Using Euclidean Tools to Construct Elliptic Objects in the Klein Disk

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Abstract

Constructions using only a straightedge and compass are basic tools in any geometer's toolbox. We show how to construct an elliptic straightedge and compass in the Klein Disk model of (single) elliptic geometry, using only a Euclidean compass and straightedge.

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1 Introduction

At the heart of Euclidean geometry is the notion of constructions using only a straightedge and compass. A *straightedge* allows you to draw the line, and hence also the line segment, between any two points, and a *compass* allows you to draw the circle centered at any point whose radius is congruent to a given line segment. We emphasize that these tools involve no measurement; a straightedge is not a ruler, nor is a compass a protractor.

Much of Euclidean geometry consists of demonstrations using a straightedge and compass, beginning with Euclid's classic construction of equilateral triangles (which is implicit in the construction of the perpendicular bisector in Section 3 below; see Figure 1). Using these tools, and these tools alone, we can copy line segments and angles (and thus duplicate triangles and other polygons), find midpoints, bisect angles—and draw spectacular patterns with overlapping circles.

A major triumph of 19th-century mathematics was the discovery of *non-Euclidean* geometries, in which one or more of Euclid's postulates fail. The geometry of (the surface of) a sphere is a special case of *elliptic* geometry, in which there are *no* parallel lines. Remarkably, elliptic geometry *does* have a notion of both a straightedge and compass. Thus, any Euclidean construction that can be done using only these tools can be converted to an elliptic construction.

In this paper, we show how to construct an elliptic straightedge and an elliptic compass, using only a Euclidean straightedge and compass. This paper is intended primarily for students of non-Euclidean geometry, who have seen Euclidean geometry and are curious about possible generalizations. But there are some surprises along the way, notably the existence of the *line of centers* (see Sections A.2 and A.3), of which we had been previously unaware.

2 The Klein Disk Model of Elliptic Geometry

Elliptic geometry is almost, but not quite, the result of replacing the Euclidean parallel postulate by its elliptic counterpart, namely the assertion that there are no parallel lines, that is, that every pair of lines must intersect. In order to obtain a consistent geometry, however, at least one other postulate must also be altered [1]. In *double* elliptic geometry, whose natural model is the sphere, lines are great circles, and intersect in *two* antipodal points, rather than one. In *single* elliptic geometry, antipodal points are identified, resulting in the projective plane.

A common model of single elliptic geometry is the Klein disk. [1] The *points* in this model are the Euclidean points in the interior of the unit disk, together with *pairs* of antipodal points on the unit circle. The *lines* in this model are arcs of Euclidean circles that intersect the unit circle in a pair of antipodal points, together with the unit circle itself. We include diameters of the unit circle in this definition by considering them to be arcs of circles of infinite radius.

[1]There is also a Klein disk model of hyperbolic geometry.

As discussed in Appendix A.1, stereographic projection maps the sphere to the equatorial plane, with the northern hemisphere mapping to the unit disk. Identifying antipodal points turns the sphere into the projective plane, whose image under stereographic projection is precisely the Klein disk, with great circles on the sphere mapping to (single) elliptic lines in the Klein disk. Since stereographic projection is a conformal map, every circle on the sphere maps to a Euclidean circle in the equatorial plane, where again we include straight lines as circles of infinite radius.

An elliptic line in the Klein disk can be constructed by connecting two antipodal points using a Euclidean circular arc, which is easily done using a (Euclidean) straightedge and compass. Can we use these tools to construct the unique elliptic line connecting two given points? That is, can we construct an elliptic straightedge from these Euclidean tools? What else can we construct with these tools? We discuss a few simple cases in Section 3, leading to constructions of elliptic lines in Section 4, angles in Section 5, and finally circles in Section 6. At this point, we have constructed an elliptic straightedge and compass, as we point out in Section 7. In order to avoid lengthy digressions, a few of the more technical details are relegated to Appendix A.

All of our constructions will be illustrated using GeoGebra [2], and in fact these constructions can be used to create a Klein disk applet in GeoGebra to allow further exploration of elliptic geometry.

3 Basic Procedures

We begin with some elementary Euclidean constructions with straightedge and compass. Recall that a straightedge can be used to draw either the line or the line segment between two given points, while a compass can be used to draw the circle centered at a given point whose radius is congruent to a given line segment. We reiterate that these tools involve no measurement.

The Euclidean Perpendicular Bisector To find the perpendicular bisector of Euclidean line segment AB, construct a Euclidean circle centered on one endpoint, A, whose radius is AB, so that it intersects the other endpoint, B, as shown in Figure 1. Repeat for the other endpoint, drawing a Euclidean circle centered at point B intersecting point A. The resulting circles intersect at points C and D. Next, draw the Euclidean line connecting points C and D. Line CD, as shown in Figure 1, is the perpendicular bisector of segment AB, as can be seen by examining congruent triangles.

