Hyperbolic Lunes

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Abstract

The formula for the area of a hyperbolic triangle in terms of its angle defect is derived using *hyperbolic lunes*, in analogy with the argument using (elliptic) lunes to express the area of an elliptic triangle in terms of its angle excess. Several pedagogical features of this construction are then discussed.

1 Introduction

A standard topic in non-Euclidean geometry is the fact that triangles in hyperbolic geometry have angle sum less than π, and further that their area is proportional to this difference. Similarly, in elliptic geometry, which includes spherical geometry, triangles have angle sum greater than π, and their area is again proportional to this difference.

However, the derivations of these results differ considerably in the two cases. Many treatments of non-Euclidean geometry emphasize hyperbolic geometry, sometimes to the exclusion of elliptic geometry altogether. Of the two, hyperbolic geometry is indeed much closer to Euclidean geometry, in the sense that only the parallel postulate needs to be changed. Nonetheless, spherical geometry is more familiar to students, and of course has historical relevance to cartography and navigation.

In hyperbolic geometry, area is typically (see for example [1]) investigated by using side-angle-side congruence (SAS) to establish *equivalence* of regions that can be decomposed into congruent triangles, and then relating equivalence to angle sums. This derivation is rather lengthy, and not entirely straightforward to generalize to elliptic geometry. However, there is an easy derivation of the area formula in spherical geometry, using lunes.

Why not go the other way?

One method of doing so is presented below. This idea is certainly not original to the author, although the presentation here may be new. Among other goals, this paper is intended as a resource for those who might wish to bring this approach into the classroom.

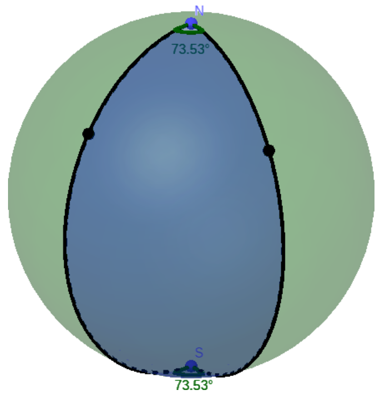

Figure 1: A spherical lune.

A quick summary of the spherical case is given in Section 2, although in the setting of single elliptic geometry rather than double elliptic (that is, spherical) geometry. The hyperbolic case is discussed in Section 3, and further discussion of the pedagogical aspects of this approach appears in Section 5.

2 Elliptic Lunes

A spherical *lune* consists of an "orange-peel" segment between two intersecting (spherical) line segments (not lines) connecting a pair of antipodal points, as shown in Figure 1. The *angle* of a spherical lune is either of the two congruent interior angles at the intersection points. There are of course *two* lunes shown in Figure 1, although we will henceforth restrict angles to lie between 0 and π, inclusive. The area A_S of a spherical lune with angle α is clearly the fraction $\alpha/2\pi$ of the area of the sphere, so

$$A_S(\alpha) = \frac{\alpha}{2\pi} 4\pi r^2 = 2\alpha r^2 \tag{1}$$

where r denotes the radius of the sphere.

It is well known that the area of a spherical triangle can be determined by considering the overlapping spherical lunes formed at its vertices. Here, we give that argument a twist by working in the Klein disk model of elliptic geometry. [1]

The Klein disk model of single elliptic geometry is obtained from the sphere using stereographic projection from the north pole into the plane through the equator, and then considering only the southern hemisphere. Equivalently, the Klein disk represents the real projective plane, obtained by identifying antipodal points on the sphere. The *points* of this geometric model are therefore either points interior to the disk bounded by the equator or *pairs* of antipodal points on the equator. Elliptic *lines* in this model are arcs of Euclidean circles that intersect the equator in antipodal (Euclidean) points. Since stereographic projection is a conformal mapping, angles in this model are Euclidean.

