The Geometry of Relativity

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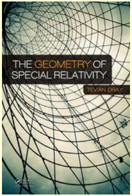
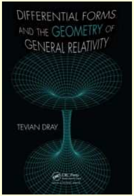
Books

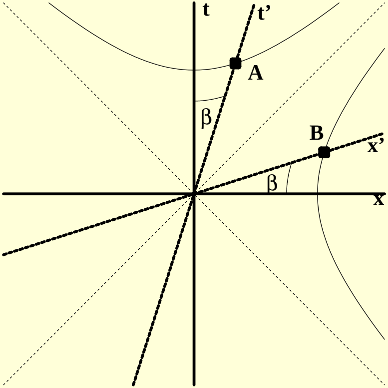


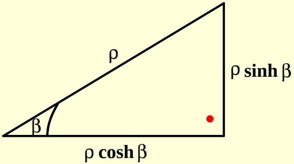
The Geometry of Special Relativity
Tevian Dray
A K Peters/CRC Press 2012
ISBN: 978-1-4665-1047-0
http://physics.oregonstate.edu/coursewikis/GSR

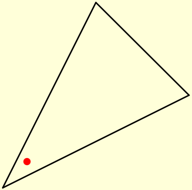


Differential Forms and the Geometry of General Relativity
Tevian Dray
A K Peters/CRC Press 2014
ISBN: 978-1-4665-1000-5
http://physics.oregonstate.edu/coursewikis/GDF
http://physics.oregonstate.edu/coursewikis/GGR

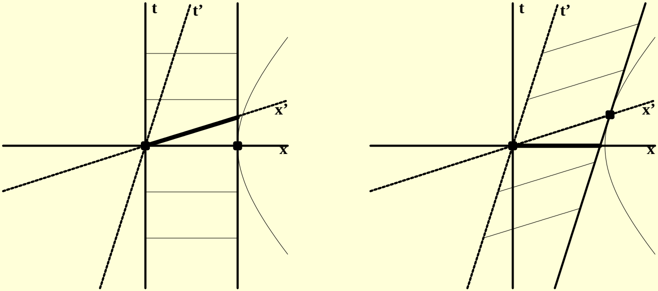
Trigonometry







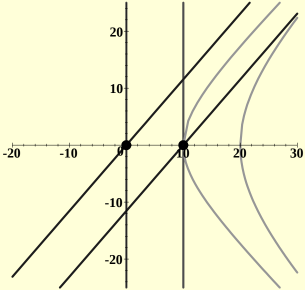
Length Contraction



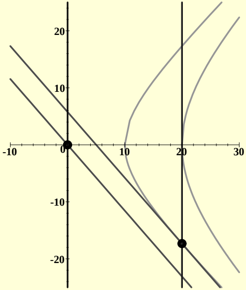
$$\ell' = \frac{\ell}{\cosh\beta}$$

Paradoxes

A 20 foot pole is moving towards a 10 foot barn fast enough that the pole appears to be only 10 feet long. As soon as both ends of the pole are in the barn, slam the doors. How can a 20 foot pole fit into a 10 foot barn?



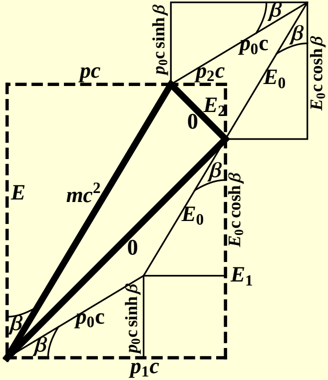
barn frame

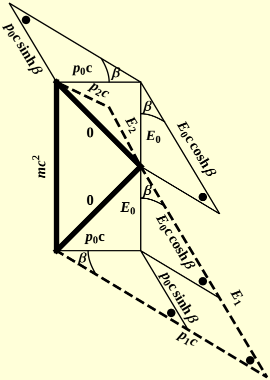


pole frame

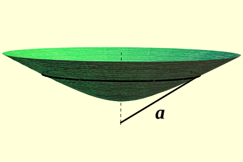
Relativistic Mechanics

A pion of (rest) mass m and (relativistic) momentum $p = \frac{3}{4}mc$ *decays into 2 (massless) photons. One photon travels in the same direction as the original pion, and the other travels in the opposite direction. Find the energy of each photon.* $[E_1 = mc^2,\ E_2 = \frac{1}{4}mc^2]$

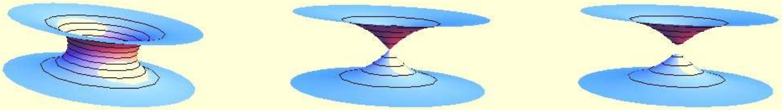




Line Elements

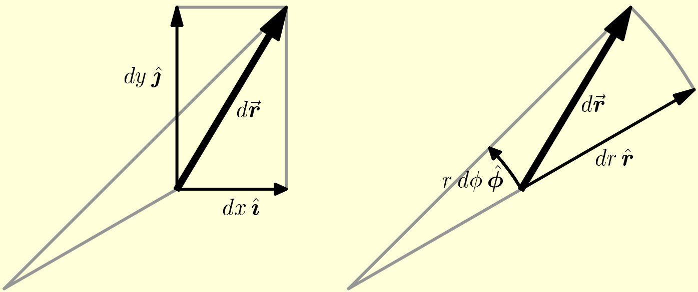


$dr^2 + r^2\, d\phi^2$ $\qquad$ $d\theta^2 + \sin^2\theta\, d\phi^2$ $\qquad$ $d\beta^2 + \sinh^2\beta\, d\phi^2$