The Center of a Euclidean Circle The center of a Euclidean circle has the property that it is the intersection of all perpendicular bisectors of chords of the circle. A chord is a segment intersecting any two points on the perimeter of the circle. Two chords are therefore enough to locate the center of the circle. Draw any two chords, such as segments AB and CD in Figure 3. (You can choose B and C to be the same point, but this is not necessary.) Next,

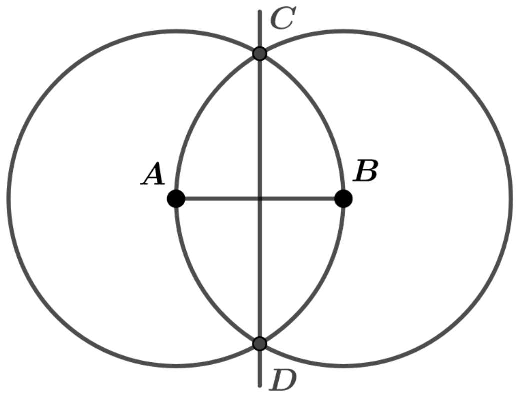

Figure 1: The line connecting the intersections of a circle centered on point A passing through point B and a circle centered on point B passing through point A is the perpendicular bisector of segment AB.

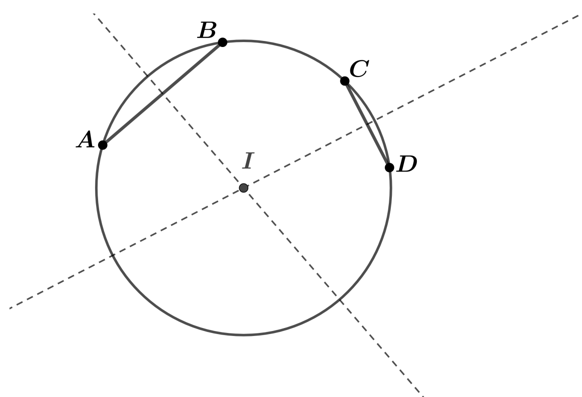

Figure 2: Point I, the intersection of the perpendicular bisectors of segments AB and CD, is the center of the circle.

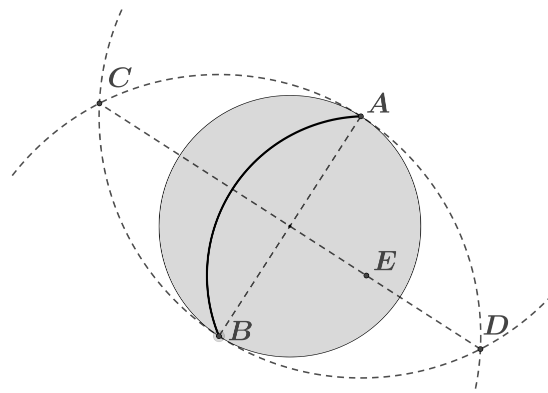

Figure 3: The elliptic line connecting points A and B is an arc of a Euclidean circle centered on point E which lies on line CD.

draw the perpendicular bisector of each chord as explained in Section 3. The intersection of the two perpendicular bisectors is the center of the circle, as shown in Figure 2.

Antipodal Points Given a point A on the perimeter of the unit disk, its antipodal point B can be constructed by first constructing the center of the disk, O, then drawing the Euclidean line through O and B. The (other) intersection of this line with the perimeter of the Klein disk is the antipodal point.

Tangent Lines To find the Euclidean tangent line to a Euclidean circle (such as an elliptic line not passing through the origin) at a given point, first locate the center of the elliptic line using Section 3. Next, construct the segment from the center to the given point, and extend it an equal distance on the other side of the circle by drawing a circle centered on the chosen point and intersecting the center of the elliptic line. Now bisect the extended radius using Section 3; the perpendicular bisector is tangent to the original circle. This construction can be seen in Figure 6 below, where the resulting tangent line to circle ACB at C is line EG.

4 Constructing Elliptic Lines

In this section, we show how to construct the unique elliptic line through two given points, thus constructing an elliptic straightedge (and showing that the analog of Euclid's first postulate holds). We also construct perpendicular lines.

Elliptic Lines Through Antipodal Points Elliptic lines in the Klein disk are arcs of Euclidean circles, whose endpoints must intersect the perimeter of the Klein disk at two antipodal points. The goal of this section is to describe a method of drawing elliptic lines through a given point on the perimeter of the Klein disk using Euclidean tools.

Since an elliptic line connects two antipodal points, start by picking a point A somewhere on the perimeter of the Klein disk, as illustrated in Figure 3. To find its antipodal point

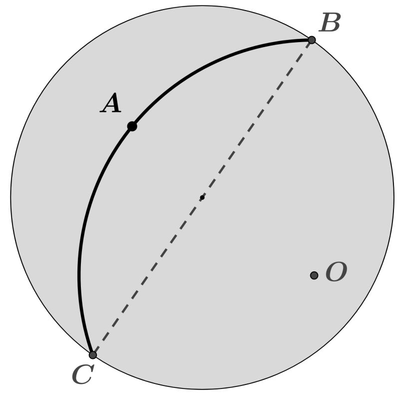

Figure 4: The arc centered on point O connecting points A, B, and C is an elliptic line.

B, follow the procedure in Section 3. Next, construct the perpendicular bisector CD of Euclidean segment AB using Section 3.

