Learning Progressions in Mathematics and Physics: An Example for Partial Derivatives

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We describe the creation of a learning progression about partial derivatives that extends from lower-division multivariable calculus through upper-division physics courses for majors. This work necessitated three modifications to the definition of a learning progression as described in the literature. The first modification is the need to replace the concept of an upper anchor with concept images specific to different (sub)disciplines. The second modification is that rich interconnections between ideas is the hallmark of an expert-like concept image. The final modification is using representations in several ways to support the development of translational fluency in emerging experts. These theoretical changes are supported by examples of research and curriculum in the use of differentials in thermodynamics.

Keywords: Learning progression/trajectory, partial derivatives, multiple representations, representational fluency, thermodynamics.

Learning Progressions

Science education has recently focused on describing learning progressions (LPs) for content that spans multiple years of instruction (Duschl, Schweingruber, & Shouse, 2007; Lemke & Gonzales, 2006); a similar idea, known as a learning trajectory, has been used in mathematics education (Clements & Sarama, 2004, p. 83). Though many of the LPs described in the literature have focused on K-12 instruction, there are science topics at the university level for which a similar model may prove valuable for educators. LPs are typically characterized by a sequence of qualitatively different levels of knowledge and skills. One goal in the development of LPs is to refocus instruction from concepts that are less consequential to those that are more central to mathematics and science (Plummer, 2012). In particular, LPs are not based solely on a logical analysis of mathematics and science ideas—they are sequences that are supported by research on learners' ideas and skills.

Although the research literature includes various definitions for what constitutes a learning progression (Lemke & Gonzales, 2006; Sikorski & Hammer, 2010; Sikorski, Winters, & Hammer, 2009), the National Research Council defines a learning progression to be "the successively more sophisticated ways of thinking about a topic" (Duschl et al., 2007). The range of content addressed by an LP is defined by a lower anchor, which is grounded in the prior ideas that students bring to the classroom, and by an upper anchor, which is grounded in the knowledge and practices of experts. These anchors are identified by research on the thinking of both novices and experts. An LP hypothesizes pathways that students may follow through content, pathways that are then tested empirically (Corcoran, Mosher, & Rogat, 2009). Individual students might follow one of many such pathways, which may be influenced by a variety of factors, including the educational environment.

Some have noted limitations with learning progressions. LPs tend to place students in definite levels of sophistication, when students might in fact give different answers to different questions, making it difficult to place students on a single level. LPs also tend to identify only one scientifically correct upper anchor. We agree with the assessment of Sikorski and Hammer (2010, p.1037) that "rather than describe students as 'having' or 'not having' a particular level of knowledge" recent learning research "conceptualizes students' knowledge as manifold, context-sensitive, and coupled to and embedded in the social and physical environment." In this paper, we describe a perspective on learning progressions that embraces this manifold view of knowledge by incorporating the idea that it is a learner's *concept image* (Tall & Vinner, 1981) that progresses in a way that broadens or enriches a learner's understanding of a topic.

In the next section, we describe three implications of thinking about LPs in terms of concept images: (1) upper anchors must be generalized in a way that allows experts from different content areas to be different from each other, (2) the strength of the interconnections within an individual's concept image are indicative of expertise, and (3) the role that representations and representational fluency play in illuminating the LP must be elaborated. Then, we illustrate our suggested theoretical changes with an example from an LP we are developing on student understanding of partial derivatives, spanning the collegiate curriculum from lower-division multivariable calculus courses through upper-division physics courses in thermodynamics.

Theoretical Additions to Learning Progressions

Experts' Concept Images as "the" Upper Anchor

Interviews with faculty experts (Kustusch, Roundy, Dray, & Manogue, 2012, 2014; Roundy, Weber, et al., 2015) have demonstrated, for example, that physicists and engineers have several ways of reasoning about small quantities that are not shared by mathematicians. These studies, along with our own internal group discussions, have shown that the ways in which experts approach complex problems vary from person to person and from field to field—mathematics experts and physics experts are not the same! We identify the rich and varied understandings of experts with the *concept image* of Tall and Vinner (1981, p.152), *i.e.,* "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes." Thus, we see the goal of an LP not as a definite, idealized upper anchor, but rather as a richer understanding more akin to the concept images of experts from varied fields.

Connections as Indicative of Expertise

Hiebert and Carpenter (1992, p.67) suggest that understanding a mathematical idea requires it to be part of an internal network and that "the degree of understanding is determined by the number and strength of the internal connections." From a concept image perspective, we view a learning progression as describing the enrichment and the increased interconnectivity of a learner's concept image. As developing professionals, middle-division students need to develop such connections rapidly. Yet Browne (2002) found that middle-division students tend not to go back and forth between elements of a concept image spontaneously. To help students increase the strength of their connections, our LP emphasizes opportunities for students to translate between such elements.