[1]Do not confuse this Klein disk with the Beltrami–Klein model of *hyperbolic* geometry, which is also,

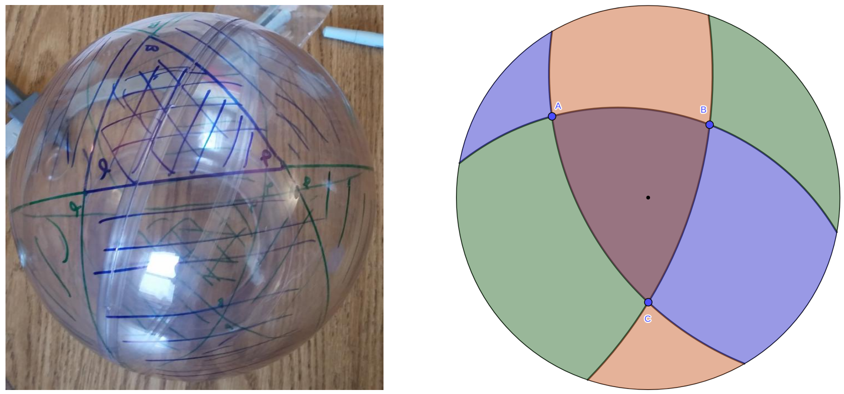

Figure 2: An elliptic triangle covered with lunes on a Lénárt sphere (left) and in the Klein disk (right).

The Klein disk is not orientable, and "wraps around;" points near the boundary are very close to points directly opposite them.

A lune in the Klein disk is the same as in spherical geometry—except that the two antipodal points now represent a single point. The right-hand image in Figure 2 shows an elliptic triangle in the Klein disk covered by three overlapping lunes, with the overlap being the triangle itself. Each lune starts at a vertex of the triangle, proceeds through the triangle to the edge of the disk, wraps around to the other side, and returns to its starting point.

We don't need to know the radius of the underlying sphere in order to use Figure 2 to obtain a formula for the area of the triangle shown. Since area in the Klein disk is by definition the same as the corresponding area on the sphere, it is clear that the area $A_K(\alpha)$ of a Klein lune is proportional to its angle, α. Writing A_D for the total area of the Klein disk, we then have

$$A_K(\alpha) = \frac{\alpha}{\pi} A_D \tag{2}$$

since the entire disk is covered if $\alpha = \pi$.

We're almost there! If the interior angles of the triangle are α, β, γ, then those are also the angles of the three Klein lunes. But these lunes cover the disk, and overlap only on the triangle, leading to *two* extra copies of the triangle. We therefore have

$$A_K(\alpha) + A_K(\beta) + A_K(\gamma) = A_D + 2A_E \tag{3}$$

from which the area A_E of the triangle can be read off as

$$A_E = \frac{1}{2}\left(A_K(\alpha) + A_K(\beta) + A_K(\gamma) - A_D\right) = \frac{\alpha + \beta + \gamma - \pi}{2\pi} A_D \tag{4}$$

which is proportional to the *angle defect* $\alpha + \beta + \gamma - \pi$, as expected. If the radius r of the underlying sphere is known, then $A_D = 2\pi r^2$—the area of the northern hemisphere—so that

$$A_E = (\alpha + \beta + \gamma - \pi)\, r^2. \tag{5}$$

confusingly, sometimes referred to as the Klein disk.

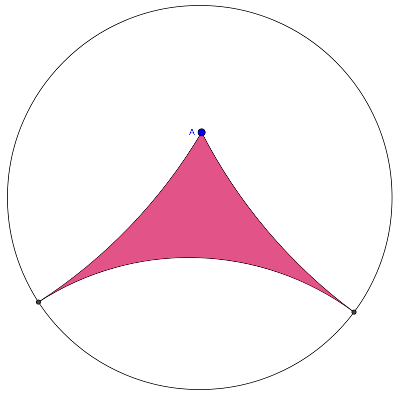

Figure 3: A hyperbolic lune.

