A visual description of (some) Lie algebras

Tevian Dray, Isabella Johnson, Corinne Manogue

Departments of Mathematics & Physics
Oregon State University

Oregon State University

Acknowledgments

Based on Isabella Johnson's senior thesis [1], a degree requirement for her Bachelor of Science in Physics degree at Oregon State University in 2020.

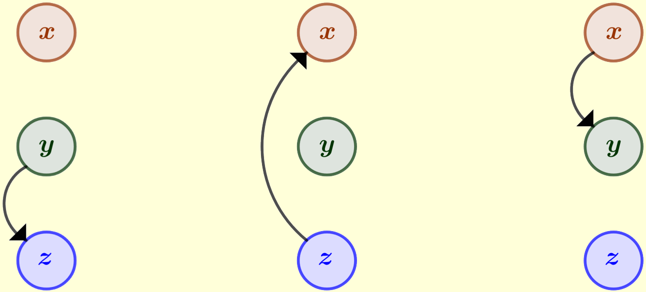
1. Isaballa Johnson, *Describing Particles with Lie and Clifford Algebras*, Senior Thesis, Department of Physics, Oregon State University, 2020.

Motivation

- Rotations $\longmapsto$ orthogonal groups $\longmapsto$ Clifford algebras.
- Unitary groups $\approx$ "complex rotation groups".
- Symmetry groups $\approx$ rotation groups over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$.
- Infinitesimal rotations $\longmapsto$ Lie algebras $\approx$ rank 2 elements of $\mathrm{C}\ell(n)$.

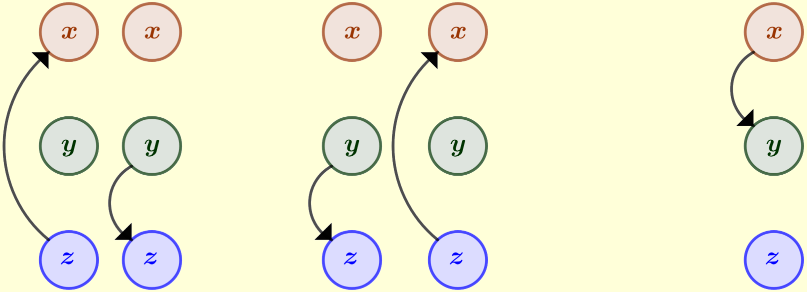
$\mathfrak{so}(3)$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} : \qquad r_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad r_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad r_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



Lie algebra

- $r_y r_x$ means do r_x first, then r_y!

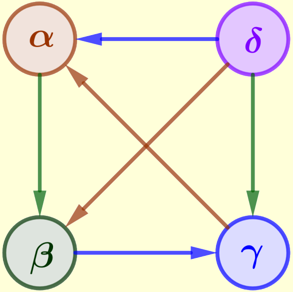


$r_y r_x : y \longmapsto z \longmapsto x$ $\quad$ $r_x r_y : x \longmapsto -z \longmapsto y$ $\quad$ $[r_x, r_y] = r_x r_y - r_y r_x = r_z$

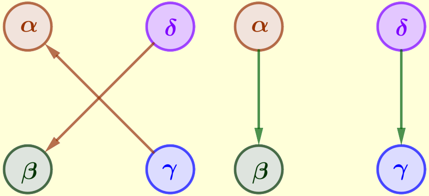
$$[r_x, r_y] = r_z, \ [r_y, r_z] = r_x, \ [r_z, r_x] = r_y$$

$\mathfrak{su}(2)$

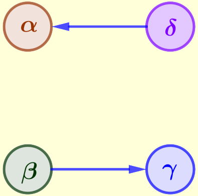
$$\begin{pmatrix} \alpha + \delta\, i \\ \beta + \gamma\, i \end{pmatrix} : \qquad s_x = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \qquad s_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad s_z = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$