Students' ability to transfer knowledge in these ways offers an important means for the empirical validation of our LP. Some of our data (Bajracharya, Emigh, & Manogue, 2017) shows that, while students readily develop a broad concept image, the separate pieces within such a concept image are not necessarily well connected. In contrast, the research discussed above indicates that experts have a rich set of tools that they can use fluently. This representational fluency is itself a key attribute of the upper anchor; achieving such fluency is one of the primary goals of our curricular materials. We regard a learning progression as leading to the enrichment of students' concept images.

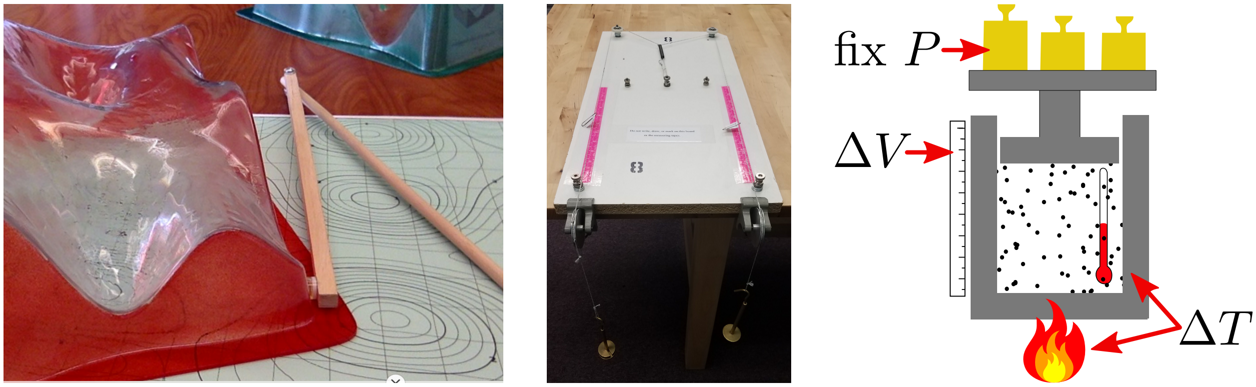
Representations and Representational Fluency

In addition to conceptual knowledge, an important aspect of a learner's concept image of a topic is knowledge of (external) representations, such as graphs, equations, experimental configurations, *etc*. Representations are tools that communicate information between learners and instructors, and that also aid learners with thinking and learning (Hutchins, 1995; Kirsh, 2010). Therefore, representations are centrally featured in our learning progression, both in our instruction and in our research about expert and student reasoning.

In our curriculum, we think about external representations in three ways: as languages for doing mathematics/physics, as disciplinary artifacts, and as pedagogical tools. First, we consider different types of representations to be different languages for doing mathematics and physics. For example, one might calculate a partial derivative at a given point in the domain from an equation, a table of data, or a contour plot. These three different ways of expressing a multivariable function have different features and therefore require different procedures for making the calculation. Starting with an equation requires acting on the equation with a differential operator, thereby transforming one algebraic expression into another, and then the evaluation of the new expression at the desired value in the domain. Starting with a table of discrete data requires reading off values, taking differences, and finding a ratio. In this case, it is necessary to include checks to insure that the differences come from a sufficiently linear regime, with the definition of "sufficiently" depending upon the experimental context (Dray, Gire, Manogue, & Roundy, 2017). We want students to be fluent with each of these representations, and also to be able to coordinate or move between representations. The language metaphor suggests that it should be possible for students to achieve some fluency with representations, which would be consistent with an interconnected concept image.

Second, we think of some external representations as disciplinary artifacts. We use the term artifact to emphasize that they are tools of cultural interest within the discipline. Continuing the metaphor of representations as language, these particular representations play the role of technical vocabulary. This distinction is particularly productive when a representation is commonly used in the professional community but is pedagogically problematic. We want students to be able to communicate with the broader community of mathematicians or physicists, so we make a point of introducing these representations in our instruction. For example, a physicist describing a thermal system might plot, on a single graph, data from two (or more) distinct processes. In cases where the resulting curves intersect, an expert interprets this plot as two smooth functions that describe two different experiments, but some students interpret such plots as a single function with a "cusp," and therefore a discontinuous first derivative (Emigh & Manogue, 2017). Plotting multiple experiments (functions) on a single set of axes is common in physics and physics courses but atypical in mathematics courses.