Since antipodal points such as A and B represent the same elliptic point, there are infinitely many elliptic lines that can be drawn through any two antipodal points. By symmetry, all such lines are Euclidean circles centered on CD, including the special case of the diameter AB, thought of as an infinite circle. Since A and B are equidistant from every point on CD, any Euclidean circle centered on CD that contains A will also contain B. Thus, the resulting arc in the unit disk is an elliptic line through the elliptic point represented by A and B; an example is shown in Figure 3. If the center (point E) is located at the center of the Klein disk, the elliptic line is along the perimeter of the Klein disk. The farther away from the center of the Klein disk point E is located, the wider the arc. If point E could be placed infinitely far away from the Klein disk, the elliptic line would be a diameter of the Klein disk, namely Euclidean segment AB.

Elliptic Lines Through an Interior Point and Antipodal Points Suppose an interior point A and a perimeter point B are given. In order to construct the elliptic line through A and B, we first construct the antipodal point C of B using Section 3. It only remains to construct the Euclidean circle containing points A, B, and C. We can find the center O of this circle using the construction in Section 3. Drawing the Euclidean arc centered at point O and passing through any one of points A, B, and C, and hence all of them, completes the construction. The resulting elliptic line is shown in Figure 4.

Elliptic Lines Through Interior Points We now consider the generic case of constructing the elliptic line connecting two given interior points, A and B, in the interior of the Klein disk. If the Euclidean line through A and B passes through the center of the Klein disk, we're done; the desired elliptic line is a diameter, and can be constructed directly with a Euclidean straightedge. So we assume without loss of generality that this is not the case. In particular, we assume that neither A nor B is the center of the Klein disk.

Our construction makes use of the remarkable property, discussed in Appendix A.2, that

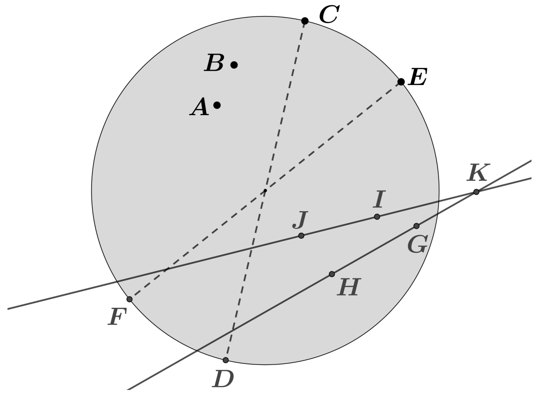

Figure 5: Points G and H are the centers of elliptic lines CAD and EAF, respectively, and points I and J are the Euclidean centers of elliptic lines CBD and EBF, respectively. The elliptic line through A and B must therefore be centered at K, the intersection of the lines of center for points A and B.

the Euclidean centers of all elliptic lines passing through a given elliptic point lie along a Euclidean line, the *line of centers*. We therefore construct *any* two elliptic lines through A, and use them to find the line of centers for A, then repeat the process for B. The Euclidean center of the desired elliptic line, containing both A and B, is the intersection of the two lines of center.

So choose two arbitrary points C and E on the perimeter, and construct their antipodal points D and E. Using chords as explained in Section 3, locate the Euclidean centers G and H of elliptic lines CAD and EAF, respectively. Repeat this construction to find the Euclidean centers I and J of elliptic lines CBD and EBF, respectively. The Euclidean line through points G and H is the line of centers for A; the Euclidean line through I and J is the line of centers for B. The intersection K of these two Euclidean lines is the desired Euclidean center of elliptic line AB, as shown in Figure 5. All that remains is to draw the Euclidean circle with center K passing through A, and hence also through B.

As can be seen in Figure 5, the Euclidean centers of elliptic lines need not lie within the Klein disk. The simplest interpretation of this result is that these centers lie in the stereographic image of the southern hemisphere; see Appendix A.1. Alternatively, our construction can be modified to use the antipodal point lying in the northern hemisphere whenever our construction appears to spill over into the southern hemisphere.

Although we started by excluding radial lines, this construction can still be used so long as neither A nor B is the center of the Klein disk. In this case, the lines of center will be parallel, intersecting only at the point at infinity—which is indeed the center of the infinite circle containing A and B. With the language we have adopted, our construction does not need to be modified for this case.

Perpendicular Elliptic Lines Finally, we show how to construct perpendicular lines. Perpendicular lines in elliptic geometry are defined the same way as in Euclidean geometry, namely as lines that meet at right angles. Angles in the Klein disk are measured with

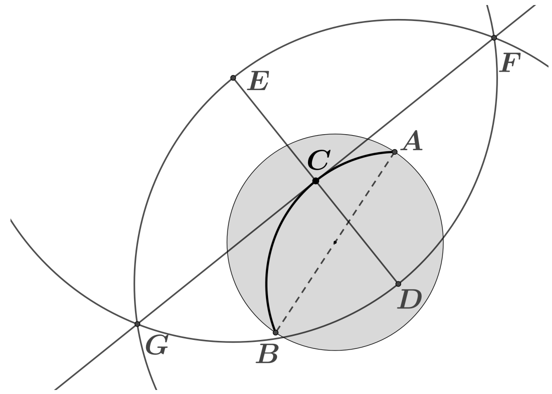

Figure 6: Line FG is perpendicular to segment ED, so an elliptic line intersecting point C perpendicular to elliptic line ACB must also be perpendicular to line FG.

