A Division Algebra Description of the Magic Square, including E_8

Tevian Dray
(joint work with Robert Wilson and Corinne Manogue)

Department of Mathematics
Oregon State University
http://www.math.oregonstate.edu/~tevian

Oregon State University

(supported by FQXi and the John Templeton Foundation)

With thanks to:

- Rob Wilson, who showed us how to get to E_8 (in 2014...);
- David Fairlie & Tony Sudbery, who got us started in the 1980s,
  Paul Davies, who believed in us from the start,
  David Griffiths, who taught us physics (and math),
  and Jim Wheeler, who explained the conformal group to us;
- Jörg Schray (Ph.D. 1994),
  Jason Janesky (1997–1998),
  Aaron Wangberg (Ph.D. 2007),
  Henry Gillow-Wiles (M.S. 2008),
  Joshua Kinkaid (M.S. 2012),
  Lida Bentz (M.S. 2017),
  and Alex Putnam (M.S. 2017),
  who taught us as much as we taught them;
- John Huerta and Susumu Okubo, who helped along the way;
- and FQXi, the John Templeton Foundation, and the Institute for Advanced Study for financial support.

References

This work: `arXiv:2204.04996 & 2204.05310`

Our group:
Fairlie & Manogue (1986, 1987), Manogue & Sudbery (1989), Schray (PhD 1994), Manogue & Schray (1993), Dray & Manogue (1998ab, 1999), Manogue & Dray (1999), Dray, Janesky, & Manogue (2000), Dray, Manogue, & Okubo (2002), Dray & Manogue (CAA 2000, CMUC 2010), Manogue & Dray (2010), Wangberg (PhD 2007), Wangberg & Dray (JMP 2013, JAA 2014), Dray, Manogue, & Wilson (CMUC 2014), Kincaid (MS 2012), Kincaid and Dray (MPLA 2014), Dray, Huerta, & Kincaid (LMP 2014)

Others:
Jordan (1933), Jordan, von Neumann, & Wigner (1934), Freudenthal (1954, 1964), Tits (1966), Vinberg (1966), Gürsey, Ramond, & Sikivie (1976), Olive & West (1983), Kugo & Townsend (1983), Günaydin & Gürsey (1987), Chung & Sudbery (1987), Goddard, Nahm, Olive & Ruegg (1987), Corrigan & Hollowood (1988), Dixon (1994), Okubo (1995), Günaydin, Koepsell, & Nicolai (2001), Barton & Sudbery (2003), Cederwall (2007), Lisi (2007, 2010), Baez & Huerta (2010), Chester, Marran, & Rios (2021), Furey (2015), Furey & Hughes (2022ab)

Division Algebras

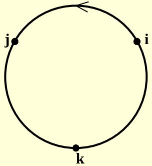
Real Numbers

$$\mathbb{R}$$

Quaternions

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$$

$$q = (x + yi) + (r + si)j$$



Complex Numbers

$$\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$$

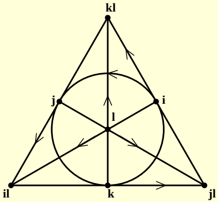
$$z = x + yi$$

Octonions

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell$$

Split Octonions

$$\mathbb{O}' = \mathbb{H} \oplus \mathbb{H}L$$



$$I^2 = J^2 = -U, \ L^2 = +U$$

Split Octonions

$$I^2 = J^2 = -U,\ L^2 = +U$$

Signature $(4,4)$:

$$x = x_1 U + x_2 I + x_3 J + x_4 K + x_5 KL + x_6 JL + x_7 IL + x_8 L \Longrightarrow$$

$$|x|^2 = x\bar{x} = (x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_5^2 + x_6^2 + x_7^2 + x_8^2)$$

Null elements:

$$|U \pm L|^2 = 0$$

Projections:

$$\left(\frac{U \pm L}{2}\right)^2 = \frac{U \pm L}{2}$$

$$(U + L)(U - L) = 0$$

Overview

- $\mathfrak{e}_{8(-24)} = \mathfrak{su}(3, \mathbb{O}' \times \mathbb{O})$
  3 × 3 matrices
- $3 \times 3 \longmapsto 2 \times 2 + 2 \times 1$
  orthogonal group + spinors
- GUT: $\mathfrak{so}(12,4) \supset \mathfrak{so}(3,1) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \otimes \mathbb{C}$
  Standard Model + Lorentz
- Albert algebras $\subset \mathfrak{e}_8$

Next talk: Standard Model

Lie Groups & Lie Algebras

Lie Group:

$$\mathrm{SO}(3) = \left\{ R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_x, R_y \right\}$$

Lie Algebra:

$$\mathfrak{so}(3) = \left\langle r_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, r_x, r_y \right\rangle$$

Properties:

$$R^\dagger = R^{-1}, \quad r_z = \frac{dR_z}{d\theta}\bigg|_{\theta=0}, \quad r_z^\dagger = -r_z \quad [r_x, r_y] = r_z$$

The Freudenthal–Tits Magic Square

Freudenthal (1964), Tits (1966):

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{c}_3$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{a}_2$	$\mathfrak{a}_2 \oplus \mathfrak{a}_2$	$\mathfrak{a}_5$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{c}_3$	$\mathfrak{a}_5$	$\mathfrak{d}_6$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Guiding Principle #1

Lie algebras are real!