Third, we use representations as pedagogical tools. In particular, we introduce some representations for their pedagogical affordances even if they are not normative (*i.e.,* used by professionals



(a) Surfaces *(b) PDM* *(c) Name the experiment*

Figure 1: Three representations with pedagogical affordances. (a) A plastic surface and matching contour map. (b) The Partial Derivative Machine (PDM), a mechanical analogue of thermodynamic systems. (c) An experiment to measure $\left(\frac{\partial V}{\partial T}\right)_p$ in which the temperature of a gas in a piston is changed using a burner, and the change in volume is measured with a ruler while the pressure is held fixed by weights on the piston.

while doing their work). For example, professionals do not make plastic surfaces (Wangberg & Johnson, 2013) to represent functions of two variables. However, these surfaces (see Figure 1a) are useful tools for helping students understand many multivariable calculus concepts, including partial derivatives, level curves, the gradient, and line/surface/volume integrals. Similarly, the Partial Derivative Machine (Figure 1b) is a mechanical system that was invented to help students understand thermal systems because the two systems have the same underlying mathematical structure (Sherer, Kustusch, Manogue, & Roundy, 2013). However, those who study thermal systems do not use Partial Derivative Machines in their research.

Example from a Partial Derivatives Learning Progression

In this section, we describe selected elements from a learning progression for partial derivatives that spans advanced undergraduate courses in mathematics and physics. We focus on partial derivatives because, to physicists, partial derivatives are physically meaningful quantities. We begin by describing an instructional activity that is part of our overall LP and that focuses on key elements of the concept image for partial derivatives. Then, we highlight several results from a research project that has informed our LP and has suggested new curricular changes.

The "Name the Experiment" Instructional Activity

A typical example of a thermodynamic system is a gas in a piston (Figure 1c). Such a system has a number of physical properties that may be measured and controlled, such as temperature T, pressure p, volume V, and entropy S. These properties are not independent, as the *state* of the system (when in equilibrium) is defined by just *two* of these quantities. Each of these four quantities—as well as any other measurable property of a gas—is referred to as a *function of state*, meaning that its value is fully determined by the *state* of the system, which itself may be determined by (*i.e.,* may be a function of) any pair of state variables. Physicists denote such dependencies by algebraic statements such as $T = T(S, V)$ which is to be interpreted as "we are currently thinking of the physical temperature T as depending on the physical quantities entropy S and volume V."

We note that this notation is not identical to the function notation commonly taught and used in mathematics.

When encountering partial derivatives in thermodynamics, students have difficulty understanding the significance of the quantity that is being held fixed—a quantity physicists denote using a subscript, as in $\left(\frac{\partial V}{\partial T}\right)_p$ to hold the pressure fixed. The quantity to be held fixed needs to be specified because it is not physically possible to "hold everything else fixed," and there is no unique pair of independent variables describing the system. Roundy, Kustusch, and Manogue (2014) introduced an instructional activity aimed at improving students' overall understanding of thermodynamic variables and what is meant by holding a variable fixed. In the activity, students are prompted to design an experiment that could be done to measure a given partial derivative. One goal of the "Name the Experiment" activity is for students to recognize an experiment as a representation of a particular partial derivative. Linking the experiment—a type of conceptual story—to the algebraic symbols goes beyond simply assigning a physics meaning to each symbol. The experimental story includes a relationship among these physical quantities over time. Figure 1c shows an example of how one could measure $\left(\frac{\partial V}{\partial T}\right)_p$ by heating a gas in a piston, while holding the pressure fixed using unchanging weights on the piston. Determining this derivative requires measuring the small changes ΔV and ΔT and then computing their ratio. This procedure reflects the ratio layer of Zandieh's (2000) framework for concept image for the derivative, as embodied in the experimental representation introduced by Roundy, Dray, Manogue, Wagner, and Weber (2015). This framework for *ordinary derivatives* is the starting point for our concept image for *partial derivatives*.

Research on Representational Fluency with Partial Derivatives

In this section, we present some of our research and describe how it has influenced our LP. This research focused on how students coordinate information from different types of representations. We gave a problem-solving task (see Figure 2a) involving the calculation of a partial derivative from a table of data and a contour graph, neither of which is sufficient on its own to solve the problem. Each of these representation types is commonly used by professional scientists; therefore this task is an appropriate probe of the students' representational fluency. This task was given as a think-aloud interview to students (N=8) who had completed an upper-division thermodynamics course (Bajracharya et al., 2017).