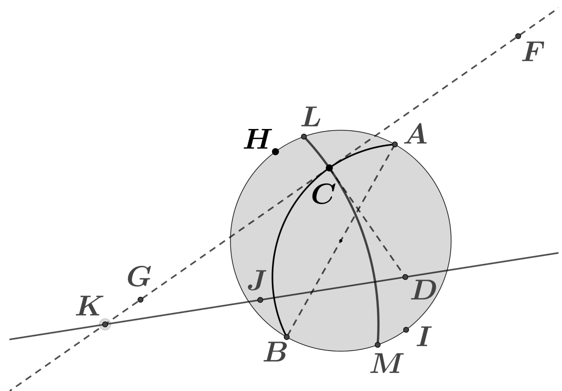

Figure 7: The construction of perpendicular elliptic line LM to the given line ACB, using the tangent line FG and the line of centers JD.

respect to the Euclidean tangent lines of the elliptic lines forming the angle. However, in Euclidean geometry, perpendicular lines intersect in exactly one point, while in elliptic geometry all lines—perpendicular or not—intersect in either one or two points, depending on the model used. In the spherical model of elliptic geometry, perpendicular lines intersect in two antipodal points, whereas in the Klein disk model, perpendicular lines intersect in only one point.

If the given elliptic line is a diameter, the techniques in Section 4 suffice to construct elliptic lines perpendicular to it. Given a point C on an elliptic line that is not a diameter, first construct its elliptic center, point D, and then construct the Euclidean tangent line to the Euclidean circle forming the elliptic line at C, using the procedure in Section 3. Since elliptic angles are measured using Euclidean tangent lines, segment CD must be tangent to the perpendicular elliptic line at point C. Thus, the center of the Euclidean circle of which the perpendicular elliptic line is an arc must lie along a Euclidean line perpendicular to CD at point C, that is, along the tangent line FG. See Figure 6.

To determine the location of the Euclidean center of the perpendicular elliptic line, we

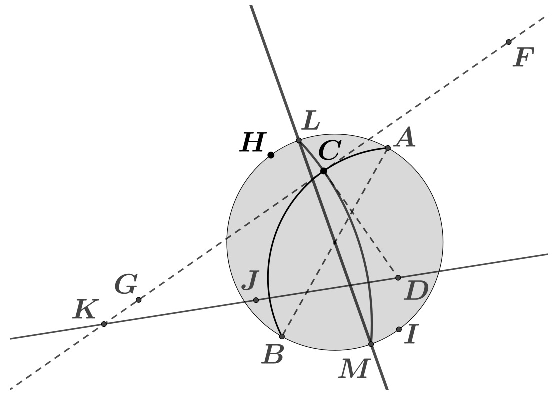

Figure 8: Euclidean line LM passes through the center of the Klein disk, so Euclidean circular arc LM is an elliptic line.

will again use the line of centers property: All elliptic lines through C are circles whose centers lie on a Euclidean line. This line of centers must contain point D, since elliptic line ACB, which contains point C, is a Euclidean arc of a circle centered at point D. So all we need to do is construct *any* other elliptic line through C, and find its Euclidean center J; the line of centers is the Euclidean line JD. The Euclidean center of the desired perpendicular line is now constrained to lie on both FG and JD, and must therefore be their intersection.

The construction just outlined is shown in Figure 7, with some steps omitted. Points H and I are arbitrary antipodal points; the elliptic line through H, C, and I (not shown) is constructed using the procedure in Section 4, with Euclidean center J. The line of centers JD is shown, as is its intersection K with the tangent line FG. Finally, the desired perpendicular elliptic line is the Euclidean circle centered at K containing C. As an additional check, constructing the Euclidean line LM (Figure 8) shows that it passes through the center of the Klein disk; the Euclidean arc LM is indeed an elliptic line.

5 Duplicating Elliptic Angles and Line Segments

In this section, we show how to duplicate an elliptic angle and, as a consequence, how to duplicate an elliptic line segment. These tools underlie the construction of an elliptic compass in the next section.

Duplicating an Elliptic Angle As noted at the beginning of Section 4, an elliptic angle is equal to the Euclidean angle formed by Euclidean lines tangent to the two elliptic lines forming the angle. Thus, to duplicate an elliptic angle, simply duplicate the corresponding Euclidean angle and then constructing elliptic lines tangent to the two Euclidean lines at the vertex of the angle.