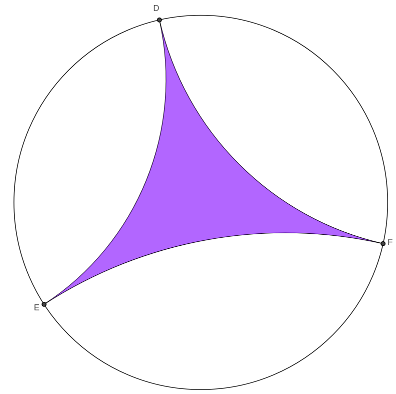

Figure 4: An ideal triangle.

3 Hyperbolic Lunes

Let's try the same approach in hyperbolic geometry. We will use the Poincaré disk model, in which the points are restricted to the interior of the disk, and lines are now arcs of Euclidean circles that intersect the (Euclidean) boundary at right angles (twice). Again, angles are Euclidean, which can be traced to the fact that the Poincaré disk is the stereographic image of the unit hyperboloid in Minkowski space [2]. The Poincaré disk model of hyperbolic geometry is perhaps best known for its appearance in several Escher drawings that seem to show repeated images that get smaller near the boundary of the disk. Once you realize that all of these objects are in fact the same size in hyperbolic geometry, you have grasped the basic nature of this model.

Unlike Euclidean geometry, with its unique parallel lines, or elliptic geometry, with no parallel lines, hyperbolic geometry has infinitely many lines through a given point that are parallel to a given line. But there are exactly two, one on each side, that are *barely parallel*, in the sense that the corresponding Euclidean arcs meet on the boundary. Each of the "legs" of the triangle shown in Figure 3 is barely parallel in this sense to the line at the bottom of the triangle. Due to the perpendicularity condition, barely parallel lines are clearly tangent to each other at the boundary.

So, as with a spherical lune, let's start at a point and go off in two (non-collinear) directions. But now the lines go off to infinity! Mind you, we know where they go—they reach the circle that bounds the Poincaré disk. Points on this circle are called *ideal points*, but are *not* part of the Poincaré disk. Ideal points can be added to any model of hyperbolic geometry; they represent the intersection points of barely parallel lines.

We can make an infinite triangle by connecting the dots, as shown in Figure 3. Any such triangle, which we will henceforth refer to as a *hyperbolic lune*, has three infinite sides. Only one of these sides is a hyperbolic line; the other two are hyperbolic rays. And two of its angles are zero! We use the remaining angle, at the vertex within the Poincaré disk, to label the hyperbolic lune.

What if all three vertices are ideal points? Such triangles, in which all three angles are

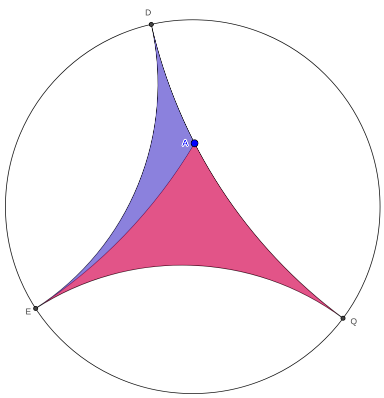

Figure 5: Supplementary lunes.

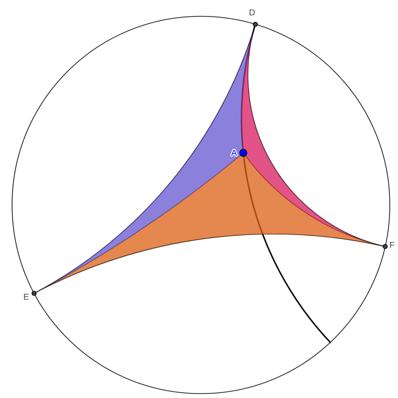

Figure 6: Combining lunes.

zero, are called *ideal triangles*, an example of which is shown in Figure 4. [2] Both hyperbolic lunes and ideal triangles can be thought of as limits to infinity of ordinary (hyperbolic) triangles. It should therefore come as no surprise that the SAS congruence property of (finite) hyperbolic triangles [1] extends to these (infinite) triangles, so long as the angle used for hyperbolic lunes is the angle in the interior of the disk. Thus, two lunes with the same (nonzero) angle are congruent, and all ideal triangles are congruent to each other. [3]

So far, so good: Just as with elliptic lunes, the area of a hyperbolic lune depends only on its angle. In the elliptic case, this result is obvious, since all points on the sphere are visibly equivalent. All points in the Poincaré disk are similarly equivalent; the disk is homogeneous and isotropic, although these properties are less obvious (unless one maps the disk to a Lorentzian hyperboloid, as in [2]).