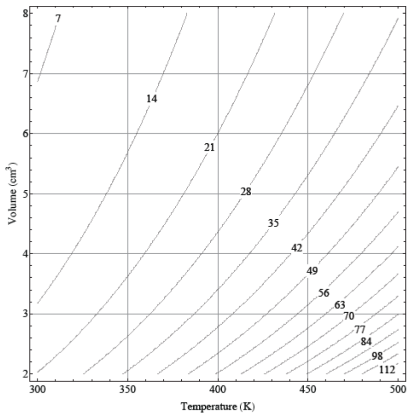
The interview task is a challenging problem with a solution requiring the coordination of many different aspects of the concept image. The analysis suggested that, in order to identify where students are having trouble, it is necessary to examine the individual steps in a solution method at a high level of detail. To facilitate our analysis, we developed a visual means of displaying these steps, which we call a transformation diagram. An example is shown in Figure 2b for one idealized solution to the interview prompt using the method of differential substitution. (Other solution methods, such as sketching a constant-pressure path on the contour map, are also valid and were attempted by students.) In the diagram, boxed items refer to individual representations, arrows refer to transformations between representations, and the transformation steps are numbered for convenience. The transformation diagram is a research tool; we do not (yet) use it as a pedagogical tool. Below, we briefly discuss the interview results pertaining to the solution shown in the diagram, and describe what these results tell us about our curriculum.

The top row of the diagram shows the three different representations of given information: a symbolic expression, a table, and a graph. Each representation gives information about a relationship between three different variables, and this information can be translated (step 1) into a purely

Evaluate $\left(\frac{\partial U}{\partial T}\right)_P$ at P = 10 atm., T = 410K using the information below.

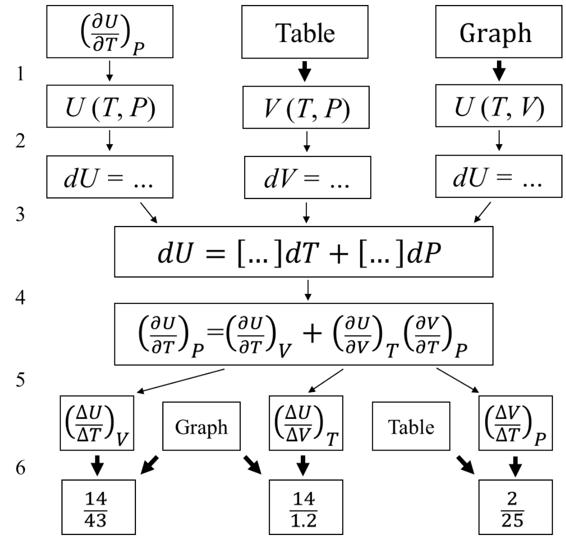
P(atm.)	T(K)	V(cm^3)
10	300	1.32
10	310	1.44
10	320	1.57
10	330	1.71
10	340	1.85
10	350	2.00
10	360	2.15
10	370	2.32
10	380	2.49
10	390	2.67
10	400	2.86
10	410	3.05
10	420	3.25
10	430	3.47
10	440	3.69
10	450	3.91
10	460	4.15
10	470	4.40

Pressure P, Temperature T, and Volume



Internal Energy $U(T, V)$.

(a) Interview Prompt



(b) Transformation Diagram

Figure 2: An interview task (a) focused on coordinating representations, and a diagram (b) showing the transformations between representations in one idealized solution.

symbolic form, such as $U(T, V)$, that explicitly identifies the dependent and independent variables associated with that information. It is then possible to determine the total differential for each representation (step 2). This pair of steps proved surprisingly challenging for some students. We believe this is partly due to the fact that jumping directly from the given information to the total differentials is too big a jump for many students to make. Experts often go through the symbolic representation mentally, but students are rarely taught to use it as an intermediate step. We aim to design instructional sequences that can leverage this result to help students identify and use such stepping stones while solving complicated problems.

Once the total differentials have been found, they can be combined using substitution to eliminate dV, which does not appear in the desired partial derivative (step 3). This expression is compared to the differential form of the multivariable chain rule, $dU = \left(\frac{\partial U}{\partial T}\right)_p dT + \left(\frac{\partial U}{\partial p}\right)_T dp$, to identify the desired partial derivative as the coefficient of dT (step 4). This pair of steps was particularly difficult for the interviewees—they were consistently unable to consolidate information from three separate representations into a single expression. This finding has suggested a new curricular goal for our LP, to help students learn when, how, and why it is necessary to consolidate information in this fashion.

Once a multivariable chain rule has been determined, each of the three new partial derivatives can be approximated (step 5) as a ratio of small changes and then read from the graph or the table (step 6). In practice, we found that few students struggled with either of these steps, once they had a symbolic expression for partial derivatives that were individually calculable from only a single representation of information. This result validates this piece of our learning progression and suggests that elements of our curriculum that focus on finding derivatives from data have been successful and should continue to feature prominently in future instruction.

Conclusion

We have described an expansion of the theory for learning progressions in undergraduate courses, illustrated by the specific example of partial derivatives in mathematics and physics. Our learning progression focuses on students' development of a rich, expert-like concept image involving multiple layers and representations, informed by extensive research on both students and experts. This perspective has led us to develop curriculum that fosters students' ability to go back and forth between many fine-grained representations fluidly and spontaneously.

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