The straightforward process for duplicating a Euclidean angle using a Euclidean straightedge and compass is illustrated in Figure 9. First, draw a circle of arbitrary radius at the original vertex (point A), and construct its intersection points with the sides of the given

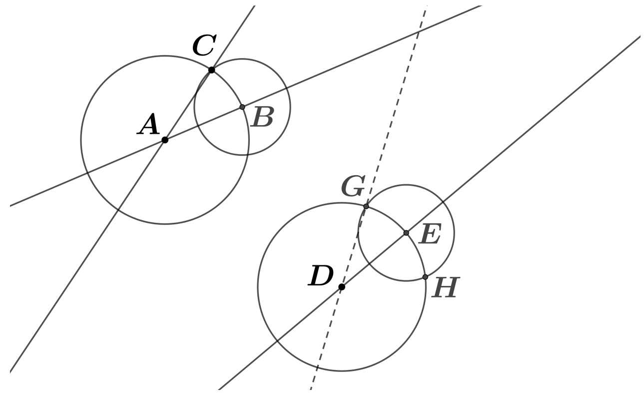

Figure 9: Duplicating a Euclidean angle using a Euclidean straightedge and compass.

angle (points B and C). Use a compass to draw a circle of the same radius at the new vertex (point D), again constructing its intersection (point E) with the given side (line a) of the angle under construction. Finally, use a compass to draw a circle at this intersection point congruent to the circle centered on point B intersecting point C. Connecting either point of intersection (point G or point H) between these last two circles with the new vertex completes the construction, as shown by the dotted line.

The duplication of an elliptic angle is therefore reduced to, first, constructing the Euclidean tangent lines to the elliptic sides of the given angle, and to the given side, then duplicating the resulting Euclidean angle, and finally constructing the elliptic line tangent to the new Euclidean side. The first step uses the procedure in Section 3, and the second step was discussed above, leaving only the construction of the final elliptic line. But that construction is essentially the same as the last step in the construction of perpendicular lines, as illustrated in Figure 7 and summarized below.

To draw the elliptic line tangent to a given Euclidean line (such as DL in Figure 7) at a given point (C), construct the line of centers (DJ) for the given point and the line (FG) perpendicular to the given line at the given point. The intersection (point K) of these two lines is the Euclidean center of the elliptic (LM) line through the given point that is tangent to the given line.

More generally, the line of centers for the new vertex can be found by drawing *any* two elliptic lines through the new vertex using the procedure in Section 4, determining their Euclidean centers using the procedure in Section 3, and then connecting these two points. By symmetry, the Euclidean centers of each elliptic side must also lie along the Euclidean line perpendicular to the corresponding Euclidean side at the vertex. Construct this perpendicular line, and intersect it with the line of centers, thus obtaining the Euclidean center of the elliptic side. Finally, the elliptic side is obtained by drawing the Euclidean arc intersecting the vertex of the angle, whose ends lie on the perimeter of the Klein disk.

We have implicitly assumed in the construction above that both the original and new vertices are interior points away from the center of the Klein disk, and that none of the elliptic sides are Euclidean lines. It is straightforward to modify the construction for these special

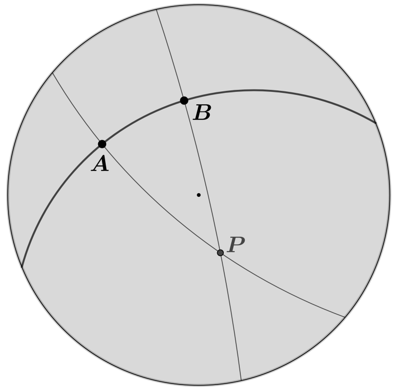

Figure 10: The pole point P of elliptic line AB is the intersection of any two elliptic lines perpendicular to AB.

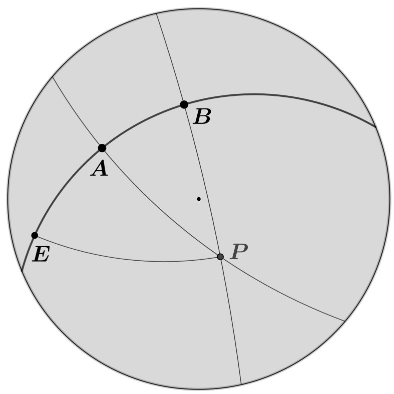

Figure 11: Using the pole P of elliptic line AB to create congruent line segments. Since the two angles at P are congruent by construction, elliptic segments EA and AB are also congruent.

cases, although the figures would look somewhat different. Most of the required changes are in fact simplifications, with an elliptic object reducing to its Euclidean counterpart. For example, if either point is the center of the Klein disk, the corresponding elliptic angle is Euclidean.

Duplicating an Elliptic Line Segment A remarkable property of elliptic geometry is the presence of poles [1]. A *pole* of an elliptic line is a point such that every line segment from the pole to the line meets the line at right angles, and all such segments are congruent. In double elliptic geometry, every line has two pole points, which can be thought of as the north and south pole for a given choice of equator. In single elliptic geometry, every line has exactly one pole point, as illustrated in Figure 10.

On a sphere, it is clear that congruent arcs along the equator correspond to equal angles at the pole, that is, to the same change in longitude. This relationship between angles and congruence holds for any elliptic line and its pole. We can therefore duplicate line segments by duplicating the polar angle.

