic *systems* [16]. The theory of evolution equations in Hilbert space can be used to obtain well-posedness results for semilinear systems of partial differential equations containing a class of vector hysteresis functionals which includes *generalized play* and continuous sums of such functionals, known as Prandtl-Ishlinskii type. These include the wave equations of the form

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\left(c\,\mathbf{v}+\mathcal{H}(\mathbf{v})\right)+\mathbf{b}\left(\mathbf{v}\right)\right)-\Delta\mathbf{v}=\mathbf{F}\ , \tag{5.1}$$

in which $c \geq 0$ and $\mathbf{b}$ is (possibly) nonlinear, as well as similar viscous damped equations of the form

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\left(c\,\mathbf{v}+\mathcal{H}(\mathbf{v})\right)-\Delta\mathbf{v}\right)-\Delta\mathbf{v}=\mathbf{F}\ . \tag{5.2}$$

The problem of coupling equations of motion with an elasto-plastic constitutive law is the subject of [13] in which the hysteresis is combined with a *scalar* quasilinear wave equation. Moreover, by developing and exploiting the dissipative effects and memory structure of very large classes of hysteresis operators, classical compactness and monotonicity methods are used very effectively.

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