Piecewise Conserved Quantities

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Personal History

- **1987:** Indo-American Fellow @ IMSc, TIFR
  (visits to Pune and RRI)
- **1988:** Returned to RRI and TIFR
- Collaborated with Paddy:
  Tevian Dray and T. Padmanabhan,
  Conserved Quantities from Piecewise Killing Vectors,
  Gen. Rel. Grav. **21**, 741–745 (1989).

  (submitted in November 1988)

Shells

- **Dray & 't Hooft (1985)**
  Gravitational shock wave of a massless particle;
  Shells of matter at horizon of Schwarzschild black hole
- **Clarke & Dray (1987)**
  Junction conditions for null hypersurfaces
- **Dray & Padmanabhan (1988)**
  Conserved quantities from piecewise Killing vectors
- **Dray & Joshi (1990)**
  Glueing Reissner-Nordstrøm spacetimes together
  Reissner–Nordström

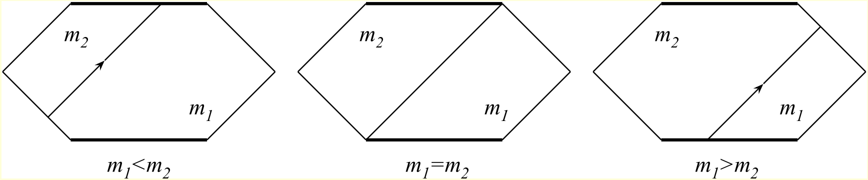
  ⋮
- **Hazboun & Dray (2010)**
  Negative-energy shells in Schwarzschild spacetimes

Signature Change

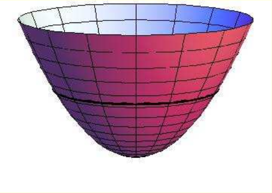
- **Dray, Manogue, & Tucker (1991); Ellis et al. (1992)**
  Scalar field in the presence of signature change
- **Dray & Hellaby (1994); Hellaby & Dray (1994)**
  Patchwork Divergence Theorem
- **Schray, Dray, Manogue, Tucker, & Wang (1996)**
  Spinors and signature change
- **Dray (1996); Dray, Ellis, Hellaby, & Manogue (1997)**
  Gravity and signature change
- **Dray (1997); Hartley, Tucker, Tuckey, & Dray (2000)**
  Tensor distributions in the presence of signature change

Gluing Spacetimes Together

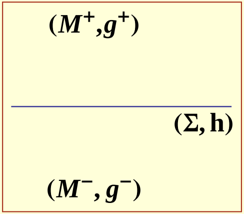
Shells:



Signature Change:



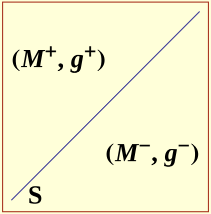
Spacelike Boundaries



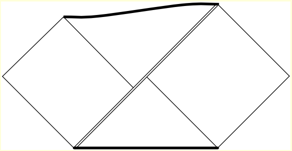
$$g_{ab} = (1-\Theta)\, g^-_{ab} + \Theta\, g^+_{ab}$$
$$[g_{ab}] = g^+_{ab}\big|_\Sigma - g^-_{ab}\big|_\Sigma = 0$$

$$\Longrightarrow R_{ab} = (1-\Theta)\, R^-_{ab} + \Theta\, R^+_{ab} + \delta_c[\Gamma^c{}_{ab}] - \delta_b[\Gamma^c{}_{ac}]$$
$$= (1-\Theta)\, R^-_{ab} + \Theta\, R^+_{ab} + \delta\rho_{ab}$$
$$(\delta_c = \delta n_c = \nabla_c \Theta)$$

Null Boundaries

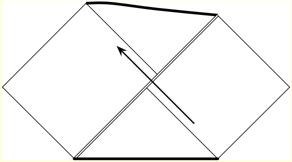


Dray & 't Hooft (1996)



$m > 0$

Hazboun & Dray (2010)



$m < 0$

Conserved Quantities

Piecewise Killing vector:

$$\xi^a = (1-\Theta)\,\xi^a_- + \Theta\,\xi^a_+$$
$$\Longrightarrow \nabla_{(a}\xi_{b)} = \left[\xi_{(a}\right]\delta_{b)}$$

Darmois junction conditions: $([h_{ij}] = 0 = [K^{ij}])$

$$\Longrightarrow [T^{ab}] = 0$$
$$\Longrightarrow \nabla_a(T^{ab}\xi_b) = (\nabla_a T^{ab})\xi_b + T^{ab}\nabla_a\xi_b$$
$$= 0 + T^{ab}\,[\xi_a]\,\delta_b$$