$\mathfrak{su}(2) \cong \mathfrak{so}(3)$



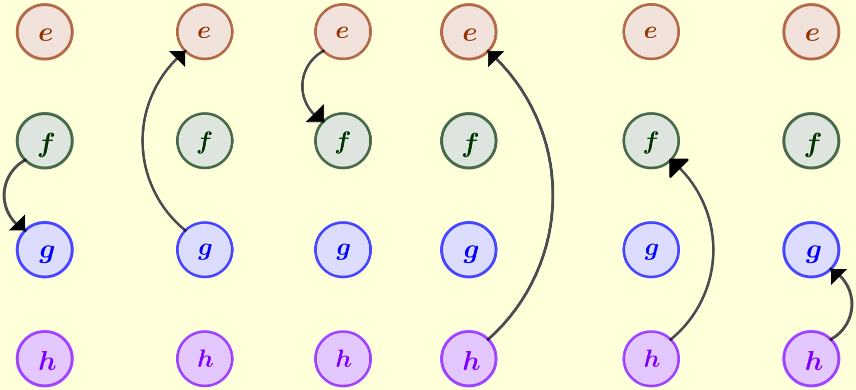
$s_x s_y : \delta \longmapsto \gamma \longmapsto \alpha; \beta \longmapsto -\alpha \longmapsto \gamma$



$[s_x, s_y] = 2s_z$

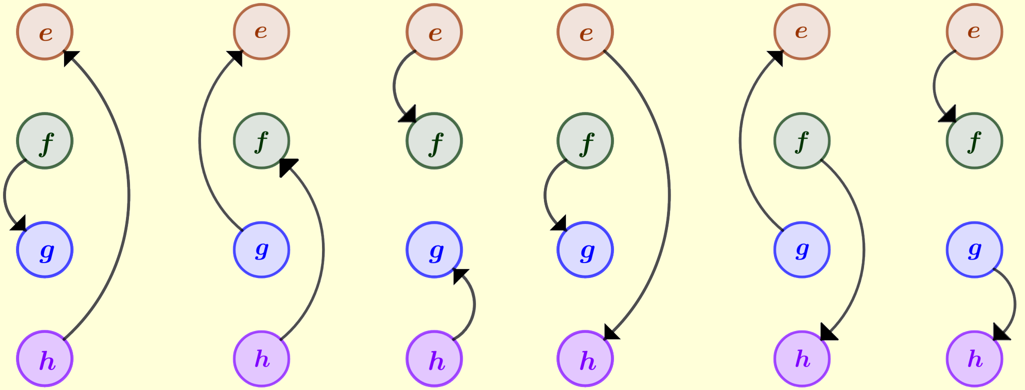
$$[s_x, s_y] = 2s_z, \ [s_y, s_z] = 2s_x, \ [s_z, s_x] = 2s_y$$