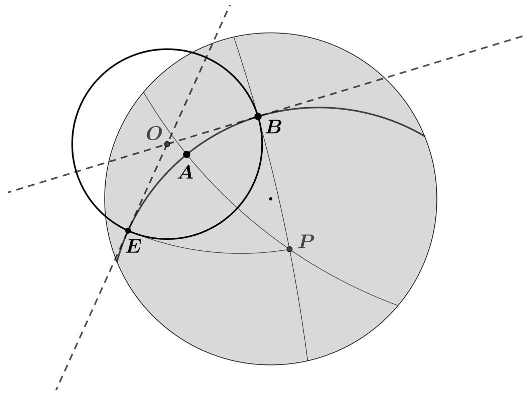

Figure 12: The circle's elliptic center is point A and its Euclidean center is point O.

Explicitly, to make a congruent copy of a given elliptic line segment, construct the elliptic lines perpendicular to the endpoints, and find their intersection, thus constructing the pole of the given line. Now duplicate the angle formed at the pole using the procedure in Section 5, thus constructing a new elliptic line through the pole. The intersection of this line with the original line forms the endpoint of the desired line segment, as shown in Figure 11.

In this case, we have merely extended the given line segment by the same amount, as that is the case we will need in the next section. All that is necessary in order to duplicate an elliptic segment elsewhere is to duplicate the angle at the appropriate pole point. Congruence still holds, for the same reason that arclength along a great circle on a sphere is proportional to the polar angle (longitude), with the ratio depending only on the radius of the sphere.

6 Constructing Elliptic Circles

In this section, we show how to construct an elliptic circle centered at a given point, whose radius is congruent to a given line segment

To draw an elliptic circle centered at a given point A and containing another given point B, first use the procedure in Section 5 to extend the line segment by an equal amount, as shown in Figure 11. As discussed in Section 5, line segments EA and AB are congruent, so point E lies on the desired circle.

We could now draw an arbitrary line through A and use the procedure in Section 5 to find its intersection with the desired circle, then construct the unique Euclidean circle through these several points. But there's an easier way.

Since EA and AB are elliptic radii of the desired circle, they must meet the circle at right angles. Thus, the Euclidean tangent lines to line EAB at E and B also meet the desired circle at right angles, and are therefore radial. The intersection of these two tangent lines, which can be constructed using the procedure in Section 3, must be the Euclidean center of the desired circle. Now that we know the Euclidean center of the circle (point O) as well as at least one point on its perimeter (points B and E), we can use a Euclidean compass to draw the circle, as shown in Figure 12. The *elliptic* center of the circle is, of course, point A.

As shown in Figure 12, the circle constructed by this method may appear to leave the Klein disk. As discussed in Appendix A.1, points outside the unit disk are the stereographic images of points in the southern hemisphere. These points should be mapped to their antipodal points in the northern hemisphere, which can be done by using a suitably scaled reflection through the origin [1].

Although we have once again implicitly assumed that the original points A and B are generic, it is again straightforward to adapt the construction to the various special cases, occasionally with simplifications.

7 Conclusion

Using a Euclidean straightedge and compass, we have constructed the elliptic line containing two given points in the Klein disk (Section 4), and the elliptic circle centered at a given elliptic point that passes through a second, given elliptic point (Section 6). We have thus succeeded in constructing an elliptic straightedge and compass using only a Euclidean straightedge and compass.

One immediate application of this construction is that Euclidean constructions using a straightedge and compass, as given for instance in Section 4.9 of [1], can be applied in elliptic geometry merely by using elliptic tools instead of their Euclidean counterparts. More generally, by building these elliptic tools as sequences of elementary Euclidean operations, they can be programmed as separate tools in, say, GeoGebra [2], thus allowing further exploration of the Klein disk.

So in this respect, at least, non-Euclidean geometries may not be so non-Euclidean after all!

Acknowledgments

This work is a significantly expanded version of a paper with the same title [3] that was submitted by GC as part of a course in non-Euclidean geometry taught by TD. That paper won the 2021 Culture of Writing Award in Mathematics, given annually at Oregon State University to the best paper in a Writing Intensive Course (WIC) in mathematics; WIC courses are a graduation requirement.

A Appendix

A.1 Stereographic Projection

Stereographic projection maps points on the surface of the unit sphere in the direction of the south pole into the equatorial plane, as shown in Figure 13. Stereographic projection is conformal and therefore preserves right angles and circles. Any circle on the sphere through

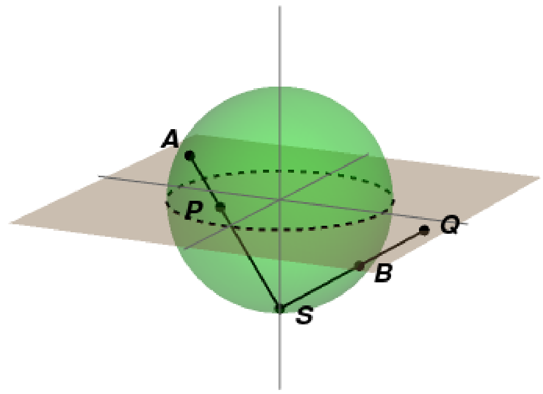

Figure 13: Stereographic projection from the south pole maps the sphere to the equatorial plane. Points in the northern hemisphere map to the interior of the unit disk, whereas points in the southern hemisphere map to its exterior.

the south pole projects to a straight line—in other words, a circle of infinite radius. The projection of the point (x, y, z) on the unit sphere is given by

$$\pi(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right). \tag{1}$$

The south pole is located at $z = -1$, so the south pole projects to the point at infinity in the one-point compactification of the plane.