(signature matters)

$\mathfrak{so}(3,1)$ has boosts and rotations

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{su}(3,\mathbb{R})$	$\mathfrak{su}(3,\mathbb{C})$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{f}_4$
$\mathbb{C}'$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{e}_{6(-26)}$
$\mathbb{H}'$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3,\mathbb{C})$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}'$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

[Barton & Sudbery (2003), Wangberg (PhD 2007),
Dray & Manogue (CMUC 2010), Wangberg & Dray (JMP 2013, JAA 2014),
Dray, Manogue, & Wilson (CMUC 2014), Wilson, Dray, & Manogue (2022)]

2 × 2 Magic Square

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{so}(2)$	$\mathfrak{so}(3)$	$\mathfrak{so}(5)$	$\mathfrak{so}(9)$
$\mathbb{C}'$	$\mathfrak{so}(2,1)$	$\mathfrak{so}(3,1)$	$\mathfrak{so}(5,1)$	$\mathfrak{so}(9,1)$
$\mathbb{H}'$	$\mathfrak{so}(3,2)$	$\mathfrak{so}(4,2)$	$\mathfrak{so}(6,2)$	$\mathfrak{so}(10,2)$
$\mathbb{O}'$	$\mathfrak{so}(5,4)$	$\mathfrak{so}(6,4)$	$\mathfrak{so}(8,4)$	$\mathfrak{so}(12,4)$

$$d = 3, 4, 6, 10$$

(1980s: Corrigan, Evans, Fairlie, Manogue, Sudbery)
(1990s: Manogue & Schray)

Unified Clifford algebra description using division algebras

[Kincaid (MS 2012), Kincaid and Dray (MPLA 2014),
Dray, Huerta, & Kincaid (LMP 2014)]

Signature matters!

Lorentz Lie algebra: $\mathfrak{so}(3,1)$ $[\det P = -(-t^2 + x^2 + y^2 + z^2)]$

$$P = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$
$$= t\,\sigma_t + x\,\sigma_x + y\,\sigma_y + z\,\sigma_z$$

group: $P \longmapsto MPM^\dagger$ algebra: $P \longmapsto AP + PA^\dagger$

Signature matters!

Lorentz Lie algebra: $\mathfrak{so}(3,1)$ $\quad [\det P = -(-t^2 + x^2 + y^2 + z^2)]$

$$P = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$
$$= t\,\sigma_t + x\,\sigma_x + y\,\sigma_y + z\,\sigma_z$$

Rotations (antihermitian!): $\quad$ (so $P \longmapsto [A, P]$)

$$A = i\sigma_x, i\sigma_y, i\sigma_z$$

Boosts (hermitian!): $\quad$ (so $P \longmapsto \{A, P\}$)

$$A = \sigma_x, \sigma_y, \sigma_z$$

Signature matters!

Lorentz Lie algebra: $\mathfrak{so}(3,1)$ $\quad [\det P = -(-t^2 + x^2 + y^2 + z^2)]$

Vector in $\mathbb{C}' \oplus \mathbb{C}$

$$P = \begin{pmatrix} Lt + Uz & 1x - iy \\ 1x + iy & Lt - Uz \end{pmatrix}$$

$$= Lt\,\sigma_t + 1x\,\sigma_x + iy\,(-i\sigma_y) + Uz\,\sigma_z$$

Rotations (antihermitian!): $\quad$ (so $P \longmapsto [A, P]$)

$$A = i\sigma_x, i\sigma_y, i\sigma_z$$

Boosts (antihermitian!): $\quad$ (so $P \longmapsto [A, P]$)

$$X_L = L\sigma_x, \quad X_{iL} = L\sigma_y, \quad D_L = L\sigma_z$$

$$\boxed{\mathfrak{so}(3,1) \cong \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2,\mathbb{C}' \otimes \mathbb{C})}$$

Summary: 2×2 Magic Square

- The algebras in the 2×2 magic square are $\mathfrak{su}(2, \mathbb{K}' \otimes \mathbb{K})$.
- Each algebra is generated by the 2×2 matrices below, with $p \in \mathbb{K}' \otimes \mathbb{K}$ and $q \in \mathrm{Im}\mathbb{K} + \mathrm{Im}\mathbb{K}'$.