$\mathfrak{so}(4)$



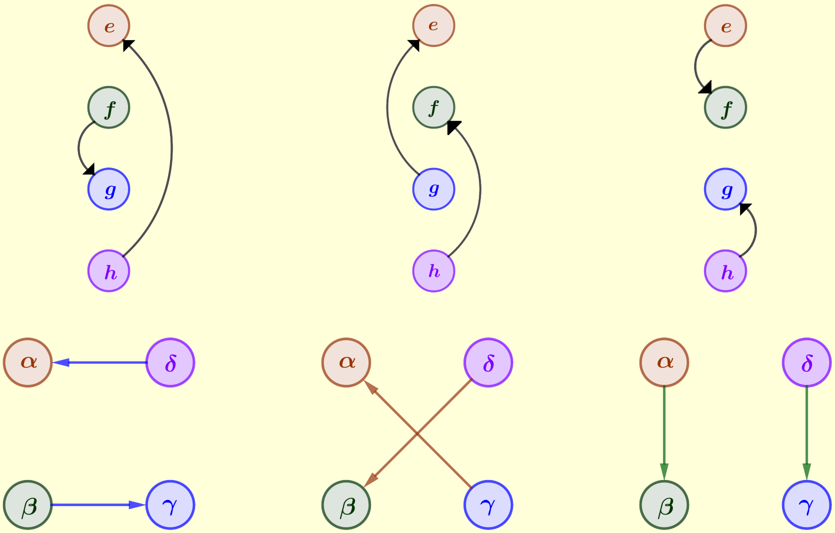
t_x t_y t_z u_x u_y u_z

$\mathfrak{so}(4)$: New Basis

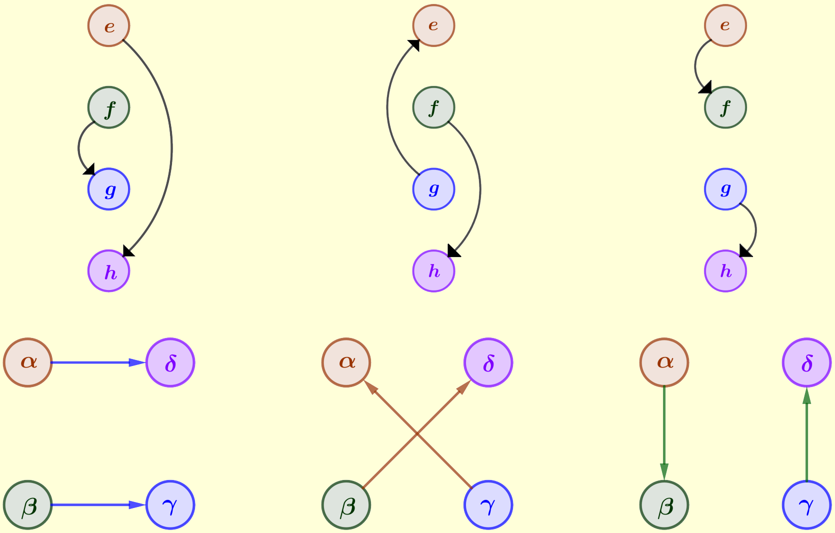


$\mathsf{t_x + u_x}$ $\mathsf{t_y + u_y}$ $\mathsf{t_z + u_z}$ $\mathsf{t_x - u_x}$ $\mathsf{t_y - u_y}$ $\mathsf{t_z - u_z}$

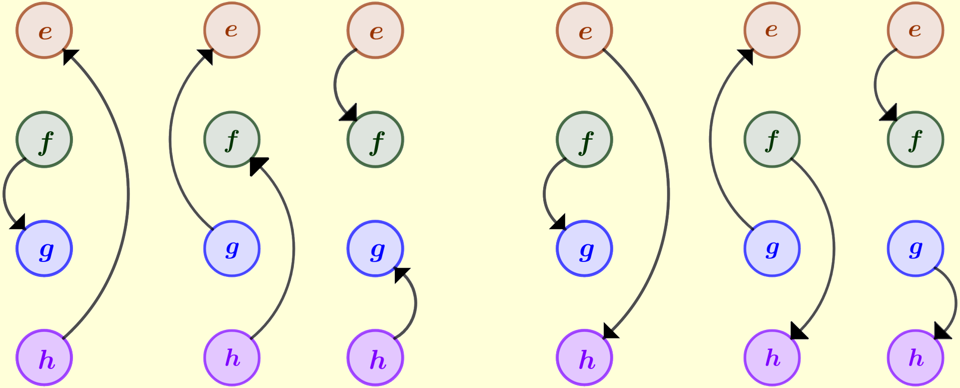
$\mathfrak{su}(2) \subset \mathfrak{so}(4)$



Another $\mathfrak{su}(2) \subset \mathfrak{so}(4)$!



$\mathfrak{so}(4) = \mathfrak{su}(2) + \mathfrak{su}(2)$



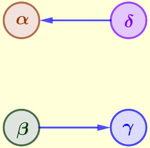
Quaternions

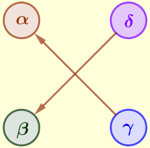
$$q = \alpha + \delta\, i + \beta\, j + \gamma\, k = (\alpha + \delta\, i) + (\beta + \gamma\, i)\, j \in \mathbb{H}$$

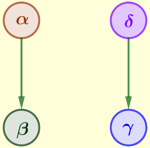
$$\longmapsto \begin{pmatrix} \alpha + \delta\, i \\ \beta + \gamma\, i \end{pmatrix} \in \mathbb{C}^2$$

$$\longmapsto \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \in \mathbb{R}^4$$

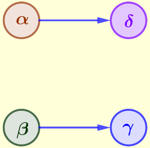
$\therefore$ quaternionic multiplication (by units) $\longmapsto \mathfrak{so}(4)$

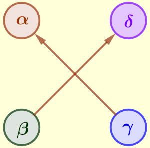


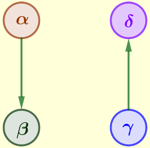




$\mathfrak{su}(2)_{\mathsf{R}}$: $q \longmapsto -qi$ $\qquad q \longmapsto -qk$ $\qquad q \longmapsto qj$







$\mathfrak{su}(2)_{\mathsf{L}}$: $q \longmapsto iq$ $\qquad q \longmapsto -kq$ $\qquad q \longmapsto jq$

SUMMARY

- Visual proof that $\mathfrak{so}(4) = \mathfrak{su}(2)_L + \mathfrak{su}(2)_R$.
- L/R correspond to quaternionic multiplication on the left/right.
- Weak symmetry, GUTs, ...
- Extend to $\mathfrak{so}(n)$, $\mathfrak{su}(n)$, ...

THE END

(Thank you!)