We are now ready to try to use hyperbolic lunes to determine the area of a triangle. Denote the area of a hyperbolic lune with (nonzero) angle α by $A_P(\alpha)$, and the area of the ideal triangle as $A_I = A_P(0)$.

Step 1: Supplementary Hyperbolic Lunes

Two supplementary lunes combine to make an ideal triangle, that is

$$A_P(\alpha) + A_P(\pi - \alpha) = A_I. \tag{6}$$

Proof: See Figure 5, which shows how to split an ideal triangle into two supplementary lunes.

Step 2: Addition of Hyperbolic Lunes

Hyperbolic lunes can be "added" in the sense that

$$A_P(\pi - \alpha) + A_P(\pi - \beta) = A_P(\pi - (\alpha + \beta)). \tag{7}$$

[2] Hyperbolic lunes are sometimes called *2/3-ideal triangles*.

[3] However, SSS congruence fails, otherwise all hyperbolic lunes would be congruent! Where does the proof fail that shows that SSS congruence follows from SAS congruence?

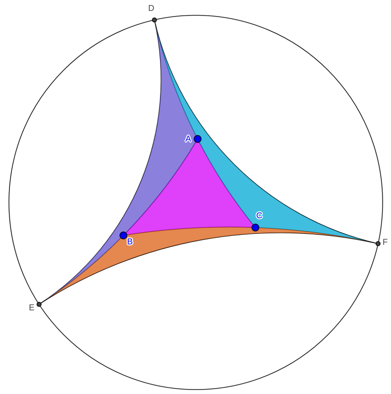

Figure 7: A hyperbolic triangle and its exterior lunes combine to make an ideal triangle.

Proof: See Figure 6. The angles α and β are *below* the marked point, separated by the vertical line. So the lune at the bottom has angle $\alpha + \beta$, and the two lunes above it have angles $\pi - \alpha$ and $\pi - \beta$, respectively. From the figure, we therefore have

$$A_P(\alpha + \beta) + A_P(\pi - \alpha) + A_P(\pi - \beta) = A_I \tag{8}$$

and using (6) leads immediately to (7).

Equation (7) says that the function $f(\alpha) = A_P(\pi - \alpha)$ must be *linear*, and furthermore that $f(0) = 0$. Thus, $A_P(\pi - \alpha) = k\alpha$, or equivalently

$$A_P(\alpha) = k(\pi - \alpha) \tag{9}$$

for some constant k. Furthermore, using (6), we also have

$$A_I = k\pi. \tag{10}$$

We are finally ready to use our lunes to determine the area of a hyperbolic triangle. Consider the triangle shown in Figure 7, together with three *exterior* hyperbolic lunes. Denoting the interior angles at A, B, C by α, β, γ, respectively, the three lunes have the supplementary angles $\pi - \alpha$, $\pi - \beta$, $\pi - \gamma$, respectively, so from the figure we see that

$$A_H + A_P(\pi - \alpha) + A_P(\pi - \beta) + A_P(\pi - \gamma) = A_I \tag{11}$$

where A_H denotes the area of the triangle. Using Step 2, we obtain the expected expression for A_H, namely

$$A_H = k(\pi - \alpha - \beta - \gamma) \tag{12}$$

thus using lunes to show that hyperbolic area is proportional to the defect.

We have therefore succeeded in replicating in hyperbolic geometry the elliptic use of lunes in Section 2 to determine the area of a triangle, despite the fact that the elliptic models are compact, whereas the total area of the Poincaré disk turns out to be infinite (as can be seen from the hyperboloid representation in [2]).