$$D_q = \begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix}, \qquad X_p = \begin{pmatrix} 0 & p \\ -\overline{p} & 0 \end{pmatrix}$$

Idea: rotations/boosts acting on $\mathbb{K}' \oplus \mathbb{K}$:

$$D_i = D_{1i};\ D_L = D_{UL};\ X_i = X_{iU};\ X_L = X_{1L}$$

- The remaining basis elements are of the form

$$D_{i,j} = \begin{pmatrix} i \circ j & 0 \\ 0 & i \circ j \end{pmatrix} = \frac{1}{2}\,[D_i, D_j]$$

where $i \circ j \doteq k$ over $\mathbb{H}$, but stands for nesting over $\mathbb{O}$.

Summary: 3×3 Magic Square

- The algebras in the 3×3 magic square are $\mathfrak{su}(3, \mathbb{K}' \otimes \mathbb{K})$.
- Each algebra is generated by the 3×3 matrices below, with $p \in \mathbb{K}' \otimes \mathbb{K}$ and $q \in \mathrm{Im}\mathbb{K} + \mathrm{Im}\mathbb{K}'$.

$$D_q = \begin{pmatrix} q & 0 & 0 \\ 0 & -q & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_q = \begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & -2q \end{pmatrix}, \quad X_p = \begin{pmatrix} 0 & p & 0 \\ -\overline{p} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Y_p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & -\overline{p} & 0 \end{pmatrix}, \quad Z_p = \begin{pmatrix} 0 & 0 & -\overline{p} \\ 0 & 0 & 0 \\ p & 0 & 0 \end{pmatrix}$$

- The remaining basis elements ~~are~~ can be chosen to be of the form

$$D_{i,j} = \begin{pmatrix} i \circ j & 0 & 0 \\ 0 & i \circ j & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $i \circ j \doteq k$ over $\mathbb{H}$, but stands for nesting over $\mathbb{O}$. **TRIALITY!**

Guiding Principle #2

The 3×3 structure is broken to 2×2.

$$\mathcal{P} = \begin{pmatrix} P & \theta \\ \theta^\dagger & n \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{P} \longmapsto \mathcal{M}\mathcal{P}\mathcal{M}^{\dagger-1} \implies P \longmapsto MPM^\dagger,\ \theta \longmapsto M\theta$$

$$\mathcal{P} \longmapsto [\mathcal{A}, \mathcal{P}] \implies P \longmapsto [A, P],\ \theta \longmapsto A\theta$$

Idea: Vector and spinor actions at same time!

Example: $\mathcal{M} \in E_6$, $\mathcal{A} \in \mathfrak{e}_6$, $\mathcal{P} \in H_3(\mathbb{O})$

Guiding Principle #2

The 3×3 structure is broken to 2×2.

$$\mathcal{P} = \begin{pmatrix} P & \theta \\ -\theta^\dagger & n \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{P} \longmapsto \mathcal{M}\mathcal{P}\mathcal{M}^{\dagger -1} \quad \Longrightarrow \quad P \longmapsto MPM^\dagger, \; \theta \longmapsto M\theta$$

$$\mathcal{P} \longmapsto [\mathcal{A}, \mathcal{P}] \quad \Longrightarrow \quad P \longmapsto [A, P], \; \theta \longmapsto A\theta$$

Idea: ~~Vector~~ **Adjoint** and spinor actions at same time!

Example: $\mathcal{M} \in E_6, \mathcal{A} \in \mathfrak{e}_6, \mathcal{P} \in \mathfrak{e}_6$

Commutators

$$2+1 \Longrightarrow \mathfrak{e}_8 = \text{adjoint} + \text{spinors}$$

Adjoint action (commutators of rotations/boosts):

$$\mathfrak{so}(12,4) \longleftrightarrow X_q, D_p, D_{p,q}$$

$$D_i = D_{1i}; \quad D_L = D_{UL}; \quad D_{i,j} = D_{i,j}$$
$$X_i = X_{iU}; \quad X_L = X_{1L}$$

Example: $[D_i, X_1] = [D_{1i}, X_{1U}] = 2X_{iU} = 2X_i$

Spinor action (possibly nested matrix multiplication):

$$\text{spinors} \longleftrightarrow Y_p, Z_q$$

$$Y_p + Z_q \longleftrightarrow \begin{pmatrix} -\overline{q} \\ p \end{pmatrix}$$

Example: $[D_i, Y_j] = -Y_k$

Subalgebras

- All algebras in both magic squares are subalgebras of $\mathfrak{e}_8$!
- $\mathfrak{e}_{8(-24)} = \mathfrak{so}(12,4) \oplus 128$.
- The 128 is a Majorana–Weyl representation of $\mathfrak{so}(12,4)$.
- The 128 contains spinor reps of each 2×2 algebra.