$\therefore$ conserved quantity if $[\xi_a] = \Xi n_a$ & $T^{ab} n_a n_b = 0$

Patchwork Divergence Theorem

Divergence Theorem, X smooth: $(m \wedge \sigma = \omega)$

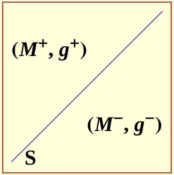
$$\mathrm{div}(X)\,\omega = \mathcal{L}_X\omega = d(i_X\omega) + i_X(d\omega)$$

$$\Longrightarrow \int_W \mathrm{div}(X)\,\omega = \oint_{\partial W} i_X\omega = \oint_{\partial W} m(X)\,\sigma$$

X piecewise smooth: (m_0 from M^- to M^+)

$$\int_W \mathrm{div}(X)\,\omega = \oint_{\partial W} m(X)\,\sigma - \int_\Sigma m_0([X])\,\sigma^0$$

Shells



Two Schwarzschild regions:

$$ds^2 = \begin{cases} -\frac{32m^3}{r} e^{-r/2m} du\, dv + r^2\, d\Omega^2 & (u \le \alpha) \\ -\frac{32m^3}{r} e^{-r/2M} dU\, dV + r^2\, d\Omega^2 & (u \ge \alpha) \end{cases}$$

$$[g] = 0 \Longrightarrow \frac{\alpha}{m} = \frac{U(\alpha)}{MU'(\alpha)} \Longrightarrow \frac{u\partial_u}{m} = \frac{U\partial_U}{M}$$

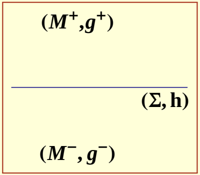
$$\xi = (1-\Theta)\,\frac{v\partial_v - u\partial_u}{4m} + \Theta\,\frac{V\partial_V - U\partial_U}{4M} \Longrightarrow [\xi] \sim \partial_V$$

$$T_{uu} = \frac{\delta}{\gamma\pi r^2}\,(M-m) \Longrightarrow T_{vv} = 0$$

Conserved quantity:

$$-\int_\Sigma \Big((1-\Theta)\; {T^t}_t + \Theta\; {T^T}_T \Big)\, dS = M - m$$

Signature Change



$$ds^2 = \begin{cases} +dt^2 + h_{ij}\,dx^i\,dx^j & (t < 0) \\ -dt^2 + h_{ij}\,dx^i\,dx^j & (t > 0) \end{cases}$$

Volume element is continuous!!

$$\rho := G_{ab}n^a n^b = \frac{1}{2}\left((K^c{}_c)^2 - K_{ab}K^{ab} - \epsilon R\right)$$

$$\text{Darmois} \Longrightarrow [R] = 0 = [K_{ab}] \text{ but } [\epsilon] \neq 0$$

$$\Longrightarrow [\rho] = -R \neq 0$$

"Energy density at change of signature"

Geometric Reasoning

Iedereen in deze kamer kan twee talen praten
(Everyone in this room is bilingual.)

Mathematics $\neq$ Physics

Mathematicians teach algebra;
Physicists do geometry!

Geometric Reasoning

Vector Calculus Bridge Project:

`http://math.oregonstate.edu/bridge`

- Differentials *(Use what you know!)*
- Multiple representations
- Symmetry *(adapted bases, coordinates)*
- Geometry *(vectors, div, grad, curl)*
- Online text (`http://math.oregonstate.edu/BridgeBook`)

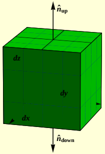
Paradigms in Physics Project:

`http://physics.oregonstate.edu/portfolioswiki`

- Redesign of undergraduate physics major *(18 new courses!)*
- Active engagement *(300+ documented activities!)*

Lorentzian Vector Calculus

Minkowski 3-space:



$$\hat{x}\cdot\hat{x} = 1 = \hat{y}\cdot\hat{y}, \qquad \hat{t}\cdot\hat{t} = -1$$

$$\vec{F} = F^x\,\hat{x} + F^y\,\hat{y} + F^t\,\hat{t}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F^x}{\partial x} + \frac{\partial F^y}{\partial y} + \frac{\partial F^t}{\partial t};$$

Divergence Theorem:

$$\int\!\!\int\!\!\int_W \vec{\nabla}\cdot\vec{F}\;\,dx\,dy\,dt = \int_{\partial W} \vec{F}\cdot\hat{n}\;\,dA$$

where $\hat{n} =$ **outward** normal in spacelike directions,
but $\hat{n} =$ **inward** normal in timelike directions!

SUMMARY

- Junction conditions at null boundary are surprising!
- Junction conditions at signature change are surprising!
- The Divergence Theorem in Minkowski space is surprising!
- All of these results follow from Patchwork Divergence Thm.

- (Working with Paddy was fun!)

THANK YOU

http://oregonstate.edu/~drayt/talks/IUCAA17pub.pdf

https://arxiv.org/abs/1701.02863