4 Why is the Area Finite?

The careful reader will have noticed a missing step in the construction just given. How do we know that k is finite? It is tempting to argue from (12) that $k < \infty$, since the triangle clearly has finite area. But this conclusion is not obvious, since the construction involved *subtracting* terms that might be infinite, which is not well defined.

It is worth pointing out that the same critique applies to the elliptic case; we have tacitly *assumed* that the (surface) area of the sphere is finite—or, equivalently, that the area of an elliptic lune is finite. This assumption is of course reasonable, both because the sphere is compact, and because we know that $A = 4\pi r^2$, which could, for instance, be derived using integration.

We could adopt the same strategy for hyperbolic lunes, and *assume* that the area of such a lune is finite. Using Step 1 above, this assumption is equivalent to assuming that the area of the ideal triangle is finite. This assumption is again reasonable, if less intuitive. One way to justify this assumption would be to use integration to determine the area of a hyperbolic lune centered at the origin. As in the elliptic case, this approach requires knowing the metric (arclength formula); the metric for the Poincaré disk can be found for instance in [2]. Although such a computation appears to involve additional structure, it is worth reiterating that some similar structure was implicitly invoked in the elliptic case.

It would be nice to have a more elementary argument about the finiteness of the area. One possibility would be to *assume* that the partial ordering provided by "area" on hyperbolic triangles extends to both hyperbolic lunes and the ideal triangle. That is, assume that when two such triangles are combined into a single triangle, the areas of the two component triangles are strictly smaller than the resulting combination, even if these areas are infinite. This assumption permits the subtraction of infinities in some cases, namely those in which a triangle is being decomposed into smaller triangles. One can then conclude that the area function has the linear form (9), and that $k(\alpha - \beta) = k\alpha - k\beta$, with both sides being infinite if k is infinite and $\alpha \neq \beta$. [4] That's enough to conclude from (12) that k must be finite—so long as one is prepared to believe that a *finite* hyperbolic triangle must have finite area.

5 Discussion

The elliptic construction given in Section 2 can, and usually is, done with spherical lunes, and makes a wonderful classroom activity. Lénárt spheres (see the left-hand image in Figure 2), for instructors fortunate enough to have access to them, are an ideal tool for visualizing the result, but any round object students can write on works fine. (Oranges are a good choice; tennis balls less so.)

As is readily apparent when working on an actual sphere (but difficult to see in Figure 2), there are now two triangles, and six lunes. However, the calculation is exactly the same as for Klein lunes, with each term multiplied by two, leading to the same formula for the area, namely (5).

[4] The crucial identity (7) can be verified directly, using SAS, without invoking the ideal triangle as in (8).

Hyperbolic lunes can be introduced immediately afterward, so long as students are already familiar with a model of hyperbolic geometry—any model, not necessarily the Poincaré disk used here. However, students will likely need significant help in constructing Figure 7 themselves; a better choice may be to present this construction as a lecture or whole-class discussion.

Finally, the demonstration that certain objects with infinite sides nonetheless have finite area is well worth discussing with the class as a whole, providing a wonderful opportunity to discuss painting infinite fences. The author's favorite version goes like this:

> A fence is built along the x-axis for $x \geq 1$, with height given by $kx^{-2/3}$ for some constant k.
>
> - What is the area of the fence? How much paint is needed to paint it?
> - If you build a big bucket as a surface of revolution that just fits the fence, what is its volume? How much paint is needed to fill it? (Don't ask how to dip the fence into the bucket...)
> - How much wood is needed to build the bucket?

Acknowledgments

The basic idea for this paper was explored in a supervised research project by a student of the author as part of a course in non-Euclidean geometry [3]. That essay, in turn, built on the outline of the argument given in [4], as also given, in slightly different form, in [5, 6, 7], where credit is given to an 1832 letter by Gauss. The treatment given here first appeared in the author's freely-available online book [8], which also contains interactive versions of some of the figures, constructed using GeoGebra [9].

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