Stereographic projection can be used to represent both single and double elliptic geometry. The sphere is a model for double elliptic geometry, so its stereographic image, namely the one-point compactification just constructed, is also a model for this geometry. A model for single elliptic geometry is the projective plane, obtained from the sphere by identifying antipodal points. Non-equatorial points can be represented by the northern hemisphere, whose stereographic image is the interior of the unit disk. The equatorial points can now be added back in as pairs of points on the unit circle, yielding precisely the Klein disk.

A.2 Line of Centers: Geometric Proof

A key building block in our construction is the fact that the Euclidean centers of the Euclidean circles representing elliptic lines through an elliptic point in the Klein disk lie along a Euclidean line. We present here a geometric proof of this remarkable fact, followed by an algebraic proof in Section A.3.

Before doing so, we first discuss one exceptional case. Elliptic lines through the origin O are diameters of the unit circle. As elsewhere in this paper, we treat Euclidean lines in the plane as circles of infinite radius; all such circles have their "center" at the single point at infinity in the one-point compactification. We could consider this point to be the "line of centers" corresponding to O, but it is easier to treat this case separately, as we have done throughout the paper. Our assertion about lines of centers is therefore the following:

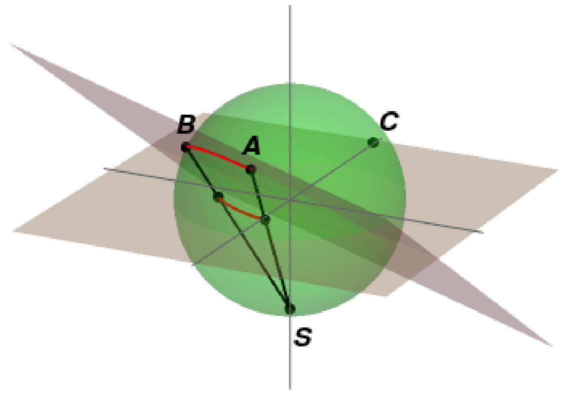

Figure 14: Elliptic segment AB on the surface of the sphere and the reflection of the south pole S through plane ABO to obtain point C.

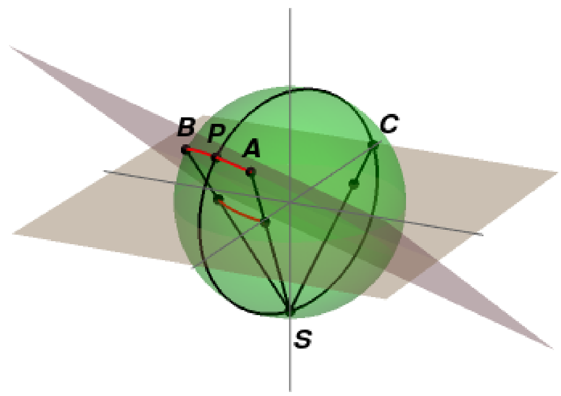

Figure 15: Circle PCS is perpendicular to the great circle AB.

> **Line of Centers:** Let A be any elliptic point in the Klein disk other than the origin. Then the Euclidean centers of all elliptic lines through A lie on a Euclidean line, referred to as the *line of centers* for A.

Consider elliptic line segment AB on the sphere, as shown in Figure 14, which also shows the stereographic projection $\pi(AB)$ of this line segment into the Klein disk. As shown in this figure, construct the plane through A, B, and the origin O, and reflect the south pole S through this plane to obtain point C.

Now let P be any point on the elliptic line segment (great circle arc) AB, and construct the Euclidean circle through P, C, S, as shown in Figure 15. Since points C and S are symmetric about plane ABO, and point P is in plane ABO, circle PCS must be perpendicular to plane ABO at point P. Therefore, the stereographic image of circle PCS is perpendicular to the stereographic image of line segment AB (Figure 16). Since circle PCS contains S, which projects to infinity, its image $\pi(PCS)$ is a Euclidean line.

Since P was an arbitrary point on elliptic line segment AB, *all* Euclidean lines from the image of point C to the image of AB are perpendicular to this latter image, which is the elliptic line segment $\pi(AB)$ in the Klein disk connecting the image points $\pi(A)$ and

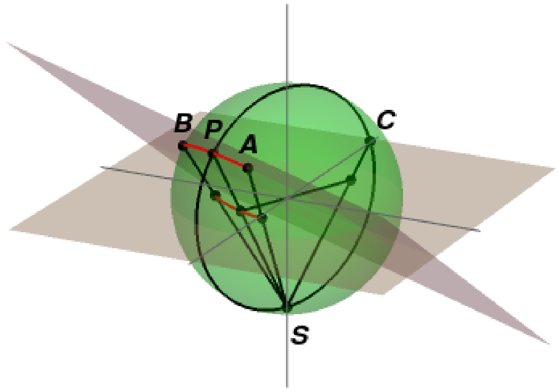

Figure 16: The stereographic image of circle PCS is perpendicular to the stereographic image of great circle AB.

