The Eigenvalue Problem for Quaternionic and Octonionic Matrices

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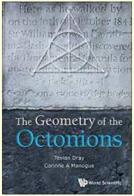
Book



The Geometry of the Octonions

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`http://octonions.geometryof.org/GO`

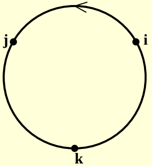
Real Numbers

$$\mathbb{R}$$

Quaternions

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$$

$$q = (x + yi) + (r + si)j$$



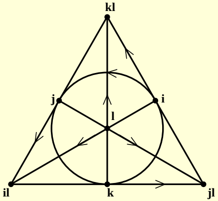
Complex Numbers

$$\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$$

$$z = x + yi$$

Octonions

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell$$



$$\boxed{i^2 = j^2 = \ell^2 = -1}$$

Cayley–Dickson (1919)

Noncommutative:

$$ji = -ij$$

Nonassociative:

$$(ij)\ell = -i(j\ell)$$

Norm:

$$|x|^2 = x\bar{x}$$

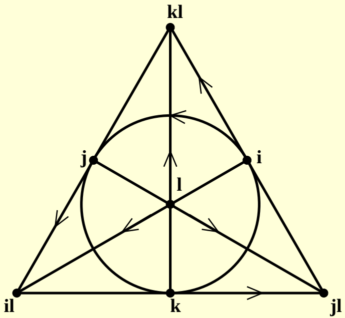
$$|x| = 0 \implies x = 0$$

Composition:

$$|xy| = |x||y|$$

Inverses (Division!):

$$x \neq 0 \implies x^{-1} = \bar{x}/|x|^2$$



The Standard Eigenvalue Problem

$$Av = \lambda v$$
$$(A^\dagger = A)$$

Reality: $\lambda \in \mathbb{R}$

Existence: $\exists n$ eigenvalues (counting multiplicity)

Orthogonality: $\lambda_1 \neq \lambda_2 \Longrightarrow v_1^\dagger v_2 = 0$

Orthonormal Basis: $\exists$ orthonormal basis of eigenvectors

Decomposition: $A = \sum \lambda_m v_m v_m^\dagger$

Reality: $(\mathbf{A}^\dagger = \mathbf{A} \implies \overline{\lambda} = \lambda)$

$$Av = \lambda v \quad \Longrightarrow \quad \overline{\lambda} v^\dagger v = (Av)^\dagger v = v^\dagger A v = v^\dagger \lambda v \neq \lambda v^\dagger v$$

$$Av = v\lambda \quad \Longrightarrow \quad \overline{\lambda}(v^\dagger v) \neq (Av)^\dagger v \neq v^\dagger (Av) \neq (v^\dagger v)\lambda$$

Orthogonality: $(\lambda_1 \neq \lambda_2 \implies \mathbf{v}_1^\dagger \mathbf{v}_2 = 0)$

$$Av_m = \lambda_m v_m \implies \lambda_1 v_1^\dagger v_2 = (Av_1)^\dagger v_2 \neq v_1^\dagger (Av_2) = \lambda_2 v_1^\dagger v_2$$

Theorem (Dray & Manogue 1998)

$v \in \mathbb{O}^3, \mathcal{A}^\dagger = \mathcal{A} \in \mathbf{h}(3,\mathbb{O}), \lambda \in \mathbb{R} \implies$

- $\exists$ **6** $(= 2 \times 3)$ *real eigenvalues* λ_m, *with* $\mathcal{A}v_m = \lambda_m v_m$;
- $(v_m v_m^\dagger) v_n = 0$ *for* $m \neq n$ *in the same "family".*

Example ($\lambda \neq \mathbb{R}$)

$$\mathbf{A} = \begin{pmatrix} 0 & -\ell \\ \ell & 0 \end{pmatrix}, \quad v = \begin{pmatrix} j \\ k\ell \end{pmatrix} \implies \mathbf{A}v = vi$$

The Right Eigenvalue Problem

$$Av = v\lambda$$
$$(A^\dagger = A)$$

Reality: Over $\mathbb{H}$, $\lambda \in \mathbb{R}$, but not over $\mathbb{O}$

Existence: 3×3 matrices over $\mathbb{O}$ have 2×3 real eigenvalues

Orthogonality: $\lambda_1 \neq \lambda_2 \Longrightarrow (v_1 v_1^\dagger) v_2 = 0$

Orthonormal Basis: $\exists$ 2 orthonormal bases of eigenvectors

Decomposition: $A = \sum \lambda_m (v_m v_m^\dagger)$ $(\times 2)$

Characteristic Equation

$$\mathcal{A} \in \mathbf{h}(3, \mathbb{O})$$

$\Longrightarrow$

$$\mathcal{A}^3 - (\mathrm{tr}\mathcal{A})\,\mathcal{A}^2 + \sigma(\mathcal{A})\,\mathcal{A} - (\det \mathcal{A})\,\mathcal{I} = 0$$

but

$$\lambda^3 - (\mathrm{tr}\mathcal{A})\,\lambda^2 + \sigma(\mathcal{A})\,\lambda - (\det \mathcal{A}) = r_m$$

- Matrix solves characteristic equation;
- Eigenvalues do not;
- $\exists\, 2$ "families" of eigenvalues.