Vector Calculus

$$ds^2 = d\vec{\mathbf{r}} \cdot d\vec{\mathbf{r}}$$



$$d\vec{\mathbf{r}} = dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{\jmath}} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}}$$

Differential Forms in a Nutshell ($\mathbb{R}^3$)

Differential forms are integrands: $(*^2 = 1)$

$$f = f \qquad \text{(0-form)}$$

$$F = \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \qquad \text{(1-form)}$$

$$*F = \vec{\mathbf{F}} \cdot d\vec{\mathbf{A}} \qquad \text{(2-form)}$$

$$*f = f\, dV \qquad \text{(3-form)}$$

Exterior derivative: $(d^2 = 0)$

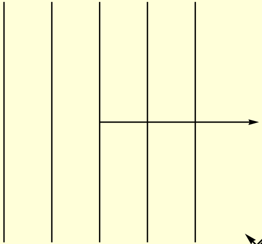
$$df = \vec{\nabla} f \cdot d\vec{\mathbf{r}}$$

$$dF = \vec{\nabla} \times \vec{\mathbf{F}} \cdot d\vec{\mathbf{A}}$$

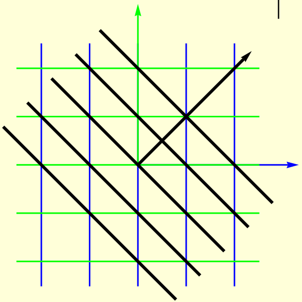
$$d{*F} = \vec{\nabla} \cdot \vec{\mathbf{F}}\, dV$$

$$d{*f} = 0$$

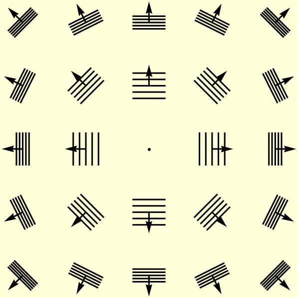
The Geometry of Differential Forms



dx



$dx + dy$



$r\,dr = x\,dx + y\,dy$

Geodesic Equation

Orthonormal basis: $d\vec{\mathbf{r}} = \sigma^i\,\hat{\mathbf{e}}_i$

Connection:

$$\omega_{ij} = \hat{\mathbf{e}}_i \cdot d\hat{\mathbf{e}}_j$$

$$d\sigma^i + \omega^i{}_j \wedge \sigma^j = 0$$

$$\omega_{ij} + \omega_{ji} = 0$$

Geodesics:

$$\vec{\mathbf{v}}\,d\lambda = d\vec{\mathbf{r}}$$

$$\dot{\vec{\mathbf{v}}} = 0$$

Symmetry:

$$d\vec{\mathbf{X}} \cdot d\vec{\mathbf{r}} = 0$$

$$\Longrightarrow \vec{\mathbf{X}} \cdot \vec{\mathbf{v}} = \text{const}$$

Einstein's Equation

Curvature:

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$

Einstein tensor:

$$\gamma^i = -\frac{1}{2}\,\Omega_{jk} \wedge *(\sigma^i \wedge \sigma^j \wedge \sigma^k)$$

$$G^i = *\gamma^i = G^i{}_j\,\sigma^j$$

$$\vec{\mathbf{G}} = G^i\hat{\mathbf{e}}_i = G^i{}_j\,\sigma^j\hat{\mathbf{e}}_i$$

$$\Longrightarrow d{*}\vec{\mathbf{G}} = 0$$

Field equation:

$$\boxed{\vec{\mathbf{G}} + \Lambda\, d\vec{\mathbf{r}} = 8\pi\vec{\mathbf{T}}}$$

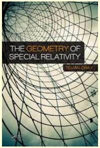
(vector valued 1-forms, not tensors)

Does it work?

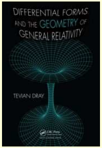
- I am a mathematician...
- There is no GR course in physics department.
  (I developed the SR course.)
- Core audience is undergraduate math and physics majors.
  (Many double majors.)
- Hartle's book:
  Perfect for physics students, but tough for math majors.
- My course: 10 weeks differential forms, then 10 weeks GR.
  (Some physics students take only GR, after "crash course".)

In this context: **YES!**

SUMMARY



http://relativity.geometryof.org/GSR
http://relativity.geometryof.org/GDF
http://relativity.geometryof.org/GGR



- Special relativity is hyperbolic trigonometry!
- Spacetimes are described by metrics!
- General relativity can be described without tensors!
- BUT: Need vector-valued differential forms...

THE END