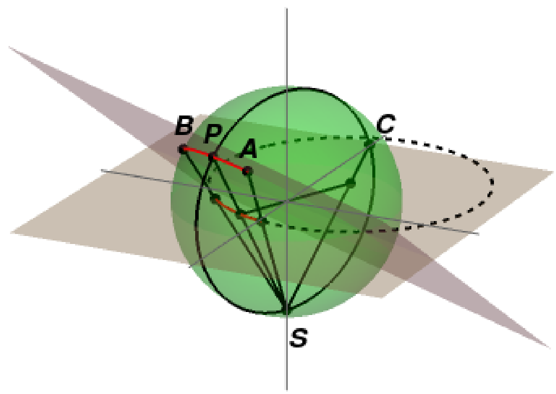

Figure 17: The image of point C is the Euclidean center of the image of great circle AB.

$\pi(B)$, and hence a Euclidean circle. The image point $\pi(C)$ is therefore the center of the this Euclidean circle, as shown in Figure 17.

Finally, since point C is reflected through plane ABO, and point A lies on plane ABO, each of the Euclidean and elliptic distances from A to C is the same as the corresponding distance from A to S. Since S is fixed, as B varies—with A also held fixed—each resulting point C must lie at the same distance from A, as shown in Figure 18. Thus, the possible points C form a circle on the sphere through point S. Since S is projected to infinity, the projection of this circle of possible points C projects to a Euclidean line of possible points $\pi(C)$.

We have therefore shown that the Euclidean centers $\pi(C)$ of all possible elliptic lines in the Klein disk through the point $\pi(A)$ lie along a Euclidean line in the equatorial plane. This line is the *line of centers* for the point $\pi(A)$ in the Klein disk.

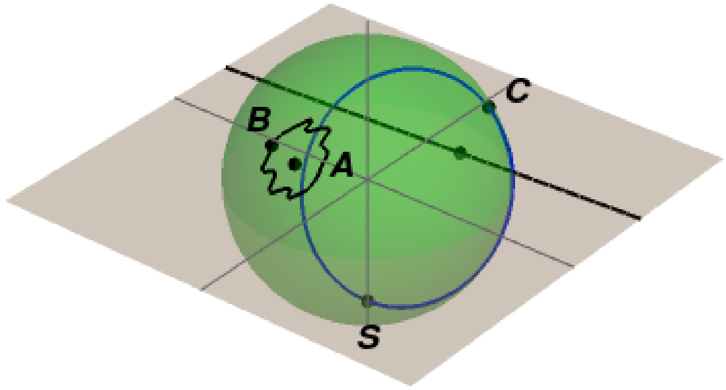

Figure 18: As point B varies (small closed locus), with A (and of course S) held fixed, all possible points C lie the same distance from point A (large circular locus), projecting to the line of centers (linear locus).

A.3 Line of Centers: Algebraic Proof

The goal of this paper is use geometric Euclidean tools to construct geometric elliptic tools in the Klein disk. Having given a purely geometric construction of the line of centers in the previous section, we now give a purely algebraic derivation of the same result.

The equation of a Euclidean circle in the plane of radius r centered at (h, k) is

$$(x-h)^2 + (y-k)^2 = r^2. \tag{2}$$

An elliptic line (not through the origin) is an arc of a circle containing the antipodal points $\pm(p, q)$, which satisfy

$$p^2 + q^2 = 1, \tag{3}$$

assuming that the Klein disk has radius 1 and is centered at the origin, $(0, 0)$. Any circle containing both of $\pm(p, q)$ must satisfy

$$(p \pm h)^2 + (q \pm k)^2 = r^2. \tag{4}$$

Adding and subtracting these two equations and using (3) yields

$$h^2 + k^2 + 1 = r^2 \tag{5}$$

and

$$ph + qk = 0, \tag{6}$$

respectively.

Consider now an elliptic line passing through point (a, b) in the Klein disk interior. We assume for simplicity that the point lies in the interior of the unit disk, and that the line does not pass through the origin; it is straightforward to handle these alternatives as special cases. Equation (2) now becomes

$$(a-h)^2 + (b-k)^2 = r^2 \tag{7}$$

and inserting (5) into (7) yields

$$(a-h)^2 + (b-k)^2 = h^2 + k^2 + 1 \tag{8}$$

which can be rearranged into a linear equation relating h and k. Explicitly, we have

$$a^2 + b^2 = 2(ah + bk) + 1 \tag{9}$$

and therefore

$$2bk = a^2 + b^2 - 2ah - 1 \tag{10}$$

so that finally

$$k = \frac{a^2 + b^2 - 1}{2b} - \frac{a}{b}h. \tag{11}$$

It follows that the centers (h, k) of all elliptic lines through a single point (a, b) lie along a Euclidean line with slope $-\frac{a}{b}$.

References

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