Characteristic Operator

$$\mathcal{A} = \begin{pmatrix} x & a & \bar{c} \\ \bar{a} & y & b \\ c & \bar{b} & z \end{pmatrix}$$

$(v \in \mathbb{O}^3,\ q \in \mathbb{O})$

$$K[v] = \mathcal{A}\big(\mathcal{A}(\mathcal{A}v)\big) - (\mathrm{tr}\mathcal{A})\,\mathcal{A}(\mathcal{A}v) + \sigma(\mathcal{A})\,\mathcal{A}v - (\det\mathcal{A})v$$

$\Longrightarrow K$ diagonal $\longmapsto$

$$K[q] = c(b(aq)) + \bar{a}(\bar{b}(\bar{c}q)) - \Big(c(ba) + (\bar{a}\bar{b})\bar{c}\Big)q$$

"Family" structure of $\mathbb{O}$

(Dray, Manogue, & Okubo 2002)

$$\mathbb{T} = \langle 1, a, b, c \rangle \subset \mathbb{O} \qquad \longleftrightarrow \begin{pmatrix} x & a & \bar{c} \\ \bar{a} & y & b \\ c & \bar{b} & z \end{pmatrix}$$

$$\Phi = \mathrm{Re}(a \times b \times c) = \tfrac{1}{2}\,\mathrm{Re}\big(a(\bar{b}c) - c(\bar{b}a)\big)$$

$$= \mathrm{Im}(a) \cdot [\mathrm{Im}(b) \times \mathrm{Im}(c)] \qquad \text{(triple product)}$$

$$\alpha = [a, b, c] = (ab)c - a(bc) \qquad \text{(associator)}$$

$$K[q] = r_m q \iff q \in \mathbb{T}_m \subset \mathbb{O}; \quad r_m^2 - 4\Phi r_m - \alpha^2 = 0$$

$$\mathbb{T}_m = \mathbb{T} s_m; \quad s_m = \frac{r_m + 4\Phi + \alpha}{2(r_m + 2\Phi)}$$

$$\mathbb{O} = \mathbb{T}s_1 \oplus \mathbb{T}s_2 \qquad (s_1 + s_2 = 1)$$

$$\mathbb{T}_2 \equiv \mathbb{T}_1\alpha \qquad (\mathbb{T}_1 \perp \mathbb{T}_2)$$

$$x, y \in \mathbb{T}_m \Longrightarrow x\bar{y} \in \mathbb{T}$$

The Jordan Eigenvalue Problem

(Dray & Manogue 1999)

$$\mathcal{A} \in \mathbf{h}(3, \mathbb{O})$$
$$\mathcal{V} \circ \mathcal{V} = \mathcal{V}$$
$$\mathcal{A} \circ \mathcal{B} = (\mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A})/2$$
$$\mathcal{A} \circ \mathcal{V} = \lambda \mathcal{V}$$

Equivalent to right eigenvalue problem over $\mathbb{H}$! $(\mathcal{V} = vv^\dagger)$

$$(vv^\dagger) \circ (vv^\dagger) = (v^\dagger v)(vv^\dagger)$$

- usual characteristic equation
- $\lambda \in \mathbb{R}$
- Cayley–Moufang plane $(\mathbb{O}\mathbb{P}^2)$
- Solutions of 10-d Dirac equation!

Octonionic projections are quaternionic!

$(a, b, c \in \mathbb{O};\ x, y, z \in \mathbb{R})$

$$\mathcal{A} = \begin{pmatrix} x & a & \bar{c} \\ \bar{a} & y & b \\ c & \bar{b} & z \end{pmatrix}$$

$$\mathcal{A}^2 = \begin{pmatrix} x^2 + |a|^2 + |c|^2 & (x+y)a + \bar{c}\bar{b} & (x+z)\bar{c} + ab \\ (x+y)\bar{a} + bc & |a|^2 + y^2 + |b|^2 & (y+z)b + \overline{ac} \\ (x+z)c + \bar{b}\bar{a} & (y+z)\bar{b} + ca & |c|^2 + |b|^2 + z^2 \end{pmatrix}$$

$$\mathcal{A}^2 = \mathcal{A} \Longrightarrow ab = (1 - x - z)\,\bar{c} \Longrightarrow [a, b, c] = 0!$$

Application: Solutions to 10-d Dirac equation (octonionic) are in fact 6-d (quaternionic), leaving room for additional symmetry.

Simultaneous Eigenstates

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -\ell \\ \ell & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

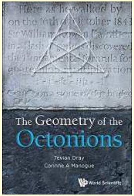
$$L_m\psi := -\frac{\hbar}{2}(\ell\sigma_m\psi)\ell$$

$$\psi = \begin{pmatrix} 1 \\ k \end{pmatrix} \Longrightarrow \begin{aligned} 2\,\mathbf{L_z}\psi &= \hbar\,\psi \\ 2\,\mathbf{L_x}\psi &= -\hbar\,\psi\,\mathbf{k} \\ 2\,\mathbf{L_y}\psi &= -\hbar\,\psi\,\mathbf{k}\ell \end{aligned}$$

"spin-up" is simultaneous eigenstate of L_x, L_y, L_z!
(but **eigenvalues** don't commute!)

SUMMARY

- Real eigenvalue problem over $\mathbf{h}(3, \mathbb{O})$ well understood;
- Always get decompositions into primitive idempotents;
- Splits octonions into two "almost quaternionic" subspaces!
- Jordan eigenvalue problem over $\mathbf{h}(3, \mathbb{O})$ well understood;
- Primitive idempotents are quaternionic! ($\mathbb{OP}^2$)
- Applications to physics: spin, Dirac equation...



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