Spinors

- Decomposition from 3×3 to 2×2 works similarly throughout the magic square.
- In each case, get subalgebra of $\mathfrak{so}(12,4)$ and appropriate (e.g. Weyl) spinor representation(s).
- No triaility in associative (sub)cases! (Need (3,3) components!)

Example:

$$\mathfrak{su}(3, \mathbb{H} \otimes \mathbb{C}') \cong \mathfrak{a}_{5(-7)} = \mathfrak{so}(5,1) \oplus \mathfrak{su}(2) \oplus \mathfrak{so}(1,1) \oplus 2 \times 8$$

Guiding Principle #3

All representations live in $\mathfrak{e}_8$!

$$\mathfrak{e}_{8(-24)} = \mathfrak{so}(12,4) \oplus \text{spinors}$$

- "Actors" and "Actees" live in same space.

Albert Algebra

Albert algebra: 3×3 *Hermitian* matrices $\mathcal{A}$ over $\mathbb{O}$.
The Albert algebra is the minimal representation of $\mathfrak{e}_6$.

$$\mathfrak{e}_{8(-24)} = \mathfrak{e}_{6(-26)} \oplus 6 \times 27 \oplus \mathfrak{sl}(3, \mathbb{R})$$

- The 6 of $\mathfrak{sl}(3, \mathbb{R})$ are "color labels": $\{I \pm IL, J \pm JL, K \pm KL\}$.
- Each 27 of $\mathfrak{e}_6$ *must be* an Albert algebra!
- $(K \pm KL)\mathcal{A}$ is *anti-Hermitian* over $\mathbb{O}' \otimes \mathbb{O}$ – and hence in $\mathfrak{e}_8$!
- Over $\mathbb{O}$, $(K \pm KL)\mathcal{I}$ is nested; really $\sim G_{K \pm KL} \in \mathfrak{g}_2'$.

[Dray, Manogue, Wilson (2023): A New Division Algebra Representation of E_6]

Labels

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\{1, U\}$	$\{\ldots, k\}$	$\{\ldots, i, j\}$	$\{\ldots, i\ell, j\ell, k\ell, \ell\}$
$\mathbb{C}'$	$\{\ldots, L\}$			
$\mathbb{H}'$	$\{\ldots, K, KL\}$			
$\mathbb{O}'$	$\{\ldots, I, IL, J, JL\}$			

New Description of $\mathfrak{e}_8$
(Wilson et al.)

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{f}_4$
$\mathbb{C}'$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{e}_{6(-26)}$
$\mathbb{H}'$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3)$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}'$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

- Everything is 3x3 “matrices” with two “labels”
- Ordinary matrices/commutators in quaternionic cases.
- Generalize commutators for double-labeled diagonal elements.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{so}(2)$	$\mathfrak{so}(3)$	$\mathfrak{so}(5)$	$\mathfrak{so}(9)$
$\mathbb{C}'$	$\mathfrak{so}(2,1)$	$\mathfrak{so}(3,1)$	$\mathfrak{so}(5,1)$	$\mathfrak{so}(9,1)$
$\mathbb{H}'$	$\mathfrak{so}(3,2)$	$\mathfrak{so}(4,2)$	$\mathfrak{so}(6,2)$	$\mathfrak{so}(10,2)$
$\mathbb{O}'$	$\mathfrak{so}(5,4)$	$\mathfrak{so}(6,4)$	$\mathfrak{so}(8,4)$	$\mathfrak{so}(12,4)$

Orthogonal Lie Algebras

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{f}_4$
$\mathbb{C}'$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{e}_{6(-26)}$
$\mathbb{H}'$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3)$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}'$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

+Spinors

- 2x2 Lie algebras are degree 2 in Clifford algebra.
- 2x2->3x3 adds spinor representations with appropriate Bott periodicity.

Type Structure

$$\begin{pmatrix} D & X & -Z^\dagger \\ -X^\dagger & \pm D & Y \\ Z & -Y^\dagger & 0 \end{pmatrix}$$

- *D*s must have both labels in the same division algebra. (We don't always write {1, *U*}).
- *X*s, *Y*s, *Z*s have one label in each division algebra.

SUMMARY

Lie algebras are real!
The 3×3 structure is broken to 2×2.
All representations live in $\mathfrak{e}_8$!

$$\mathfrak{e}_{8(-24)} = \mathfrak{so}(12,4) \oplus \text{spinors}$$

Albert algebras $\subset \mathfrak{e}_8$

- Wilson, Dray, and Manogue: An octonionic construction of E_8 ..., Innov. Incidence Geom. **20**, 611–634 (2023). arXiv.org:2204.04996
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- Dray, Manogue, and Wilson: A New ... Representation of E_7, (in preparation)